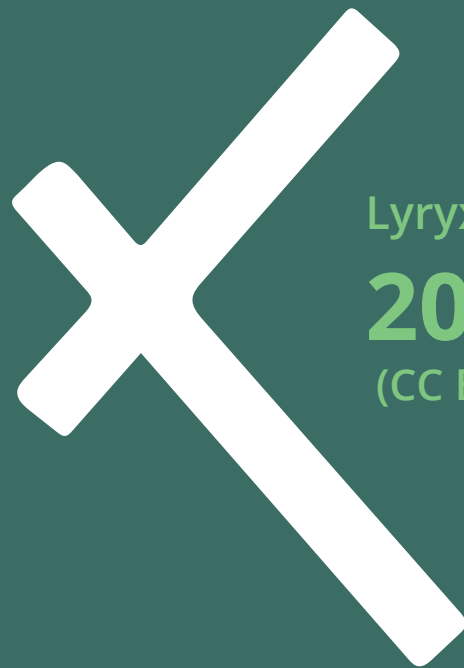


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Business Math: A-Step-by-Step Handbook

J. Olivier



Lyryx Version

2023-A-D

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BUSINESS MATH: A Step-By-Step Handbook

Version 2023 – Revision A-D

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CONTRIBUTIONS

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BUSINESS MATH: A Step-By-Step Handbook

Version 2023 – Revision A-D

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Preface

I have spoken with many math instructors and professors all across Canada. As educators, it is apparent that many of us are experiencing the same challenges with our students. These challenges include, but are not limited to:

- Mathematics has a negative perception amongst most students and is viewed as the "dreaded" course to take;
- It is commonly accepted that mathematics is one of the few courses where students make the comment of "I'm not very good at it" and people just accept it as truth; and
- The declining ability of our students with regards to fundamental mathematical skills and problem-solving ability.

Compared to when we grew up, our students now face a very different Canada. Today's society is extremely time-pressed and always on the go. Most students are not only attending college or university, but holding down full- or part-time jobs to pay their bills and tuition. There are more mature students returning from industry for upgrading and retraining who have families at home. There are increasing rates of cultural diversity and international students in the classroom. Many students do not go directly home at night and do their homework. Rather, they fit their schoolwork, assignments, and homework into their schedules where they can.

In addition to all of this, our students have grown up surrounded by technology at home, in school, and at work. It is the modern way of doing business. Microsoft Office is prevalent across most industries and companies, however LibreOffice is readily available and starting to compete with Microsoft Office. As technology continues to develop and prices become more affordable, we see more students with laptops, iPhones, and even iPads. Our students need us to embrace this technology which is a part of their life in order to help them become the leaders of tomorrow.

What Do We Need To Do?

I set out to write this textbook to answer the question, "What can we, as educators, do to help?" We can't change the way our students are or how they live. Nor can we change the skill sets they bring into our classroom. What we can do though is adapt our textbooks, resources, and the way we teach mathematics. After all, isn't it our job to find teaching strategies that meet with the needs of our students?

You may ask, 'what do students need in a textbook'? The answer requires us to listen to our students both in feedback and the questions that they pose. Term after term, year after year, do these questions sound familiar?

1. How do we approach the mathematical problem (I don't know where to start)?
2. What are the steps needed to arrive at the answer (how do I get there)?
3. Why is this material so repetitive (particularly made with respect to annuities)?
4. Why do we use algebraic symbols that have absolutely no relevance to the variable they represent?
5. How and why does a formula work?
6. Is there a quick way to locate something in the book when I need it?
7. How does this material relate to me personally and my professional career (what's in it for me)?
8. How does the modern world utilize technology to aid in math computations (does anyone do this by hand)?
9. Where are the most common errors so that I can try to avoid them?
10. Are there any shortcuts or "trade secrets" that can help me understand the concepts better?
11. How do all the various mathematical concepts fit together when we only cover each piece one at a time?
12. Is there any way to judge whether I conceptually understand the material?

The answer to each of these questions is incorporated into the content, structure, and features of this textbook. Students helped create this textbook. It is written to meet the needs of a twenty-first century student. Through the combination of all of the features of this textbook, it is my goal to enhance your classroom and increase your student success rates. I encourage you to provide feedback about your experiences and how we, as mathematical educators, can constantly improve our textbooks for our students. I look forward to hearing about your challenges and, most importantly, your novel solutions that help your students succeed. Please email Lyryx at info@lyryx.com with your suggestions, questions, comments, and input. Thank you in advance for your support and input!

Jean-Paul Olivier

About The Author

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Jean-Paul has been teaching Business and Financial Mathematics, as well as Business Statistics, Data Analytics, and Quantitative Methods since 1997. He is a dedicated instructor interested in helping his students succeed through multi-media teaching involving PowerPoints, videos, whiteboards, in-class discussions, readings, online software, and homework practice. He regularly facilitates these quantitative courses and leads a team of instructors. You may have met Jean-Paul at various mathematical and statistical symposiums held throughout Canada (and the world). He has contributed to various mathematical publications for the major publishers with respect to reviews, PowerPoint development, online algorithmic authoring, technical checks, and co-authoring of textbooks. This text represents Jean-Paul's first venture into solo authoring.

Jean-Paul resides in Winnipeg, Manitoba, Canada, along with his wife and two daughters. When he is not teaching, he loves vacationing in the warm climes of the south with his family in California, Florida, and Hawaii.

This book is dedicated to:

My wife, Pamela, who supports me in every way possible,

And to

My daughters, Gabrielle and Madison, who wanted Daddy to help others learn their numbers.

Textbook Features

How To Approach The Mathematical Problem (The PUPP Model)

Our students struggle with completing math exercises from beginning to end. They are unfamiliar with mathematical thought process and the techniques involved. A systematic approach is needed to help students learn how to think. This textbook's core design centers around a structured thinking process termed The PUPP Model (Plan, Understand, Perform, and Present). This process is found throughout the textbook in every guided example to help students develop a step-by-step problem-solving approach.

1. **Plan.** One of the most common difficulties is when students rush into calculations and have no clue what they are solving for! This step focuses and reminds students about what it is they are doing. The goal is to encourage students to recognize what question has been asked and what variable(s) need to be calculated.
2. **Understand.** It is critical to know what you are doing and how you are going to get there. Before performing calculations, this step urges students to think through the entire process required in order to logically arrive at the solution. There are two critical elements:
 - i. **What You Already Know.** Students must read through the question and assign the correct values to appropriate variables. If necessary, diagrams such as timelines are drawn to help visualize the information being provided and what to do with that information.
 - ii. **How You Will Get There.** Everybody needs a roadmap. This step focuses students on describing the steps, procedures, and formulas required to arrive at the solution.
3. **Perform.** This stage is all about the mathematics. Students execute their roadmap and perform the needed calculations and procedures to arrive at the root(s) of the problem.
4. **Present.** The final part of the model encourages students to express their solution(s) in the context of the question. It's not good enough just to calculate a number. They have to understand what it means and how it fits into the situation. This also makes it easier to see if the solution makes sense. If not, there is a higher chance of spotting an error. Additionally, visual charts and graphs assist students in understanding how the various numbers fit into the cohesive puzzle.

What Are The Steps Needed To Arrive At The Final Answer? (How It Works)

Have you sometimes known what the destination is but you just couldn't figure out how to get there? Students commonly experience this problem in financial mathematics, particularly when questions become grander in scope. Integrated into the PUPP Model is another core feature of this textbook called "How It Works". In this section, students can find:

1. Detailed, step-by-step sequential procedures to address common business mathematical problems.
2. Handy reference flowchart guides that summarize the key steps required to arrive at the solutions.
3. Where possible, quick and simple examples to further illustrate the procedure.

In every section of the textbook, the "How It Works" is introduced. These steps are then utilized in all guided examples to assist students in taking a problem from beginning to end.

The Repetitiveness Of Annuities

I have heard from instructors, professors, textbook reviewers, and students alike the same concerns about the repetitiveness in teaching the four types of annuities. It is no secret that there are only minor differences between the annuity types. Many have expressed that their courses are highly time-constrained. Other textbooks address most or all annuity types separately. Each annuity type is then solved for the five common unknown variables. This can result in up to twenty different lectures.

An overwhelming number of educators have indicated that they find it better to teach the four annuity types up front. Once students understand how to recognize the annuity type and the key differences, students can solve any annuity for any variable!

In this textbook, the four annuity types are thoroughly introduced and discussed, followed by sections discussing the solving for each formula variable. Key knowledge is offered in a single chapter. This textbook also utilizes only four annuity formulas - two for ordinary annuities and another two for annuities due. Incorporated into the design of these formulas are the mathematics needed to

address both simple and general formats. To reduce the chance of error, students are not required to use combinations of annuity formulas nor substitute solutions of one formula into another.

When solving for other annuity variables, students use their algebra skills to substitute and rearrange the four formulas. There are no long lists of formulas that are nothing more than algebraic rearrangements of existing formulas. There are two formulas for ordinary annuities and two formulas for annuities due. Clean and simple.

Relevant Algebraic Symbols

There are two key student concerns here: irrelevant algebraic symbols and algebraic symbols with multiple usages. Both of these concerns are addressed through symbol choices and formulas utilized in this textbook.

Nothing is more frustrating for students than algebraic symbols that have nothing to do with the variable. For example, it is common to let "j" represent the nominal interest rate. To students this is like apples and oranges. What makes more sense is to use relevant algebraic symbols that help students to understand the formulas being presented. Doesn't IY for interest rate per year make much more sense than "j"? This text utilizes representative singular (such as P for profit) or plural (such as PMT for annuity payments) symbols to further student understanding.

Additionally, having identical algebraic symbols with different interpretations is very confusing for students. For example, the symbol for future value (FV) is used to represent the future value of a single payment, an ordinary annuity, or an annuity due. However, many questions require students to execute combinations of single payments and annuities. This requires multiple appearances of the same FV symbol each requiring different computational formulas. Except where industry standards exist, this textbook differentiates algebraic symbols so that each symbol has only one meaning. Thus FV is for the future value of a single payment, FV_{ORD} is the future value of an ordinary annuity payment stream, and FV_{DUE} is the future value of an annuity due payment stream. This means less confusion for our students.

How And Why A Formula Works

We all agree that understanding a formula and not just memorizing it goes a long way in business mathematics. In this respect, this textbook offers annotated and detailed formulas allowing students to simultaneously visualize the formula as a whole and by its various components. Instead of cross-referencing variable definitions and concept explanations, each part of the formula is visually addressed and the elements explained as a cohesive whole.

Handbook Design

It is difficult to get students to look at and use their textbook. An easy reference handbook design forms a cornerstone to aiding our students in two ways.

1. The book offers a friendly, strong visual appeal. It is not just page after page of endless and uninviting formulas. Instead, students find modern graphics, pictures, and an appearance designed to encourage their interaction.
 2. Unlike other textbooks, every section and chapter follows an identical structure. The goal is to make material easy to find. Throughout the book there are constant guides and locators.
 - In addition to the table of contents, every chapter introduction and summary provides key topic summaries.
 - If a student needs to know the steps in solving a problem, students always find the details in the "How It Works" section. If they just need a summary or reminder, they can always find a nutshell summary at the end of each chapter.
 - If students need calculator function instructions, they always find it in the Technology section of each chapter summary.
- Call the book a reference manual if you will. It is about students finding what they need when they need it made easy.

Relevance

As educators, we constantly hear, "...and when will I ever use that in my life?" This textbook demonstrates to students both personal and professional applications in discussions, guided examples, case studies, and even their homework questions.

1. On a personal level, students are shown realistic scenarios, companies, and products that they commonly are exposed to. It is always easier to learn something when students can relate to it.

2. On a professional level, this textbook demonstrates business situations in different business fields so that students can see applicability regardless of their chosen career path. Whether students are planning on becoming accountants or marketers, there are applications to be found in this textbook.

This book offers to your student:

- Math organized by business function using products and companies that are familiar to students.
- Both business and personal application sections ranging from HELOCs and Promissory Notes to RRSP planning and purchasing vehicles.

Integration Of Technology & Resources

Texas Instruments BAI+ Calculator

One of the most widely used calculators in Canada, the Texas Instruments BAI+, is illustrated and fully integrated into this textbook.

For those instructors and professors utilizing this calculator, step-by-step calculator button solutions are provided at the bottom of relevant PUPP model examples. Improving this integration is the usage of similar formula and calculator symbols that allows students to make the easy transfer between algebraic variables and calculator buttons.

If calculators are not used in your course or you use an alternate calculator, the layout and design of the textbook strategically places these calculator solutions in an inconspicuous manner in the guided solutions. Thus, they do not interfere with your students' coverage of the material and are easily skipped.

Structured Exercises

Every section, as well as chapter end, has approximately 18 practice questions for the students. Students comment that having more than that appears daunting and discourages doing their homework. Imagine reading a section and finding eighty homework questions at the end! Solutions to all exercises are found at the back of the book. The exercises are broken into three sections:

1. **Mechanics.** This section focuses on fundamental mathematical skills and practicing of the formulas. These questions are straight-forward and may appear in a table format that encourages students to become more familiar with the formulas and how to solve them. After completing this section, your students should have the basic mathematical skills however will not be able to demonstrate adequate competency in the subject matter without continuing to the next section.
2. **Applications.** Once students has mastered the mathematics, it is time to put their skills to work and solve various business and personal applications. The questions in this section are in word format and require the students to execute their problem-solving skills. After completing this section, students will be able to demonstrate adequate competency in the subject matter.
3. **Challenge, Critical Thinking, and Other Applications.** This section raises the difficulty bar and asks challenging questions. Solutions may require multi-stage approaches or integration of different concepts. Strong problem-solving skills are needed. Questions may also take students in new directions by having them interrelate different concepts. Students completing this section demonstrate a high level of competency in the subject matter.

Lyryx Integration

This book is partnered with Lyryx Learning, a Canadian publisher of open textbooks and developer of educational software including online formative homework. Delivered in a positive learning environment, students can repeatedly try randomly generated questions, each time the answers are carefully analyzed and personalized formative feedback is provided to guide them in their learning.

Students are awarded partial marks reflecting not only their efforts on correct answers, but also for “consistent” answers which are incorrect but correctly deduced from previous incorrect answers; this avoids penalizing the student again and helps them to identify exactly where the actual error was committed. Moreover students feel no threat in attempting a question again, as each time only a better mark is recorded, encouraging them to learn by practice.

A proven suggestion is to create a series of 10 or so assignments for a normal semester with a final grade weight of around 10%-20%, which is enough incentive to encourage the students to complete this necessary practice. You may want to only count the best 9 allowing for student schedule flexibility, but on the other hand leaving them open for 2 weeks or so already provides much flexibility to complete them.

Your role as an instructor is to be enthusiastic about Lyryx! Make it a part of your course and your lectures. Bring it into your classroom. Take advantage of its features. Use the software to help identify student problem areas and generate practice questions for in-class hands-on activities. Review your student performance before tests and use Lyryx to pinpoint key review topics.

Lyryx not only provides excellent online assessment, but also timely support and services to students and instructor all along the course (see Page ii), visit them directly at <http://lyryx.com> or contact them by email info@lyryx.com.

Identification of Common Errors & Finding Shortcuts

Built-in to the design of this book are the standard elements that make students aware of the common errors (Things to Watch Out For) as well as some tips and tricks (Paths To Success) along the way. What makes these different in this book is that they are an integrated component of the textbook and not separated into text boxes that are generally ignored by our students.

How Do All the Pieces Fit Together?

Helping your students integrate their concepts is accomplished in two main ways:

1. **End of Chapter Exercises.** These questions allow students to explore the various concepts outside of their chapter section. Where possible, various concepts are combined in questions that allow students to integrate their acquired knowledge.
2. **Short Case Studies.** A fictional company called Lightning Wholesale is used throughout this textbook. Almost every chapter features a short case study showing how a single business applies all the various mathematical concepts into its daily operations. Students can learn how various chapter elements combine together into business scenarios.

Am I Understanding the Material? (Give It Some Thought)

The students have read the material, but did they take the time to understand the material? Throughout the textbook, after concepts have been introduced, is a question and answer section called "Give It Some Thought". This is not a mathematical computation section. It features conceptually based questions whereby no calculations are required. Instead, this section offers students periodic checkpoints to determine if they are understanding the relationships of various variables and the concepts being presented.

A Note on Rounding Rules

For some, the method by which a textbook performs rounding in its calculations plays a very important role in the delivery of business math. This textbook addresses the issue of rounding in as simplified manner as possible.

For the instructor:

- The rules utilized in this textbook follow a very basic premise: no rounding until the final answer is achieved unless there is a logical reason to round an interim solution.
- All interim solutions display up to six decimal places, limited only by the graphical display capabilities of the BAII+ calculator. Thus, what the student sees on the screen of the calculator is exactly what is presented in the examples.
- Unless there is a standard for rounding the final answer (such as currency), the solution displays up to six decimals in decimal format and four decimals in percentage format.

For the student:

- An introduction to the global rounding rules is presented in the appendix to Chapter 1.
- The basic principles of rounding are discussed in Chapter 2.
- Any nuances, textbook standards, or industry standards are presented as needed throughout the book as various topics are introduced. These key rules are summarized in Appendix C.

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Chapter 1: Succeeding In Business Mathematics

(How To Use This Textbook)

Math!

You may really dread the “M” word. Not only that, but this course is business mathematics, which some people say is even harder. Is that true, or is it just a rumour?

Take a look at your world from a different perspective—notice that math is everywhere! Before you went to bed last night, you performed some math and solved a complex algebraic equation (and you probably didn’t even realize it!). That equation helped you to figure out what time to get up this morning so that you could arrive at your destination on time. You factored in variables such as the required arrival time, commuting time (including the unknown variables of traffic and weather delays), morning routine time, and even snooze button time. When you solved this complex algebraic equation, you then set your alarm clock.

From the moment you woke up this morning, numbers have appeared everywhere. Perhaps you figured out how many calories are in your breakfast cereal. Maybe your car’s gas tank was low, so you calculated whether you had enough gas for your commute today. You checked your bank account online to see if you had enough money to pay the bills. At Tim Hortons you figured out what size of coffee you could buy to go with your donut using only the \$1.80 in change in your pocket. I hope you get my point.

Students ask, “But why do I want to learn about business math?” The answer is simply this: money. Our world revolves around money. We all work to earn money. We need money to purchase our necessities, such as homes, transportation, food, and utilities. We also need money to enjoy the pleasures of life, including vacations, entertainment, and hobbies. We need money to retire. And businesses need money to survive and prosper.

Quite simply, business math teaches you monetary concepts and how to make smart decisions with your money.

Outline of Chapter Topics

- 1.1: What Is Business Math?
- 1.2: How Can I Use the Features of This Book to My Advantage?
- 1.3: Where Do We Go from Here?

1.1: What Is Business Math?

Business math is the study of mathematics required by the field of business. By the fact that you are reading this textbook, you must be interested in a business field such as accounting, marketing, human resources, or economics. Regardless of your path, you cannot avoid dealing with money and numbers.

Both personally and in your career you certainly use elementary arithmetic such as addition, subtraction, multiplication, and division. However, there is a whole field of mathematics that deals specifically with money. You will work with taxes, gross earnings, product prices, and currency exchange; you will be offered loans, lines of credit, mortgages, leases, savings bonds, and other financial tools. Do you know what these are and how these financial tools can maximize your earnings and minimize your costs? Do you have what it takes to execute smart monetary decisions both personally and for your business? Do you know how interest works and how it gets calculated? If you can answer “yes” to these questions, then you are already off to a great start. If not, by the end of this textbook you will have a better understanding of all of these topics and more.

How Do I Learn about Business Math?

Let’s be realistic. In some areas of life and business, you can achieve a reasonable degree of understanding just by reading. However, reading about business mathematics without doing it would be disastrous. To succeed, you must follow a structured approach:

1. Always read the content prior to your professor covering the topic in class.
2. Attend class, ask questions, and explore the topic to advance your understanding.
3. Do the homework and assignments—you absolutely must practice, practice, and practice!
4. Seek help immediately when you need it.

Learning mathematics is like constructing a building. Each floor of the building requires the floor below it to be completed first. In mathematics, each section of a textbook requires the concepts and techniques from the sections that preceded it. If you have trouble with a concept, you must fix it NOW before it causes a large ripple effect on your ability to succeed in subsequent topics.

So the bottom line is that you absolutely cannot replace this approach—you must follow it.

1.2: How Can I Use The Features Of This Book To My Advantage?

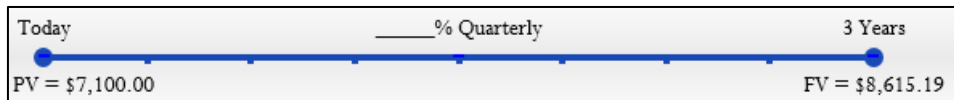
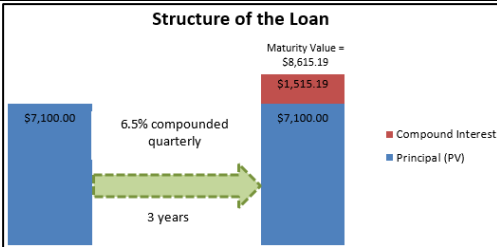
I wrote this textbook with the help of student input specifically for students like you. It is designed to address common student needs in mathematics. Let’s examine each of those needs and learn how you benefit from the features in this textbook.

I Don't Know How to Solve a Math Problem

Do you find it difficult to start a math problem? What do you do first? What is the next step? The reason this is challenging to some students is that they have never been shown problem-solving techniques. This textbook extensively uses an author-created design called the PUPP model in all guided examples. Successfully working through a problem involves four steps: Plan, Understand, Perform, and Present (PUPP). The figure on the next page shows the PUPP model at work. This model provides a structured problem-solving approach that will aid you not only in mathematics but in any problem-solving situation you may encounter.

Example 9.5A: Straightforward Interest Rate Calculation—FV and PV Known

When Sandra borrowed \$7,100 from Sanchez, she agreed to reimburse him \$8,615.19 three years from now including interest compounded quarterly. What interest rate is being charged?

Plan	Find the nominal quarterly compounded rate of interest (IY).														
Understand	What You Already Know Step 1: The present value, future value, term, and compounding are known, as illustrated in the timeline. CY = quarterly = 4 Term = 3 years 	How You Will Get There Step 2: Calculate N using Formula 9.2. Step 3: Substitute into Formula 9.3 and rearrange for <i>i</i>. Step 4: Substitute into Formula 9.1 and rearrange for IY.													
	Step 2: $N = 4 \times 3 = 12$ Step 3: $\\$8,615.10 = \\$7,100(1 + i)^{12}$ $1.213394 = (1 + i)^{12}$ $1.213394^{\frac{1}{12}} = 1 + i \quad 0.016249 = i$ Step 4: $0.016249 = \frac{IY}{4} \quad IY = 0.064996 = 0.065 \text{ or } 6.5\%$	Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>12</td><td>Answer: 6.499709</td><td>-7100</td><td>0</td><td>8615.10</td><td>4</td><td>4</td></tr></table>	N	I/Y	PV	PMT	FV	P/Y	C/Y	12	Answer: 6.499709	-7100	0	8615.10	4
N	I/Y	PV	PMT	FV	P/Y	C/Y									
12	Answer: 6.499709	-7100	0	8615.10	4	4									
Perform															
Present	Sanchez is charging an interest rate of 6.5% compounded quarterly on the loan to Sandra. 														

To understand each of these steps, let's plan a vacation.

- Plan.** If you are heading out on vacation, the first thing you want to know is where you are going! Otherwise, how would you get there? In mathematics, you need to figure out what variable you are solving for before you can proceed. This step focuses your problem solving on the end goal. Many students mistakenly rush through this step and start punching numbers on a calculator or plugging numbers into formulas without understanding what it is they are doing. Spend plenty of time on this step, because the remaining three steps do not help at all if you solve for the wrong variable!
- Understand.** In planning your vacation, once you figure out where you want to go you need to gather information about your destination. There are things you already know, but there are also things that you do not know, so you need a plan to figure these out. This step of the PUPP model helps your problem solving in two ways:
 - First, you need to assess what information has already been provided to you. You need to assign values to variables. You may need to draw diagrams to understand how all the numbers fit together.
 - Second, you create a plan by which you will solve your problem. Your roadmap identifies how you go from Point A (some variable is unknown) to Point B (knowing what the variable is).
- Perform.** Once your vacation is planned, all that is left is to go on vacation! In other words, execute your plan. This step involves all of the mechanics and computations of your roadmap that you developed in the previous step.
- Present.** After you return from vacation, you tell everyone about what you just did. In mathematics, this means you must explain solutions in the context of the question asked. If you just calculated $x = \$700$, what exactly does that represent and how should it be interpreted? Calculating a solution without understanding it is not good enough. A further benefit of this last step is that you can detect errors more easily when you have to explain the solution. For example, read this presentation statement: "If an investor puts \$1,000 into a savings account that earns 10% interest, I have calculated that

after one year the investor has \$700 in her savings account.” Did that make sense? If an account is earning interest, the balance should have become bigger, not smaller! Clearly, the solution of $x = \$700$ must be an error!

Once you repeatedly practise this model and ingrain it in your thinking process, you have a structured approach to solving not just math problems but any life problem. When you are first learning the PUPP model you will need to write out every step of the model. As you keep practising, you'll find that the model becomes a way of thinking. You can never skip one of the steps, but you won't have to write them all out anymore because you'll execute some steps with rapid-fire thought!

I Can't Find What I Want When I Need It

When practising your business math through homework and assignments, you will need to go back to look up various formulas or techniques or to review concepts. The challenge of most textbooks is in finding what you need when you need it buried in the vast pages of the textbook.

Browse the table of contents of this book and notice the functional layout. This textbook incorporates a handbook design that aids in finding information. When you need to look something up, you will notice the following:

- Every part of every section in every chapter is designed and laid out using an identical structure. While not every part is required in every section, the order is always as follows:
 - Concept introduction
 - Concept fundamentals and characteristics
 - Formula development and explanation
 - Steps and procedures required to solve problems
 - Important concept elements and technology use
 - Common pitfalls and shared tips
 - Conceptual understanding questions
 - Guided examples
 - Homework exercises
- The start of every chapter includes a miniature table of contents allowing you to locate what you need in the chapter without having to flip back to the book's main table of contents.
- End-of-chapter summaries deliver exactly that—each chapter in a nutshell. They contain key concept reviews, key terms, algebraic symbol definitions, formula summaries, and technology discussions on calculator usage. Links to other chapters are also provided as needed. If you need to look something up, this is the place to start!

Are There Any Steps to Help Me Remember?

Students have always asked if they need to follow a logical sequence of steps to solve a problem. The answer, of course, is YES! A section called "How It Works" helps you with this by providing step-by-step procedures, techniques, and formulas you need to solve the problem.

Can This Book Help Me Understand the Formulas Better?

You can't avoid formulas—this is math, after all. What helps, though, is to understand what the formulas do and how each component comes together to produce the solution. If you can understand, you do not need to memorize! This book lays formulas out visually, clearly labelling all symbols. In addition, each component is fully explained (see below). You are not required to cross-reference paragraph-style explanations to formulas, as the visual layout makes it all clear. The goal is for you to understand what exactly the formula does and how, so that you understand enough to recognize when solutions make sense or might be wrong.

N is Net Price: The **net price** is the price of the product after the discount is removed from the list price. It is a dollar amount representing what price remains after you have applied the discount.

Formula 6.1 – Single Discount:
$$N = L \times (1 - d)$$

L is List Price: The **list price** is the normal or regular dollar price of the product before any discounts. It is the **Manufacturer's Suggested Retail Price (MSRP)** - a price for a product that has been published or advertised in some way), or any dollar amount before you remove the discount.

d is Discount Rate: The discount rate represents the percentage (in decimal format) of the list price that is deducted.

Additionally, you will not find any of those archaic algebraic symbols in this book. Representative symbols make the algebra easier to remember and understand. Those ancient mathematicians just loved to speak Greek (literally!), but this text speaks modern-day English and understands that we have technology here in the twenty-first century! We'll use symbols like N for net price, L for list price, P for profit, and E for expenses, letters that actually remind you of the variable they stand for!

Does Anyone Actually Do Any of This in the Real World?

As you read through the guided examples and do your homework, pause to consider how the question applies to you. Does it demonstrate a business application you can use to be a smarter business professional? How can you extend the question to similar situations? This textbook shows you examples from both business and personal situations that you either have already encountered or probably will encounter in the future. (Remember: Math is all around you!)

- **Business.** Why do business programs require a business math course in the first term or two? How is it relevant to your future? This textbook provides numerous examples of applications in marketing, finance, economics, accounting, and more. Once you complete this textbook, you should see clearly how business mathematics is relevant and important to your chosen future career.
- **Personal.** You will see companies and products from your everyday life. Real-world situations show you how to put a few extra dollars in your pocket by making smart mathematical choices. What if you could save \$100 per month simply by making smarter choices? Over 40 years at a very conservative rate of interest, that is approximately \$100,000 in your pocket. You will also find life lessons. Are you ready to start planning your RRSP? Would you like to know the basics on how to do this? If you are thinking of buying a home, do you know how the numbers work on your mortgage? To learn how to buy a car at the lowest cost, read on!

Are There Any Secrets?

There are never any secrets. However, I will point out some tricks of the trade and commonly made mistakes throughout this book.

TOP SECRET

1. **Paths To Success.** This feature is designed to make you feel like part of the “in” crowd by letting you take advantage of shortcuts or tricks of the trade, such as some easy way to remember a particular technique or formula. It helps you see how some of the math could be made a little simpler or performed in a different way.
2.  **Things to Watch Out For.** Awareness of common pitfalls and sources of error in mathematics reduces your chances of making a mistake. A reminder also helps when important concepts are about to be used, reused, or combined in a novel way. It gives you a “heads up” when needed.

Is There Any Way to Know That I'm "Getting It"?

Mastering business math requires two key skills:

1. Executing the required techniques and calculations
2. Understanding and successfully integrating various mathematical concepts. (For example, when a variable changes, can you estimate how this affects the final result?)



The second skill is much harder to acquire than the first. A feature called **Give It Some Thought**, poses scenarios and questions for which no calculations are required. You need to visualize and conceptualize various mathematical theorems. Ultimately, you perform a self-test to see if you are “getting it.”

In Today's World, Who Does This By Hand Anymore?

That is a perfectly valid question. You are right: Most people do not do this work by hand and instead use mathematical tools such as calculators, spreadsheet software, and apps. However, as the saying goes, you must learn to walk before you can run. The learning of mathematics requires pencil and paper first. Once you master these, you can then employ technology to speed up the process. This basic version textbook incorporates one technology to aid you in your application of business mathematics:

1. **The Calculator.** Integrated throughout this book are instructions for one of the financial calculators most widely used in Canada—the Texas Instruments BAII Plus. Calculator functions with step-by-step button sequences are presented with guided examples illustrating how you solve the math problem not only by hand but also using the financial calculator.

Is There Any Way I Can Assess My Ability?

At the end of almost every section of this book you will find approximately 18 practice questions covering the concepts specifically introduced in that section, with final solutions listed to all questions at the end of this textbook. These questions are divided into three categories to help develop your mathematical skills and assess your abilities:

1. **Mechanics.** This group of questions focuses on fundamental mathematical skills and formula usage. They are your first taste of using the section formulas and working with the mathematical concepts at an introductory level. Successful completion of the Mechanics section means that you:
 - a. Have at least a basic understanding of the variables at play,

- b. Show rudimentary application of concepts, and
- c. Can calculate solutions using the formulas.

Do not stop here, though! You do not yet have enough skills to meet the basic requirements of most business math courses. Once you have mastered the Mechanics section, you must progress to the next section, where you learn to apply your knowledge.

- 2. **Applications.** These questions are more typical of business math course expectations (check with your instructor). The key difference from the Mechanics section is that you must now execute your problem-solving skills. These questions require you to determine the unknown variable and figure out how the various pieces of the puzzle come together in the solution. Successful completion of these questions means that you
 - a. Understand concepts at a satisfactory level,
 - b. Can problem solve typical business math applications, and
 - c. Can successfully integrate concepts at an acceptable performance level.
- 3. **Challenge, Critical Thinking, and Other Applications.** These questions put your knowledge and understanding to the test. The difficulty bar is set high, as these questions commonly require multistep solutions with advanced problem-solving techniques. As well, success might require you to integrate formulas, concepts, or procedures in novel ways. You will be challenged! Successful completion of these questions means that you
 - a. Understand concepts at an above-average level,
 - b. Can problem solve in difficult situations, and
 - c. Can successfully integrate concepts at a higher level of thinking.

How Do I Put It All Together?

You may have noticed in your previous studies that many math textbooks teach one topic at a time. However, they do not necessarily help you see how all the little topics come together and interrelate. This leaves you wondering if there is a master plan.

Throughout this textbook, a single fictional company called Lightning Wholesale illustrates such interconnections. Case studies appear at the end of Chapters 3 through 15. Even if your instructor does not assign these case studies or work through them in class, completing them benefits you in three ways:

- 1. You learn how to integrate chapter concepts into a cohesive whole.
- 2. You perform large multitask problems that reflect real-world problem solving. This will make you a much more valuable employee when you graduate.
- 3. You see how business mathematics works throughout an organization.

1.3: Where Do We Go From Here?

At this point, I hope that you can see that all the features of this textbook are built for you! They are here to help you learn, understand, and perform math. This book gives you every opportunity to succeed!

Now that you know what business mathematics is and why it is important to you, let's get going! Complete the exercises below. Then look at your course syllabus and read your first section. Remember to take advantage of all the features of this textbook along the way.



Engage Active Learning App!

Lyryx Engage is an active learning app designed to increase student engagement in reading linear algebra material. The content is "chunked" into small blocks, each with an interactive assessment activity to promote comprehension.

Visit us at <https://lyryx.com> to register and access Lyryx Engage!



Lyryx Online and Text Exercises!

Lyryx offers formative online exercises! The educational software captures instructors' experience using algorithms to analyze and evaluate students' work, identifying common errors and providing personalized feedback. Students are encouraged to try these randomized problems repeatedly, learning from their mistakes until mastery.

Lyryx also offers end-of-chapter exercises for Lyryx users.

Visit us at <https://lyryx.com> to register and access Lyryx exercises!

Appendix 1: Rounding Rules

How and when to round always is a source of confusion. This Appendix provides you with some rules for rounding your answers. We start with some general guidelines that apply globally throughout the entire textbook, then comment at the rules that apply to some specific sections of text.

Global Rounding Rules

1. All interim solutions never get rounded unless there is a logical reason or business process forcing it to be rounded.
2. When writing nonterminating decimals in this textbook, up to six decimals are written. The horizontal line format is used for repeating decimals. If the number is not a final solution, then it is assumed that all decimals or as many as possible are being carried forward.
3. All final numbers are rounded to four decimals in decimal format and two decimals in percent format unless instructions indicate otherwise.
4. Final solutions are rounded according to common business practices, practical limitations, or specific instructions.
5. Zeroes not required at the end of decimals are generally not written unless required to meet a rounding standard or to visually line up a sequence of numbers.

Topic Specific Rounding Rules

Some specific topics have their own rounding standards. Until you learn about these topics, it does not make any sense to put those standards here. However, the standards are introduced with the relevant topic. They are also summarized into Appendix C: Rounding Rules.

Chapter 2: Back To The Basics

(Shoulder Check)

Where can you go in life and not be exposed to numbers and mathematics? Whether you are figuring out the price of a product (including shipping) on eBay, or balancing your bank accounts, you use your elementary mathematical skills from both your primary and secondary education.

Think about the math you perform every day:

- At the grocery store, you compare products to calculate the best value. One brand of potato chips retails for \$3.99 for 300 g, while the equally satisfying brand beside it is priced at \$3.49 for 250 g. Which offers the better value?
- As the host for a large gathering, you are preparing a homemade lasagna and need to triple the original recipe, which calls for $1\frac{2}{3}$ cups of tomato sauce. In the expanded recipe, how many cups of tomato sauce do you need?
- If you are a sports fan, you know plenty of statistics about your favourite players and teams. Many come in percentage form, such as three-point throws for NBA stars or save percentages for NHL goaltenders. What exactly do those percentages mean?
- Many employers pay out bonuses. Perhaps in your company managers get twice as large a bonus as employees. Your company has five managers and 25 employees. If it announces a \$35,000 total bonus, what is your share as an employee?

Mathematics and numbers surround you in the business world, where you must read many numerical reports, interpret how the numbers fit together, and create your own reports showing such metrics as sales and profit projections.

Away from work, you must manage your income and pay your bills. This is a mathematical problem you likely solve on a daily basis, ensuring that the money flowing out of your bank account does not exceed the money flowing in. To purchase groceries, vacations, or entertainment, you need numbers.

This chapter gives you a refresher on your basic mathematical skills, which are essential for success in later chapters. Some instructors will review this chapter with you, while others will leave it up to you to complete this chapter independently. Either way, this chapter is important and should be used to test your basic abilities. For example, it would be unfortunate if you got a leasing calculation wrong because you made an error by breaking an order of operation rule even though you understood basic leasing concepts.

Approach this chapter with confidence, and if you encounter any difficulties ensure that you master the concepts before moving on to future chapters.

Outline of Chapter Topics

- 2.1: Order of Operations (Proceed in an Orderly Manner)
- 2.2: Fractions, Decimals, and Rounding (Just One Slice of Pie, Please)
- 2.3: Percentages (How Does It All Relate?)
- 2.4: Algebraic Expressions (The Pieces of the Puzzle)
- 2.5: Linear Equations—Manipulating and Solving (Solving the Puzzle)
- 2.6: Natural Logarithms (How Can I Get that Variable Out of the Exponent?)

2.1: Order of Operations

(Proceed In An Orderly Manner)

You have just won \$50,000 in a contest. Congratulations! But before you can claim it, you are required to answer a mathematical skill-testing question, and no calculators are permitted. After you hand over your winning ticket to the redemption agent, she hands you your time-limited skill-testing question: $2 \times 5 + 30 \div 5$. As the clock counts down, you think of the various possibilities. Is the answer 8, 14, 16, or something else altogether? Would it not be terrible to lose \$50,000 because you cannot solve the question! If you figured out the solution is 16, you are on the right path. If you thought it was something else, this is a great time to review order of operations.

The Symbols

While some mathematical operations such as addition use a singular symbol (+), there are other operations, like multiplication, for which multiple representations are acceptable. With the advent of computers, even more new symbols have crept into mathematical symbology. The table below lists the various mathematical operations and the corresponding mathematical symbols you can use for them.

Mathematical Operation	Symbol or Appearance	Comments
Brackets	() or [] or {}	In order, these are known as round, square, and curly brackets.
Exponents	2^3 or $2^{\wedge}3$	- In 2^3, 3 is the exponent; an exponent is always written as a superscript. - On a computer, the exponent is recognized with the ^ symbol; 2^3 is represented by $2^{\wedge}3$.
Division	/ or $\div$ or $\frac{4}{2}$	In order, these are known as the slash, divisor, and divisor line. Note in the last example that the division is represented by the horizontal line.
Multiplication	$\times$ or * or • or 2(2) or (2)(2)	In order, these are known as times, star, or bullet. The last two involve implied multiplication, which is when two terms are written next to each other joined only by brackets.
Addition	+	There are no other symbols.
Subtraction	–	There are no other symbols.

You may wonder if the different types of brackets mean different things. Although mathematical fields like calculus use specialized interpretations for the different brackets, business math uses all the brackets to help the reader visually pair up the brackets. Consider the following two examples:

$$\text{Example 1: } 3 \times (4 / (6 - (2 + 2))) + 2$$

$$\text{Example 2: } 3 \times [4 / \{6 - (2 + 2)\} + 2]$$

Notice that in the second example you can pair up the brackets much more easily, but changing the shape of the brackets did not change the mathematical expression. This is important to understand when using a calculator, which usually has only round brackets. Since the shape of the bracket has no mathematical impact, solving example 1 or example 2 would involve the repeated usage of the round brackets.

BEDMAS

In the section opener, your skill-testing question was $2 \times 5 + 30 \div 5$. Do you just solve this expression from left to right, or should you start somewhere else? To prevent any confusion over how to resolve these mathematical operations, there is an agreed-upon sequence of mathematical steps commonly referred to as BEDMAS. **BEDMAS** is an acronym for **B**rackets, **E**xponents, **D**ivision, **M**ultiplication, **A**ddition, and **S**ubtraction.



How It Works

Step 1: Brackets must be resolved first. As brackets can be nested inside of each other, you must resolve the innermost set of brackets first before proceeding outwards to the next set of brackets. When resolving a set of brackets, you must perform the mathematical operations within the brackets by following the remaining steps in this model (EDMAS). If there is more than one set of brackets but the sets are not nested, work from left to right and top to bottom.

Step 2: If the expression has any exponents, you must resolve these next. Remember that an exponent indicates how many times you need to multiply the base against itself. For example, $2^3 = 2 \times 2 \times 2$. More review of exponents is found in Section 2.4.

Step 3: The order of appearance for multiplication and division does not matter. However, you must resolve these operations in order from left to right and top to bottom as they appear in the expression.

Step 4: The last operations to be completed are addition and subtraction. The order of appearance doesn't matter; however, you must complete the operations working left to right through the expression.



Important Notes

Before proceeding with the Texas Instruments BAII Plus calculator, you must change some of the factory defaults, as explained in the table below. To change the defaults, open the Format window on your calculator. If for any reason your calculator is reset (either by removing the battery or pressing the reset button), you must perform this sequence again.

Buttons Pushed	Calculator Display	What It Means
2nd Format	DEC=2.00	You have opened the Format window to its first setting. DEC tells your calculator how to round the calculations. In business math, it is important to be accurate. Therefore, we will set the calculator to what is called a floating display, which means your calculator will carry all of the decimals and display as many as possible on the screen.
9 Enter	DEC=9.	The floating decimal setting is now in place. Let us proceed.
↓	DEG	This setting has nothing to do with business math and is just left alone. If it does not read DEG, press 2nd Set to toggle it.
↓	US 12-31-1990	Dates can be entered into the calculator. North Americans and Europeans use slightly different formats for dates. Your display is indicating North American format and is acceptable for our purposes. If it does not read US, press 2nd Set to toggle it.
↓	US 1,000	In North America it is common to separate numbers into blocks of 3 using a comma. Europeans do it slightly differently. This setting is acceptable for our purposes. If your display does not read US, press 2nd Set to toggle it.
↓	Chn	There are two ways that calculators can solve equations. This is known as the Chain method, which means that your calculator will simply resolve equations as you punch it in without regard for the rules of BEDMAS. This is not acceptable and needs to be changed.
2nd Set	AOS	AOS stands for Algebraic Operating System. This means that the calculator is now programmed to use BEDMAS in solving equations.
2nd Quit	0.	Back to regular calculator usage.

Also note that on the BAII Plus calculator you have two ways to key in an exponent:

1. If the exponent is squaring the base (e.g., 3^2), press $3 \times^2$. It calculates the solution of 9.
2. If the exponent is anything other than a 2, you must use the y^x button. For 2^3 , you press $2 y^x 3 =$. It calculates the solution of 8.



Things To Watch Out For

Negative Signs. Remember that mathematics uses both positive numbers (such as $+3$) and negative numbers (such as -3). Positive numbers do not need to have the $+$ sign placed in front of them since it is implied. Thus $+3$ is written as just 3 . Negative numbers, though, must have the negative sign placed in front of them. Be careful not to confuse the terminology of a negative number with a subtraction or minus sign. For example, $4 + (-3)$ is read as "four plus negative three" and not "four plus minus three." To key a negative number on a calculator, enter the number first followed by the $\pm$ button, which switches the sign of the number.

The Horizontal Divisor Line. One of the areas in which people make the most mistakes involves the "hidden brackets." This problem almost always occurs when the horizontal line is used to represent division. Consider the following mathematical expression:

$$(4 + 6) \div (2 + 3)$$

If you rewrite this expression using the horizontal line to represent the divisor, it looks like this:

$$\frac{4 + 6}{2 + 3}$$

Notice that the brackets disappear from the expression when you write it with the horizontal divisor line because they are implied by the manner in which the expression appears. Your best approach when working with a horizontal divisor line is to reinsert the brackets around the terms on both the top and bottom. Thus, the expression looks like this:

$$\frac{(4 + 6)}{(2 + 3)}$$

Employing this technique will ensure that you arrive at the correct solution, especially when using calculators.

**TOP
SECRET**

Paths To Success

Hidden and Implied Symbols. If there are hidden or implied symbols in the expressions, your first step is to reinsert those hidden symbols in their correct locations. In the example below, note how the hidden multiplication and brackets are reinserted into the expression:

$$4 \left[\frac{3 + 2^2 \times 3}{(2 + 8) \div 2} \right] \text{ transforms into } 4 \times \left[\frac{\{3 + 2^2 \times 3\}}{\{(2 + 8) \div 2\}} \right]$$

Once you have reinserted the symbols, you are ready to follow the BEDMAS model.

Calculators are not programmed to be capable of recognizing implied symbols. If you key in " $3(4 + 2)$ " on your calculator, failing to input the multiplication sign between the " 3 " and the " $(4 + 2)$," you get a solution of 6 . Your calculator ignores the " 3 " since it doesn't know what mathematical operation to perform on it. To have your calculator solve the expression correctly, you must punch the equation through as " $3 \times (4 + 2) =$ ". This produces the correct answer of 18 .

Simplifying Negatives. If your question involves positive and negative numbers, it is sometimes confusing to know what symbol to put when simplifying or solving. Remember these two rules:

Rule #1: A pair of the same symbols is always positive. Thus " $4 + (+3)$ " and " $4 - (-3)$ " both become " $4 + 3$."

Rule #2: A pair of the opposite symbols is always negative. Thus " $4 + (-3)$ " and " $4 - (+3)$ " both become " $4 - 3$."

A simple way to remember these rules is to count the total sticks involved, where a " $+$ " sign has two sticks and a " $-$ " sign has one stick. If you have an odd number of total sticks, the outcome is a negative sign. If you have an even number of total sticks, the outcome is a positive sign. Note the following examples:

$$\begin{aligned} 4 + (-3) &= \rightarrow 3 \text{ total sticks is odd and therefore simplifies to negative} \rightarrow 4 - 3 = 1 \\ (-2) \times (-2) &= \rightarrow 2 \text{ total sticks is even and therefore simplifies to positive} \rightarrow (-2) \times (-2) = +4 \end{aligned}$$

Example 2.1A: Solving Expressions Using BEDMAS

Evaluate each of the following expressions.

- a. $2 \times 5 + 30 \div 5$
 b. $(6 + 3)^2 + 18 \div 2$
 c. $4 \times \left[\frac{\{3 + 2^2 \times 3\}}{\{(2 + 8) \div 2\}} \right]$

Plan	You need to evaluate each of the expressions. This means you must solve each expression.																											
Understand	What You Already Know You are provided with the mathematical expressions in a formula format. These expressions are ready for you to solve.	How You Will Get There Employ the knowledge of BEDMAS to solve each operation in its correct order.																										
Perform	a.	<table><tr><td>$2 \times 5 + 30 \div 5$</td><td>Step 1: No brackets Step 2: No exponents Step 3: Working left to right, resolve the multiplication first</td></tr><tr><td>$10 + 30 \div 5$</td><td>Step 3: Now resolve the division</td></tr><tr><td>$10 + 6$</td><td>Step 4: Perform the remaining addition</td></tr><tr><td>16</td><td>Final solution</td></tr></table>	$2 \times 5 + 30 \div 5$	Step 1: No brackets Step 2: No exponents Step 3: Working left to right, resolve the multiplication first	$10 + 30 \div 5$	Step 3: Now resolve the division	$10 + 6$	Step 4: Perform the remaining addition	16	Final solution	c.	<table><tr><td>$4 \times \left[\frac{\{3 + 2^2 \times 3\}}{\{(2 + 8) \div 2\}} \right]$</td><td>Step 1: Start with the innermost set of brackets (2 + 8) and perform EDMAS. In this bracket, you have only an addition to resolve.</td></tr><tr><td>$4 \times \left[\frac{\{3 + 2^2 \times 3\}}{\{10 \div 2\}} \right]$</td><td>The innermost brackets are complete, so you now drop them. Step 1: You still have two sets of inner {} brackets. Start with the top one and perform EDMAS.</td></tr><tr><td>$4 \times \left[\frac{\{3 + 4 \times 3\}}{\{10 \div 2\}} \right]$</td><td>Exponent resolved.</td></tr><tr><td>$4 \times \left[\frac{\{3 + 12\}}{\{10 \div 2\}} \right]$</td><td>Multiplication performed.</td></tr><tr><td>$4 \times \left[\frac{15}{\{10 \div 2\}} \right]$</td><td>Addition performed. You no longer need the brackets, so you drop them. Step 1: Now do the bottom brackets, performing EDMAS. You have only one division to resolve.</td></tr><tr><td>$4 \times \left[\frac{15}{5} \right]$</td><td>Division completed. You no longer need the brackets, so you drop them. Step 1: One last set of brackets to go! Only a division is left.</td></tr><tr><td>4×3</td><td>Division performed. You no longer need the brackets, so you drop them. Step 2: No exponents Step 3: Perform the multiplication</td></tr><tr><td>12</td><td>Step 4: Not needed Final solution</td></tr></table>	$4 \times \left[\frac{\{3 + 2^2 \times 3\}}{\{(2 + 8) \div 2\}} \right]$	Step 1: Start with the innermost set of brackets (2 + 8) and perform EDMAS. In this bracket, you have only an addition to resolve.	$4 \times \left[\frac{\{3 + 2^2 \times 3\}}{\{10 \div 2\}} \right]$	The innermost brackets are complete, so you now drop them. Step 1: You still have two sets of inner {} brackets. Start with the top one and perform EDMAS.	$4 \times \left[\frac{\{3 + 4 \times 3\}}{\{10 \div 2\}} \right]$	Exponent resolved.	$4 \times \left[\frac{\{3 + 12\}}{\{10 \div 2\}} \right]$	Multiplication performed.	$4 \times \left[\frac{15}{\{10 \div 2\}} \right]$	Addition performed. You no longer need the brackets, so you drop them. Step 1: Now do the bottom brackets, performing EDMAS. You have only one division to resolve.	$4 \times \left[\frac{15}{5} \right]$	Division completed. You no longer need the brackets, so you drop them. Step 1: One last set of brackets to go! Only a division is left.	4×3	Division performed. You no longer need the brackets, so you drop them. Step 2: No exponents Step 3: Perform the multiplication	12	Step 4: Not needed Final solution
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	Calculator Instructions a. $2 \times 5 + 30 \div 5 =$ b. $(6 + 3) \times 2 + 18 \div 2 =$ c. $4 \times ((3 + 2 \times 2 \times 3) \div ((2 + 8) \div 2)) =$																											
Present	The final solutions for the expressions are: a. 16 b. 90 c. 12																											



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2.2: Fractions, Decimals, & Rounding

(Just One Slice Of Pie, Please)

Your local newspaper quotes a political candidate as saying, “The top half of the students are well-educated, the bottom half receive extra help, but the middle half we are leaving out.”¹ You stare at the sentence for a moment and then laugh. To halve something means to split it into two. However, there are three halves here! You conclude that the speaker was not thinking carefully.

In coming to this conclusion, you are applying your knowledge of fractions. In this section, you will review fraction types, convert fractions into decimals, perform operations on fractions, and also address rounding issues in business mathematics.

Types of Fractions

To understand the characteristics, rules, and procedures for working with fractions, you must become familiar with fraction terminology. First of all, what is a fraction? A **fraction** is a part of a whole. It is written in one of three formats:

$$1/2 \text{ or } \frac{1}{2} \text{ or } \frac{1}{2}$$

Each of these formats means exactly the same thing. The number on the top, side, or to the left of the line is known as the **numerator**. The number on the bottom, side, or to the right of the line is known as the **denominator**. The slash or line in the middle is the **divisor line**. In the above example, the numerator is 1 and the denominator is 2. There are five different types of fractions, as explained in the table below.

Fraction	Terminology	Characteristics	Result of Division*
$\frac{2}{5}$	Proper	The numerator is smaller than the denominator.	Answer is between 0 and 1
$\frac{5}{2}$	Improper	The numerator is larger than the denominator.	Answer is greater than 1
$3\frac{2}{5}$	Compound	A fraction that combines an integer with either a proper or improper fraction. When the division is performed, the proper or improper fraction is <i>added</i> to the integer.	Answer is greater than the integer
$3\frac{2/5}{7}$	Complex	A fraction that has fractions within fractions, combining elements of compound, proper, or improper fractions together. It is important to follow BEDMAS in resolving these fractions.	Answer varies depending on the fractions involved
$\frac{1}{2}$ and $\frac{2}{4}$	Equivalent	Two or more fractions of any type that have the same numerical value upon completion of the division. Note that both of these examples work out to 0.5.	Answers are equal

*Assuming all numbers are positive.



How It Works

First, focus on the correct identification of proper, improper, compound, equivalent, and complex fractions. In the next section, you will work through how to accurately convert these fractions into their decimal equivalents.

Equivalent fractions require you to either solve for an unknown term or express the fraction in larger or smaller terms.

¹ Neal, Marcia. Candidate for the 3rd Congressional District Colorado State Board of Education, as quoted in Perez, Gayle. 2008. "Retired School Teacher Seeks State Board Seat." *Pueblo Chieftain*.

Solving For An Unknown Term. These situations involve two fractions where only one of the numerators or denominators is missing. Follow this four-step procedure to solve for the unknown:

Step 1: Set up the two fractions.

Step 2: Note that your equation contains two numerators and two denominators. Pick the pair for which you know both values.

Step 3: Determine the multiplication or division relationship between the two numbers.

Step 4: Apply the same relationship to the pair of numerators or denominators containing the unknown.

Assume you are having a party and one of your friends says he would like to eat one-third of the pizza. You notice the pizza has been cut into nine slices. How many slices would you give to your friend?

Step 1: Your friend wants one out of three pieces. This is one-third. You want to know how many pieces out of nine to give him. Assign a meaningful variable to represent your unknown, so have s represent the number of slices to give; you need to give him s out of 9 pieces, or $s/9$.

$$\frac{1}{3} = \frac{s}{9}$$

Step 2: Work with the denominators 3 and 9 since you know both of them.

Step 3: Take the larger number and divide it by the smaller number. We have $9 \div 3 = 3$. Therefore, the denominator on the right is three times larger than the denominator on the left.

Step 4: Take the 1 and multiply it by 3 to get the s . Therefore, $s = 1 \times 3 = 3$.

$$\frac{1 \times 3}{3 \times 3} = \frac{3}{9}$$

You should give your friend three slices of pizza.

Expressing The Fraction In Larger Or Smaller Terms. When you need to make a fraction easier to understand or you need to express it in a certain format, it helps to try to express it in larger or smaller terms. To express a fraction in larger terms, multiply both the numerator and denominator by the same number. To express a fraction in smaller terms, divide both the numerator and denominator by the same number.

- Larger terms: $\frac{2}{12}$ expressed with terms twice as large would be $\frac{2 \times 2}{12 \times 2} = \frac{4}{24}$
- Smaller terms: $\frac{2}{12}$ expressed with terms half as large would be $\frac{2 \div 2}{12 \div 2} = \frac{1}{6}$

When expressing fractions in higher or lower terms, you do not want to introduce decimals into the fraction unless there would be a specific reason for doing so. For example, if you divided 4 into both the numerator and denominator of $\frac{2}{12}$, you would have $\frac{0.5}{3}$, which is not a typical format. To find numbers that divide evenly into the numerator or denominator (called factoring), follow these steps:

- Pick the smallest number in the fraction.
- Use your multiplication tables and start with $1 \times$ before proceeding to $2 \times$, $3 \times$, and so on. When you find a number that works, check to see if it also divides evenly into the other number.

For example, if the fraction is $\frac{12}{18}$, you would factor the numerator of 12. Note that $1 \times 12 = 12$; however, 12 does not divide evenly into the denominator. Next you try 2×6 and discover that 6 does divide evenly into the denominator. Therefore, you reduce the fraction to smaller terms by dividing by 6, or $\frac{12 \div 6}{18 \div 6} = \frac{2}{3}$.



Things To Watch Out For

With complex fractions, it is critical to obey the rules of BEDMAS. As suggested in Section 2.1, always reinsert the hidden symbols before solving. Note in the following example that an addition sign and two sets of brackets were hidden: You

should rewrite $3 \frac{2/5}{7}$ as $3 + \left[\frac{(2/5)}{7} \right]$ before you attempt to solve with BEDMAS.

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What do you do when there is a negative sign in front of a fraction, such as $-\frac{1}{2}$? Do you put the negative with the numerator or the denominator? The common solution is to multiply the numerator by negative 1, resulting in $\frac{(-1) \times 1}{2} = \frac{-1}{2}$. In the special case of a compound fraction, multiply the entire fraction by -1 . Thus, $-1\frac{1}{2} = (-1) \times \left(1 + \frac{1}{2}\right) = -1 - \frac{1}{2}$.

Example 2.2A: Identifying Types of Fractions

Identify the type of fraction represented by each of the following:

- a. $\frac{2}{3}$ b. $6\frac{7}{8}$ c. $12\frac{\frac{4}{3}}{6\frac{4}{5}}$ d. $\frac{15}{11}$ e. $\frac{5}{6}$ f. $\frac{3}{4} \& \frac{9}{12}$

Plan	For each of these six fractions, identify the type of fraction.	
Understand	What You Already Know There are five types of fractions, including proper, improper, compound, complex, or equivalent	How You Will Get There Examine each fraction for its characteristics and match these characteristics with the definition of the fraction.
Perform	a. $\frac{2}{3}$	The numerator is smaller than the denominator. This matches the characteristics of a proper fraction.
	b. $6\frac{7}{8}$	This fraction combines an integer with a proper fraction (since the numerator is smaller than the denominator). This matches the characteristics of a compound fraction.
	c. $12\frac{\frac{4}{3}}{6\frac{4}{5}}$	There are lots of fractions involving fractions nested inside other fractions. The fraction as a whole is a compound fraction, containing an integer with a proper fraction (since the numerator is smaller than the denominator). Within the proper fraction, the numerator is an improper fraction ($\frac{4}{3}$) and the denominator is a compound fraction containing an integer and a proper fraction ($6\frac{4}{5}$). This all matches the definition of a complex fraction: nested fractions combining elements of compound, proper, and improper fractions together.
	d. $\frac{15}{11}$	The numerator is larger than the denominator. This matches the characteristics of an improper fraction.
	e. $\frac{5}{6}$	The numerator is smaller than the denominator. This matches the characteristics of a proper fraction.
	f. $\frac{3}{4} \& \frac{9}{12}$	There are two proper fractions here that are equal to each other. If you were to complete the division, both fractions calculate to 0.75. These are equivalent fractions.
Present	Of the six fractions examined, there are two proper fractions (a and e), one improper fraction (d), one compound fraction (b), one complex fraction (c), and one equivalent fraction (f).	

Example 2.2B: Working with Equivalent Fractions

- a. Solve for the unknown term x : $\frac{7}{12} = \frac{49}{x}$ b. Express this fraction in lower terms: $\frac{5}{50}$

Plan	a. Find the value of the unknown term, x . b. Take the proper fraction and express it in a lower term.

Understand	What You Already Know The needed fractions in a ready-to-solve format are provided.	How You Will Get There a. Apply the four-step technique to solving equivalent fractions. The first step has already been done for you, in that the equation is already set up. b. Find a common divisor that divides evenly into both the numerator and denominator. As only 1 and 5 go into the number 5, it makes sense that you should choose 5 to divide into both the numerator and denominator. Note that 5 factors evenly into the denominator, 50, meaning that no remainder or decimals are left over.
Perform	a. Step 2: You have both of the numerators, so work with that pair. Step 3: Take the larger number and divide by the smaller number, or $49 \div 7 = 7$. Therefore, multiply the fraction on the left by 7 to get the fraction on the right. Step 4: Applying the same relationship, $12 \times 7 = 84$. b. $\frac{5 \div 5}{50 \div 5} = \frac{1}{10}$	
Present	a. The unknown denominator on the right is 84, and therefore $\frac{7}{12} = \frac{49}{84}$. b. In lower terms, $\frac{5}{50}$ is expressed as $\frac{1}{10}$.	

Converting to Decimals

Although fractions are common, many people have trouble interpreting them. For example, in comparing $\frac{27}{37}$ to $\frac{57}{73}$, which is the larger number? The solution is not immediately apparent. As well, imagine a retail world where your local Walmart was having a $\frac{3}{20}$ th off sale! It's not that easy to realize that this equates to 15% off. In other words, fractions are converted into decimals by performing the division to make them easier to understand and compare.



How It Works

The rules for converting fractions into decimals are based on the fraction types.

Proper and Improper Fractions. Resolve the division. For example, $\frac{3}{4}$ is the same as $3 \div 4 = 0.75$. As well, $\frac{5}{4} = 5 \div 4 = 1.25$.

Compound Fractions. The decimal number and the fraction are joined by a hidden addition symbol. Therefore, to convert to a decimal you need to reinsert the addition symbol and apply BEDMAS:

$$3\frac{4}{5} = 3 + 4 \div 5 = 3 + 0.8 = 3.8$$

Complex Fractions. The critical skill here is to reinsert all of the hidden symbols and then apply the rules of BEDMAS:

$$2\frac{\frac{11}{4}}{1\frac{1}{4}} = 2 + \left[\frac{(11 \div 4)}{(1 + 1 \div 4)} \right] = 2 + \left[\frac{(11 \div 4)}{(1 + 0.25)} \right] = 2 + \left[\frac{2.75}{1.25} \right] = 2 + 2.2 = 4.2$$

Example 2.2C: Converting Fractions to Decimals

Convert the following fractions into decimals:

a. $\frac{2}{5}$ b. $6\frac{7}{8}$ c. $12\frac{\frac{9}{2}}{1\frac{2}{10}}$

Plan	Take these fractions and convert them into decimal numbers.	
Understand	What You Already Know The three fractions are provided and ready to convert.	How You Will Get There a. This is a proper fraction requiring you to complete the division. b. This is a compound fraction requiring you to reinsert the hidden addition symbol and then apply BEDMAS. c. This is a complex fraction requiring you to reinsert all hidden symbols and apply BEDMAS.
Perform	a. $\frac{2}{5} = 2 \div 5 = 0.4$ b. $6\frac{7}{8} = 6 + 7 \div 8 = 6 + 0.875 = 6.875$ c. $12\frac{\frac{9}{2}}{1\frac{2}{10}} = 12 + \left[\frac{9 \div 2}{1 + 2 \div 10} \right] = 12 + \left[\frac{9 \div 2}{1 + 0.2} \right] = 12 + \left[\frac{4.5}{1.2} \right] = 12 + 3.75 = 15.75$	
Present	In decimal format, the fractions have converted to 0.4, 6.875, and 15.75, respectively.	

Rounding Principle

Your company needs to take out a loan to cover some short-term debt. The bank has a posted rate of 6.875%. Your bank officer tells you that, for simplicity, she will just round off your interest rate to 6.9%. Is that all right with you? It shouldn't be!

What this example illustrates is the importance of rounding. This is a slightly tricky concept that confuses most students to some degree. In business math, sometimes you should round your calculations off and sometimes you need to retain all of the digits to maintain accuracy.

**How It Works**

To round a number off, you always look at the number to the right of the digit being rounded. If that number is 5 or higher, you add one to your digit; this is called *rounding up*. If that number is 4 or less, you leave your digit alone; this is called *rounding down*.

For example, if you are rounding 8.345 to two decimals, you need to examine the number in the third decimal place (the one to the right). It is a 5, so you add one to the second digit and the number becomes 8.35.

For a second example, let's round 3.6543 to the third decimal place. Therefore, you look at the fourth decimal position, which is a 3. As the rule says, you would leave the digit alone and the number becomes 3.654.

Nonterminating Decimals

What happens when you perform a calculation and the decimal doesn't terminate?

1. You need to assess if there is a pattern in the decimals:

- **The Nonterminating Decimal without a Pattern:** For example, $\frac{6}{17} = 0.352941176 \dots$ with no apparent ending decimal and no pattern to the decimals.
 - **The Nonterminating Decimal with a Pattern:** For example, $\frac{2}{11} = 0.18181818 \dots$ endlessly. You can see that the numbers 1 and 8 repeat. A shorthand way of expressing this is to place a horizontal line above the digits that repeat. Thus, you can rewrite $0.18181818 \dots$ as $0.\overline{18}$.
2. You need to know if the number represents an interim or final solution to a problem:
- **Interim Solution:** You must carry forward all of the decimals in your calculations, as the number should not be rounded until you arrive at a final answer. If you are completing the question by hand, write out as many decimals as possible; to save space and time, you can use the shorthand horizontal bar for repeating decimals. If you are completing the question by calculator, store the entire number in a memory cell.
 - **Final Solution:** To round this number off, an industry protocol or other clear instruction must apply. If these do not exist, then you would make an arbitrary rounding choice, subject to the condition that you must maintain enough precision to allow for reasonable interpretation of the information.



Important Notes

To assist in your calculations, particularly those that involve multiple steps to resolve, your calculator has 10 memory cells. Your display is limited to 10 digits, but when you store a number into a memory cell the calculator retains all of the decimals associated with the number, not just those displaying on the screen. Your calculator can, in fact, carry up to 13 digit positions. It is strongly recommended that you take advantage of this feature where needed throughout this textbook.

Let's say that you just finished keying in $\frac{6}{17}$ on your calculator, and the resultant number is an interim solution that you need for another step. With 0.352941176 on your display, press STO followed by any numerical digit on the keypad of your calculator. STO stands for *store*. To store the number into memory cell 1, for example, press STO 1. The number with 13 digits is now in permanent memory. If you clear your calculator (press CE/C) and press RCL # (where # is the memory cell number), you will bring the stored number back. RCL stands for *recall*. Press RCL 1. The stored number 0.352941176 reappears on the screen.

Example 2.2D: Rounding Numbers

Convert the following to decimals. Round each to four decimals or use the repeating decimal notation.

- a. $\frac{6}{13}$ b. $\frac{4}{9}$ c. $\frac{4}{11}$ d. $\frac{3}{22}$ e. $5\frac{1/7}{10/27}$

Plan	Convert each of the fractions into decimal format, then round any final answer to four decimals or use repeating decimal notation.	
Understand	What You Already Know The fractions and clear instructions on how to round them have been provided.	How You Will Get There To convert the fractions to decimals, you need to complete the division by obeying the rules of BEDMAS. Watch for hidden symbols and adhere to the rules of rounding.

Perform	a. $\frac{6}{13} = 0.461538$. The fifth decimal is a 3, so round down. The answer is 0.4615. b. $\frac{4}{9} = 0.444444$. Note the repeating decimal of 4. Using the horizontal bar, write $0.\overline{4}$. c. $\frac{4}{11} = 0.363636$. Note the repeating decimals of 3 and 6. Using the horizontal bar, write $0.\overline{36}$. d. $\frac{3}{22} = 0.136363$. Note the repeating decimals of 3 and 6 after the 1. Using the horizontal bar, write $0.1\overline{36}$. e. $5 \frac{1/7}{10/27} = 5 + \frac{(1 \div 7)}{(10 \div 27)} = 5 + \frac{0.142857}{0.370} = 5 + 0.385714 = 5.385714$. Since the fifth digit is a 1, round down. The answer is 5.3857.
Present	According to the rounding instructions, the solutions are 0.4615, $0.\overline{4}$, $0.\overline{36}$, $0.1\overline{36}$, and 5.3857, respectively.

Rounding Rules

One of the most common sources of difficulties in math is that different people sometimes use different standards for rounding. This seriously interferes with the consistency of final solutions and makes it hard to assess accuracy. So that everyone arrives at the same solution to the exercises/examples in this textbook, these rounding rules apply throughout the book:

- Never round an interim solution unless there is a logical reason or business process that forces the number to be rounded. Here are some examples of logical reasons or business processes indicating you should round:
 - You withdraw money or transfer it between different bank accounts. In doing so, you can only record two decimals and therefore any money moving between the financial tools must be rounded to two decimals.
 - You need to write the numbers in a financial statement or charge a price for a product. As our currency is in dollars and cents, only two decimals can appear.
- When you write nonterminating decimals, show only the first six (or up to six) decimals. Use the horizontal line format for repeating decimals. If the number is not a final solution, then assume that all decimals or as many as possible are being carried forward.
- Round all final numbers to six decimals in decimal format and four decimals in percentage format unless instructions indicate otherwise.
- Round final solutions according to common business practices, practical limitations, or specific instructions. For example, round any final answer involving dollar currency to two decimals. These types of common business practices and any exceptions are discussed as they arise at various points in this textbook.
- Generally avoid writing zeros, which are not required at the end of decimals, unless they are required to meet a rounding standard or to visually line up a sequence of numbers. For example, write 6.340 as 6.34 since there is no difference in interpretation through dropping the zero.

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Does your final solution vary from the actual solution by a small amount? Did the question involve multiple steps or calculations to get the final answer? Were lots of decimals or fractions involved? If you answer yes to these questions, the most common source of error lies in rounding. Here are some quick error checks for answers that are "close":

- Did you remember to obey the rounding rules laid out above? Most importantly, are you carrying decimals for interim solutions and rounding only at final solutions?
- Did you resolve each fraction or step accurately? Check for incorrect calculations or easy-to-make errors, like transposed numbers.
- Did you break any rules of BEDMAS?



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2.3: Percentages

(How Does It All Relate?)

Your class just wrote its first math quiz. You got 13 out of 19 questions correct, or $\frac{13}{19}$. In speaking with your friends Sandhu and Illija, who are in other classes, you find out that they also wrote math quizzes; however, theirs were different. Sandhu scored 16 out of 23, or $\frac{16}{23}$, while Illija got 11 out of 16, or $\frac{11}{16}$. Who achieved the highest grade? Who had the lowest? The answers are not readily apparent, because fractions are difficult to compare.

Now express your grades in percentages rather than fractions. You scored 68%, Sandhu scored 70%, and Illija scored 69%. Notice you can easily answer the questions now. The advantage of percentages is that they facilitate comparison and comprehension.

Converting Decimals to Percentages

A **percentage** is a part of a whole expressed in hundredths. In other words, it is a value out of 100. For example, 93% means 93 out of 100, or $\frac{93}{100}$.



The Formula

% is Percentage: This is the decimal expressed as a percentage. It must always be written with the percent (%) symbol immediately following the number.

100 is Conversion Factor: A percentage is always expressed in hundredths.

Formula 2.1 - Percentage Conversion: $\% = \text{dec} \times 100$

dec is Decimal Number: This is the decimal number needing to be converted into a percentage.



How It Works

Assume you want to convert the decimal number 0.0875 into a percentage. This number represents the dec variable in the formula. Substitute into Formula 2.1:

$$\% = 0.0875 \times 100 = 8.75\%$$



Important Notes

You can also solve this formula for the decimal number. To convert any percentage back into its decimal form, you need to perform a mathematical opposite. Since a percentage is a result of multiplying by 100, the mathematical opposite is achieved by dividing by 100. Therefore, to convert 81% back into decimal form, you take $81\% \div 100 = 0.81$.



Things To Watch Out For

Your Texas Instruments BAII Plus calculator has a % key that can be used to convert any percentage number into its decimal format. For example, if you press 81 and then %, your calculator displays 0.81.

While this function works well when dealing with a single percentage, it causes problems when your math problem involves multiple percentages. For example, try keying $4\% + 3\% =$ into the calculator using the % key. This should be the same as $0.04 + 0.03 = 0.07$. Notice, however, that your calculator has 0.0412 on the display.

Why is this? As a business calculator, your BAII Plus is programmed to take portions of a whole. When you key 3% into the calculator, it takes 3% of the first number keyed in, which was 4%. As a formula, the calculator sees $4\% + (3\% \times 4\%)$. This works out to $0.04 + 0.0012 = 0.0412$.

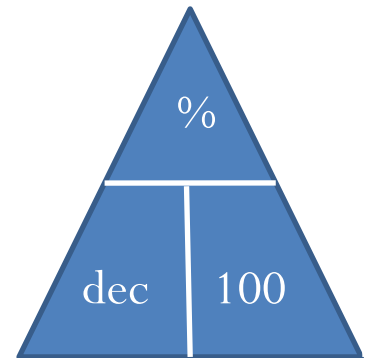
To prevent this from happening, your best course of action is not to use the % key on your calculator. It is best to key all percentages as decimal numbers whenever possible, thus eliminating any chance that the % key takes a portion of your whole. Throughout this textbook, all percentages are converted to decimals before calculations take place.

**TOP
SECRET**

Paths To Success

When working with percentages, you can use some tricks for remembering the formula and moving the decimal point.

Remembering the Formula. When an equation involves only multiplication of all terms on one side with an isolated solution on the other side, use a mnemonic called the triangle technique. In this technique, draw a triangle with a horizontal line through its middle. Above the line goes the solution, and below the line, separated by vertical lines, goes each of the terms involved in the multiplication. The figure to the right shows how Formula 2.1 would be drawn using the triangle technique.



To use this triangle, identify the unknown variable, which you then calculate from the other variables in the triangle:

1. Anything on the same line gets multiplied together. If solving for %, then the other variables are on the same line and multiplied as $\text{dec} \times 100$.
2. Any pair of items with one above the other is treated like a fraction and divided. If solving for dec, then the other variables are above/below each other and are divided as $\frac{\%}{100}$.

Moving the Decimal. Another easy way to work with percentages is to remember that multiplying or dividing by 100 moves the decimal over two places.

1. If you are multiplying by 100, the decimal position moves two positions to the right.

$$0.73 \times 100 = 0.73 = 73\%$$

2 positions to the right
2. If you are dividing by 100, the decimal position moves two positions to the left (see Figure 2.5).

$$73\% \div 100 = 73. = 0.73$$

2 positions to the left

Example 2.3A: Working with Percentages

Convert (a) and (b) into percentages. Convert (c) back into decimal format.

- a. $\frac{3}{8}$ b. 1.3187 c. 12.399%

Plan

For questions (a) and (b), you need to convert these into percentage format. For question (c), you need to convert it back to decimal format.

	What You Already Know	How You Will Get There
Understand	a. This is a fraction to be converted into a decimal, or dec. b. This is dec. c. This is %.	a. Convert the fraction into a decimal to have dec. Then apply Formula 2.1: $\% = \text{dec} \times 100$ to get the percentage. b. As this term is already in decimal format, apply Formula 2.1: $\% = \text{dec} \times 100$ to get the percentage. c. This term is already in percentage format. Using the triangle technique, calculate the decimal number through $\text{dec} = \frac{\%}{100}$.
Perform	a. $\frac{3}{8} = 3 \div 8 = 0.375$ b. $\% = 1.3187 \times 100$ c. $\text{dec} = \frac{12.399\%}{100} = 0.12399$	$\% = 0.375 \times 100$ $\% = 37.5\%$ $\% = 131.87\%$
Present	In percentage format, the first two numbers are 37.5% and 131.87%. In decimal format, the last number is 0.12399.	

Rate, Portion, Base

In your personal life and career, you will often need to either calculate or compare various quantities involving fractions. For example, if your income is \$3,000 per month and you can't spend more than 30% on housing, what is your maximum housing dollar amount? Or perhaps your manager tells you that this year's sales of \$1,487,003 are 102% of last year's sales. What were your sales last year?



The Formula

Rate is The Relationship: The **rate** is the decimal form expressing the relationship between the portion and the base. Convert it to a percentage if needed by applying Formula 2.1. This variable can take on any value, whether positive or negative.

Portion is The Part Of The Quantity: The portion represents the part of the whole. Compare it against the base to assess the rate.

Formula 2.2 – Rate, Portion, Base:

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

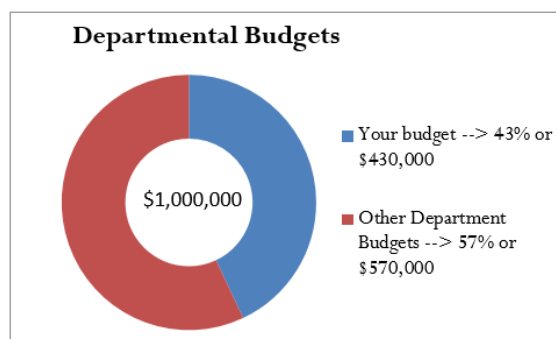
Base is The Entire Quantity: The base is the entire amount or quantity that is of concern. It represents a whole, standard, or benchmark that you assess the portion against.



How It Works

Assume that your company has set a budget of \$1,000,000. This is the entire amount of the budget and represents your *base*. Your department gets \$430,000 of the budget—this is your department's part of the whole and represents the *portion*. You want to know the relationship between your budget and the company's budget. In other words, you are looking for the *rate*.

- Apply Formula 2.2, where $\text{rate} = \frac{\$430,000}{\$1,000,000}$.
- Your budget is 0.43, or 43%, of the company's budget.



Important Notes

There are three parts to this formula. Mistakes commonly occur through incorrect assignment of a quantity to its associated variable. The table below provides some tips and clue words to help you make the correct assignment.

Variable	Key Words	Example
Base	of	If your department can spend 43% of the company's total budget of \$1,000,000, what is your maximum departmental spending?
Portion	is, are	If your department can spend 43% of the company's total budget of \$1,000,000, what is your maximum departmental spending?
Rate	%, percent, rate	If your department can spend 43% of the company's total budget of \$1,000,000, what is your maximum departmental spending?



Things To Watch Out For

In resolving the rate, you must express all numbers in the same units—you cannot have apples and oranges in the same sequence of calculations. In the above example, both the company's budget and the departmental budget are in the units of dollars. Alternatively, you would not be able to calculate the rate if you had a base expressed in kilometres and a portion expressed in metres. Before you perform the rate calculation, express both in kilometres or both in metres.



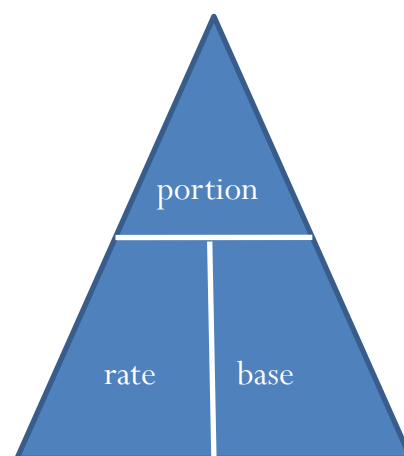
Paths To Success

Formula 2.2 is another formula you can use the triangle technique for. You do not need to memorize multiple versions of the formula for each of the variables. The triangle appears to the right.

Be very careful when performing operations involving rates, particularly in summing or averaging rates.

Summing Rates. Summing rates requires each rate to be a part of the same whole or base. If Bob has 5% of the kilometres travelled and Sheila has 6% of the oranges, these are not part of the same whole and cannot be added. If you did, what does the 11% represent? The result has no interpretation. However, if there are 100 oranges of which Bob has 5% and Sheila has 6%, the rates can be added and you can say that in total they have 11% of the oranges.

Averaging Rates. Simple averaging of rates requires each rate to be a measure of the same variable with the same base. If 36% of your customers are female and 54% have high income, the average of 45% is meaningless since each rate measures a



different variable. Recall that earlier in this chapter you achieved 68% on your test and Sandhu scored 70%. However, your test involved 19 questions and Sandhu's involved 23 questions. These rates also cannot be simply averaged to 69% on the reasoning that $(68\% + 70\%) / 2 = 69\%$, since the bases are not the same. When two variables measure the same characteristic but have different bases (such as the math quizzes), you must use a weighted-average technique, which will be discussed in Section 3.1.

When can you average rates? Hypothetically, assume Sandhu achieved his 70% by writing the same test with 19 questions. Since both rates measure the same variable and have the same base, the simple average of 69% is now calculable.



Give It Some Thought

Consider the following situations and select the best answer without performing any calculations.

- If the rate is 0.25%, in comparison to the base the portion is
 - just a little bit smaller than the base.
 - a lot smaller than the base.
 - just a little bit bigger than the base.
 - a lot bigger than the base.
- If the portion is \$44,931 and the base is \$30,000, the rate is
 - smaller than 100%.
 - equal to 100%.
 - larger than 100% but less than 200%.
 - larger than 200%.
- If the rate is 75% and the portion is \$50,000, the base is
 - smaller than \$50,000.
 - larger than \$50,000.
 - the same as the portion and equal to \$50,000.

Solutions:
 1. b (0.25% is 0.0025, resulting in a very small portion)
 2. c (the portion is larger than the base, but not twice as large)
 3. b (the portion represents 75% of the base, meaning the base must be larger)

Example 2.3B: Rate, Portion, Base

Solve for the unknown in the following three scenarios.

- If your total income is \$3,000 per month and you can't spend more than 30% on housing, what is the maximum amount of your total income that can be spent on housing?
- Your manager tells you that 2024 sales are 102% of 2023 sales. The sales for 2024 are \$1,487,003. What were the sales in 2023?
- In a large city, total commercial real estate sales in the first quarter of 2020 were \$1.28 billion. The industrial, commercial, and institutional (ICI) land sector in the city had sales of \$409.6 million. What percentage of commercial real estate sales is accounted for by the ICI land sector?

Plan

- You are looking for the maximum amount of your income that can be spent on housing.
- You need to figure out the sales for 2023.
- You must determine the percentage of commercial real estate sales accounted for by the ICI land sector in the city.

Understand	What You Already Know - Look for key words in the question: "what is the maximum amount" and "of your total income." The total income is the base, and the maximum amount is the portion. Base = \$3,000 Rate = 30% Portion = maximum amount - Look for key words in the question: "sales for 2024 are \$1,487,003" and "of 2023 sales." The 2024 sales is the portion, and the 2023 sales is the base. Portion = \$1,487,003 Rate = 102% Base = 2023 sales - Look for key words in the question: "of commercial real estate sales" and "are accounted for by the ICI land sector." The commercial real estate sales are the base, and the ICI land sector sales are the portion. Base = \$1.28 billion Portion = \$409.6 million Rate = percentage	How You Will Get There - Apply Formula 2.2, but rearrange using the triangle technique to have Portion = Rate $\times$ Base. - Apply Formula 2.2, but rearrange using the triangle technique to have Base = Portion / Rate. - Apply Formula 2.2, Rate = Portion / Base.
Perform	- Portion = $30\% \times \\$3,000 = 0.3 \times \\$3,000 = \\$900$ - Base = $\frac{\\$1,487,003}{102\%} = \frac{\\$1,487,003}{1.02} = \\$1,457,846.08$ - Rate = $\frac{\\$409.6 \text{ million}}{\\$1.28 \text{ billion}} = \frac{\\$409,600,000}{\\$1,280,000,000} = 0.32 \text{ or } 32\%$	
Present	- The maximum you can spend on housing is \$900 per month. 2023 sales were \$1,457,846.08. - The ICI land sector accounted for 32% of commercial real estate sales in the city for the first quarter of 2020.	



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2.4: Algebraic Expressions

(The Pieces Of The Puzzle)

If you are like most Canadians, your employer pays you biweekly. Assume you earn \$12.00 per hour. How do you calculate your pay cheque every pay period? Your earnings are calculated as follows:

$$\$12.00 \times (\text{Hours worked during the biweekly pay period})$$

The hours worked during the biweekly pay period is the unknown variable. Notice that the expression appears lengthy when you write out the explanation for the variable. Algebra is a way of making such expressions more convenient to manipulate. To shorten the expression, making it easier to read, algebra assigns a letter or group of letters to represent the variable. In this case, you might choose h to represent "hours worked during the biweekly pay period." This rewrites the above expression as follows:

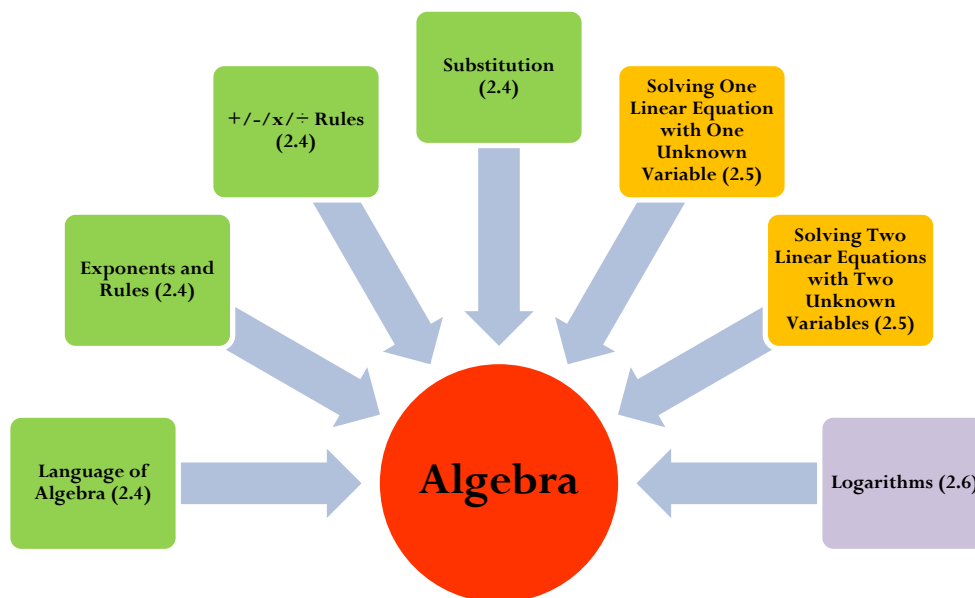
$$\$12.00 \times h \text{ or } \$12h$$

Unfortunately, the word *algebra* makes many people's eyes glaze over. But remember that algebra is just a way of solving a numerical problem. It demonstrates how the pieces of a puzzle fit together to arrive at a solution.

For example, you have used your algebraic skills if you have ever programmed a formula into Microsoft Excel. You told Excel there was a relationship between cells in your spreadsheet. Perhaps your calculation required cell A3 to be divided by cell B6 and then multiplied by cell F2. This is an algebraic equation. Excel then took your algebraic equation and calculated a solution by automatically substituting in the appropriate values from the referenced cells (your variables).

As illustrated in the figure to the right, algebra involves integrating many interrelated concepts. This figure shows only the concepts that are important to business mathematics, which this textbook will present piece by piece. Your understanding of algebra will become more complete as more concepts are covered over the course of this book.

This section reviews the language of algebra, exponent rules, basic operation rules, and substitution. In Section 2.5 you will put these concepts to work in solving one linear equation for one unknown variable along with two linear equations with two unknown variables. Finally, in Section 2.6 you will explore the concepts of logarithms and natural logarithms.



The Language of Algebra

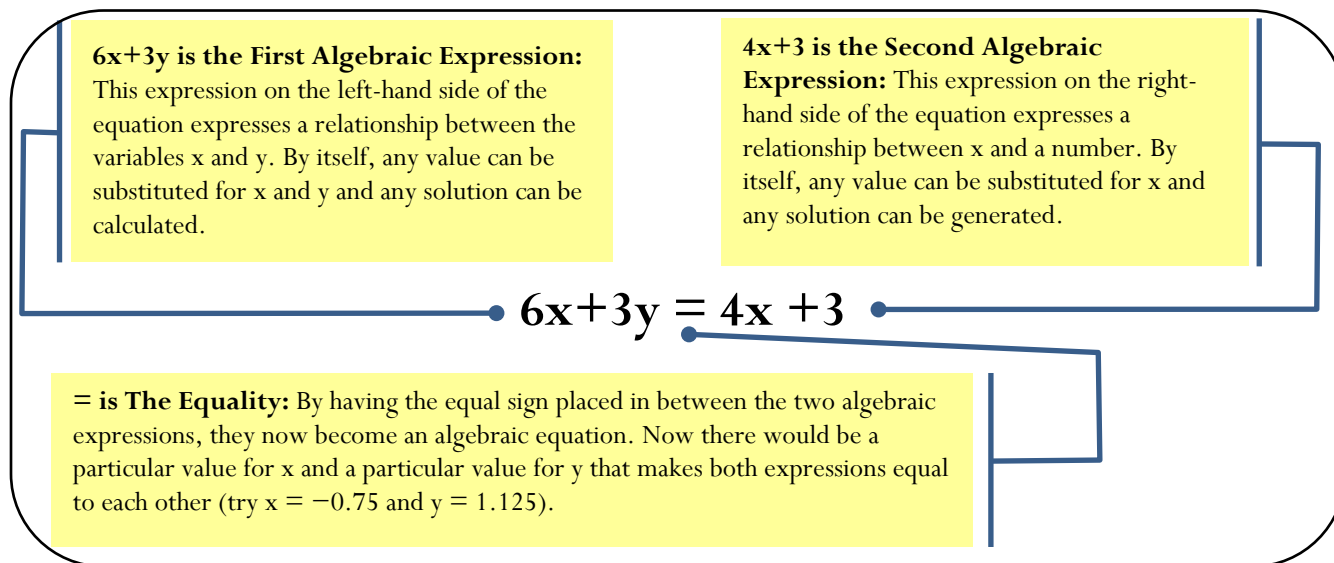
Understanding the rules of algebra requires familiarity with four key definitions.

Algebraic Expression

A mathematical **algebraic expression** indicates the relationship between and mathematical operations that must be conducted on a series of numbers or variables. For example, the expression $\$12h$ says that you must take the hourly wage of \$12 and multiply it by the hours worked. Note that the expression does not include an equal sign, or " $=$ ". It only tells you what to do and requires that you substitute a value in for the unknown variable(s) to solve. There is no one definable solution to the expression.

Algebraic Equation

A mathematical **algebraic equation** takes two algebraic expressions and equates them. This equation can be solved to find a solution for the unknown variables. Examine the following illustration to see how algebraic expressions and algebraic equations are interrelated.



Term

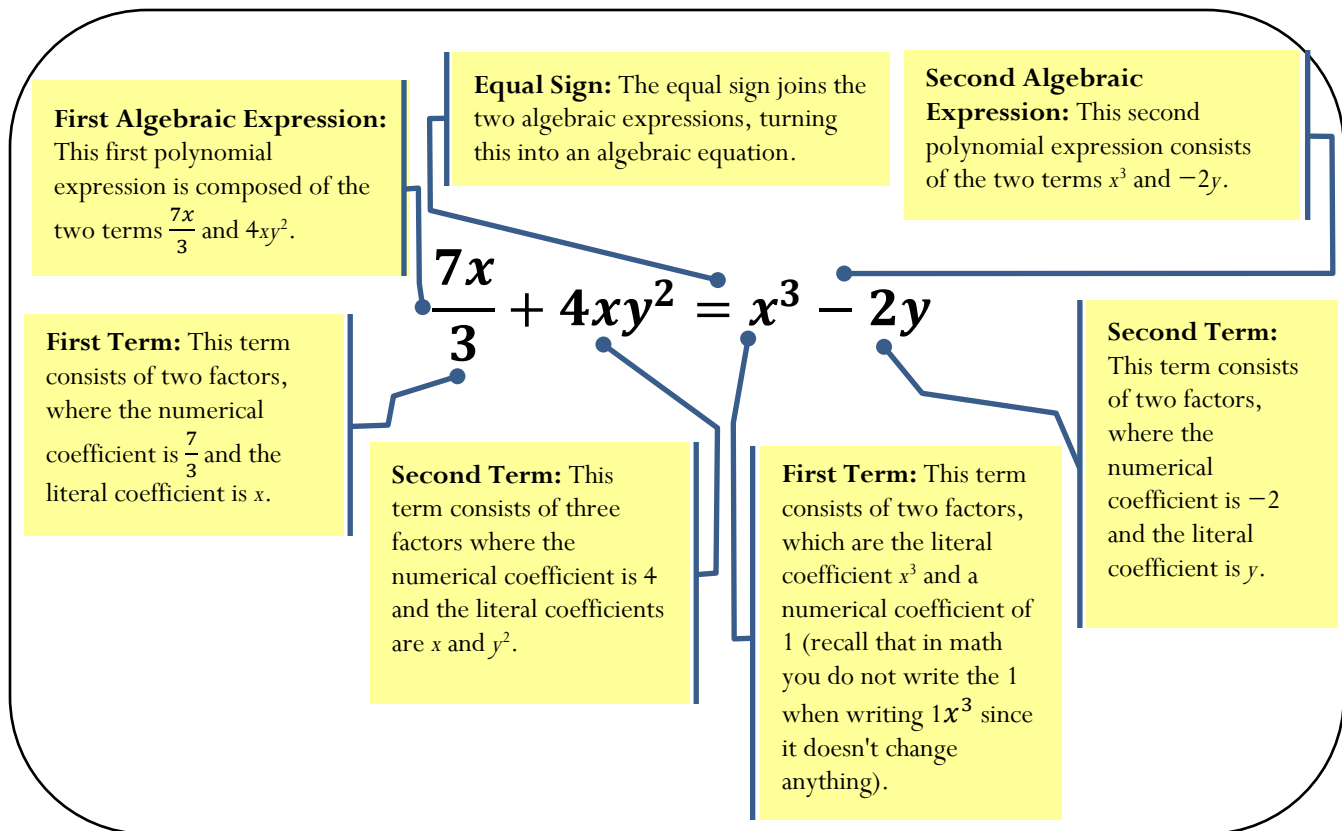
In any algebraic expression, **terms** are the components that are separated by addition and subtraction. In looking at the example above, the expression $6x + 3y$ is composed of two terms. These terms are “ $6x$ ” and “ $3y$.” A **nomial** refers to how many terms appear in an algebraic expression. If an algebraic expression contains only one term, like “\$12.00 h ,” it is called a **monomial**. If the expression contains two terms or more, such as “ $6x + 3y$,” it is called a **polynomial**.

Factor

Terms may consist of one or more **factors** that are separated by multiplication or division signs. Using the $6x$ from above, it consists of two factors. These factors are “6” and “ x ”; they are joined by multiplication.

- If the factor is numerical, it is called the **numerical coefficient**.
- If the factor is one or more variables, it is called the **literal coefficient**.

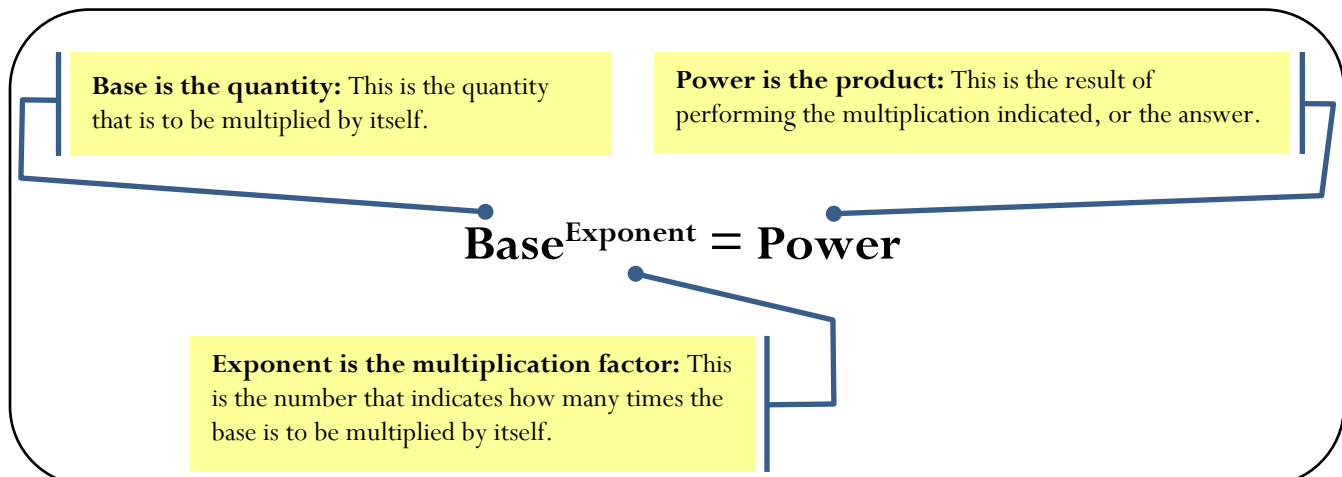
The next graphic shows how algebraic expressions, algebraic equations, terms, and factors all interrelate within an equation.



Exponents

Exponents are widely used in business mathematics and are integral to financial mathematics. When applying compounding interest rates to any investment or loan, you must use exponents (see Chapter 9 and onwards).

Exponents are a mathematical shorthand notation that indicates how many times a quantity is multiplied by itself. The format of an exponent is illustrated below.



Assume you have $2^3 = 8$. The exponent of 3 says to take the base of 2 multiplied by itself three times, or $2 \times 2 \times 2$. The power is 8. The proper way to state this expression is "2 to the exponent of 3 results in a power of 8."



How It Works

Many rules apply to the simplification of exponents, as shown in the table below.

Rule	Illustration	Explanation
1. Multiplication	$y^a \times y^b = y^{a+b}$	If the bases are identical, add the exponents and retain the unchanged base.
2. Division	$\frac{y^a}{y^b} = y^{a-b}$	If the bases are identical, subtract the exponents and retain the unchanged base.
3. Raising powers to exponents	$(y^b z^c)^a = y^{b \times a} z^{c \times a}$ or $\left(\frac{y^b}{z^c}\right)^a = \frac{y^{b \times a}}{z^{c \times a}}$	If a <i>single</i> term is raised to an exponent, each factor must retain its base and you must multiply the exponents for each by the raised exponent. Note that if the expression inside the brackets has more than one term, such as $(y^b + z^c)^a$, which has two terms, you cannot multiply the exponent a inside the brackets.
4. Zero exponents	$y^0 = 1$	Any base to a zero exponent will always produce a power of 1. This is explained later in this section when you review the concept of algebraic division.
5. Negative exponents	$y^{-a} = \frac{1}{y^a}$	The negative sign indicates that the power has been shifted between the numerator and denominator. It is commonly used in this textbook to simplify appearances. Note that on the BAII Plus calculator, keying in a negative exponent requires you to key in the value of y , press y^x , enter the value of a , press $\pm$, and then press $=$ to calculate.
6. Fractional exponents	$y^{\frac{a}{b}} = \sqrt[b]{y^a}$	A fractional exponent is a different way of writing a radical sign. Note that the base is first taken to the exponent of a , then the root of b is found to get the power. For example, $2^{4/2} = \sqrt[2]{2^4} = \sqrt[2]{16} = 4$. This is the same as $2^{4/2} = 2^2 = 4$. To key in a fractional exponent on the BAII Plus calculator, input the value of y , press y^x , open a set of brackets, key in $a \div b$, close the brackets, and press $=$ to calculate.



Important Notes

Recall that mathematicians do not normally write the number 1 when it is multiplied by another factor because it doesn't change the result. The same applies to exponents. If the exponent is a 1, it is generally not written because any number multiplied by itself only once is the same number. For example, the number 2 could be written as 2^1 , but the power is still 2. Or take the case of $(yz)^a$. This could be written as $(y^1 z^1)^a$, which when simplified becomes $y^{1 \times a} z^{1 \times a}$ or $y^a z^a$. Thus, even if you don't see an exponent written, you know that the value is 1.

Example 2.4A: Exponents in Algebra

Simplify the following expressions:

a. $h^3 \times h^6$ b. $\frac{h^{14}}{h^8}$ c. $\left[\frac{hk^5m^3}{n^4}\right]^3$ d. 1.49268^0 e. $\frac{x^2y^4}{xy^{-2}}$ f. $6^{3/5}$

Plan

You have been asked to simplify the expressions. Notice that expressions (d) and (f) contain only numerical coefficients and therefore can be resolved numerically. All of the other expressions involve literal coefficients and require algebraic skills to simplify.

Understand	What You Already Know You have been provided with the expressions, and you have six rules for simplifying exponents in algebra.	How You Will Get There a. This expression involves multiplying two powers with the same base. Apply Rule #1. b. This expression involves dividing two powers with the same base. Apply Rule #2. c. This expression involves a single term, with products and a quotient all raised to an exponent. Apply Rule #3. d. This power involves a zero exponent. Apply Rule #4. e. This expression involves multiplication, division, and negative exponents. Apply Rules #1, #2, and #5. f. This power involves a fractional exponent. Apply Rule #6.
Perform	a. $h^3 \times h^6 = h^{3+6} = h^9$ b. $\frac{h^{14}}{h^8} = h^{14-8} = h^6$ c. $\left[\frac{hk^5m^3}{n^4} \right]^3 = \frac{h^{1 \times 3} k^{5 \times 3} m^{3 \times 3}}{n^{4 \times 3}} = \frac{h^3 k^{15} m^9}{n^{12}}$ d. $1.49268^0 = 1$ e. $\frac{x^2 y^4}{x y^{-2}} = \frac{x^2 y^4 y^2}{x^1} \text{ (Rule \#5)} = \frac{x^2 y^{4+2}}{x^1} \text{ (Rule \#1)} = \frac{x^2 y^6}{x^1} = x^{2-1} y^6 \text{ (Rule \#2)} = x y^6$ f. $6^{3/5} = 2.930156$	Calculator Instructions d. $1.49268 \ y^x \ 0 =$ f. $6 \ y^x \ (\ 3 \div 5 \) =$
Present	Here are the simplified solutions: a. h^9 b. h^6 c. $\frac{h^3 k^{15} m^9}{n^{12}}$ d. 1 e. xy^6 f. 2.930156	

Addition and Subtraction

Simplification of unnecessarily long or complex algebraic expressions is always preferable to increase understanding and reduce the chances of error. For example, assume you are a production manager looking to order bolts for a product that you make. Your company makes three products, all in equal quantity. Product A requires seven bolts, Product B requires four bolts, and Product C requires fourteen bolts. If q represents the quantity of products required, you need to order $7q + 4q + 14q$ bolts. This expression requires four calculations to solve every time (each term needs to be multiplied by q and you then need to add everything together). With the algebra rules that follow, you can simplify this expression to $25q$. This requires only one calculation to solve. So what are the rules?



How It Works

In math, terms with the same literal coefficients are called **like terms**. Only terms with the identical literal coefficients may be added or subtracted through the following procedure:

Step 1: Simplify any numerical coefficients by performing any needed mathematical operation or converting fractions to decimals. For example, terms such as $\frac{1}{2}y$ should become $0.5y$.

Step 2: Add or subtract the numerical coefficients of like terms as indicated by the operation while obeying the rules of BEDMAS.

Step 3: Retain and do not change the common literal coefficients. Write the new numerical coefficient in front of the retained literal coefficients.

From the previous example, you require $7q + 4q + 14q$ bolts. Notice that there are three terms, each of which has the same literal coefficient. Therefore, you can perform the required addition.

Step 1: All numerical coefficients are simplified already. Skip to step 2.

Step 2: Take the numerical coefficients and add the numbers: $7 + 4 + 14$ equals 25.

Step 3: Retain the literal coefficient of q . Put the new numerical coefficient and literal coefficient together. Thus, $25q$.

Therefore $7q + 4q + 14q$ is the same as $25q$.



Things To Watch Out For

A common mistake in addition and subtraction is combining terms that do not have the same literal coefficient. You need to remember that the literal coefficient *must* be identical. For example, $7q$ and $4q$ have the identical literal coefficient of q . However, $7q$ and $4q^2$ have different literal coefficients, q and q^2 , and cannot be added or subtracted.



Paths To Success

Remember that if you come across a literal coefficient with no number in front of it, that number is assumed to be a

1. For example, x has no written numerical coefficient, but it is the same as $1x$. Another example would be $\frac{x}{4}$ is the same as $\frac{1x}{4}$ or $\frac{1}{4}x$.

On a similar note, mathematicians also don't write out literal coefficients that have an exponent of zero. For example, $7x^0$ is just $7(1)$ or 7 . Thus, the literal coefficient is always there; however, it has an exponent of zero. Remembering this will help you later when you multiply and divide in algebra.



Give It Some Thought

Examine the following algebraic expressions and indicate how many terms can be combined through addition and subtraction. No calculations are necessary. Do not attempt to simplify.

1. $\frac{3}{2}x + 4x^2 - 10x - 2y + \frac{x}{3}$

2. $23g^2 - \frac{17g^2}{5} + g^4 + g^2 - \frac{2}{3}g^2 - 0.15g + g^3$

Solutions:
1. Three terms (all of the x)
2. Four terms (all of the g^2)

Example 2.4B: Addition and Subtraction in Algebra

Simplify the following three algebraic expressions.

a. $9x + 3y - \frac{7}{2}x + 4y$

b. $P(1 + 0.11 \times \frac{121}{365}) + \frac{15P}{1 + 0.11 \times \frac{36}{365}}$

c. $x(1 + \frac{0.1}{4})^3 + \frac{x}{(1 + \frac{0.1}{4})^4} - \frac{3x}{(1 + \frac{0.1}{4})^2}$

Plan You have been asked to simplify the three algebraic expressions. Notice that each term of the expressions is joined to the others by addition or subtraction. Therefore, apply the addition and subtraction rules to each expression.

Understand **What You Already Know**
The three algebraic expressions have already been supplied.

How You Will Get There
Applying the three steps for addition and subtraction you should simplify, combine, and write out the solution.

Perform	a.		
	$9x + 3y - \frac{7}{2}x + 4y$	Step 1: Simplify the numerical coefficients.	
	$9x + 3y - 3.5x + 4y$	Step 2: Combine the numerical coefficients of like terms. You have two terms with x from which you need to take $9 - 3.5 = 5.5$. You also have two terms with y from which you need to take $3 + 4 = 7$.	
	$5.5x + 7y$	Step 3: Write the numerical coefficients in front of the unchanged literal coefficients.	
	b.		
	$P \left(1 + 0.11 \times \frac{121}{365} \right) + \frac{15P}{1 + 0.11 \times \frac{36}{365}}$	Step 1: Simplify the numerical coefficients.	
	$1.036465P + \frac{15}{1.010849}P$	Step 1: Continue to simplify the second term.	
	$1.036465P + 14.839006P$	Step 2: Combine the numerical coefficients by performing the addition.	
	$15.875472P$	Step 3: Write the numerical coefficient in front of the unchanged literal coefficient.	
	c.		
	$x \left(1 + \frac{0.1}{4} \right)^3 + \frac{x}{\left(1 + \frac{0.1}{4} \right)^4} - \frac{3x}{\left(1 + \frac{0.1}{4} \right)^2}$	Step 1: Simplify the numerical coefficients.	
	$1.076890x + \frac{1}{1.103812}x - \frac{3}{1.050625}x$	Step 1: Continue to simplify the second and third terms.	
	$1.076890x + 0.905950x - 2.855443x$	Step 2: Combine the numerical coefficients through the specified operations.	
	$-0.872602x$	Step 3: Write the numerical coefficient in front of the unchanged literal coefficient.	
Present	The algebraic expressions simplify as follows: a. $5.5x + 7y$ b. $15.15566P$ c. $-0.872602x$ Notice how much easier these are to work with than the original expressions.		

Multiplication

Whether you are multiplying a monomial by another monomial, a monomial by a polynomial, or a polynomial by another polynomial, the rules for multiplication remain the same.



How It Works

Follow these steps to multiply algebraic expressions:

Step 1: Check to see if there is any way to simplify the algebraic expression first. Are there any like terms you can combine? For example, you can simplify $(3x + 2 + 1)(x + x + 4)$ to $(3x + 3)(2x + 4)$ before attempting the multiplication.

Step 2: Take *every* term in the first algebraic expression and multiply it by *every* term in the second algebraic expression. This means that numerical coefficients in both terms are multiplied by each other, and the literal coefficients in both terms are multiplied by each other. It is best to work methodically from left to right so that you do not miss anything. Working with the example, in $(3x + 3)(2x + 4)$ take the first term of the first expression, $3x$, and multiply it by $2x$ and then by 4 . Then move to the second term of the first expression, 3 , and multiply it by $2x$ and then by 4 (see the figure).

$$(3x + 3)(2x + 4)$$

$$(3x)(2x) + (3x)(4) + 3(2x) + 3(4)$$

It becomes: $6x^2 + 12x + 6x + 12$

Step 3: Perform any final steps of simplification by adding or subtracting like terms as needed. In the example, two terms contain the literal coefficient x , so you simplify the expression to $6x^2 + 18x + 12$.



Important Notes

If the multiplication involves more than two expressions being multiplied by each other, it is easiest to work with only one pair of expressions at a time starting with the leftmost pair. For example, if you are multiplying $(4x + 3)(3x)(9y + 5x)$, resolve $(4x + 3)(3x)$ first. Then take the solution, keeping it in brackets since you have not completed the math operation, and multiply it by $(9y + 5x)$. This means you are required to repeat step 2 in the multiplication procedure until you have resolved all the multiplications.



Things To Watch Out For

The negative sign causes no end of grief for a lot of people when working with multiplication. First, if a numerical coefficient is not written explicitly, it is assumed to be 1. For example, look at $2(4a + 6b) - (2a - 3b)$. This is the same as $2(4a + 6b) + (-1)(2a - 3b)$.

When you multiply a negative through an expression, all signs in the brackets will change. Continuing with the second term in the above example, $-(2a - 3b)$ becomes $-2a + 3b$. The expression then looks like $2(4a + 6b) - 2a + 3b$.

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The order in which you write the terms of an algebraic expression does not matter so long as you follow all the rules of BEDMAS. For example, whether you write 3×4 or 4×3 , the answer is the same because you can do multiplication in any order. The same applies to $4 + 3 - 1$ or $3 - 1 + 4$. Now let's get more complex. Whether you write $3x^2 + 5x - 4$ or $5x - 4 + 3x^2$, the answer is the same as you have not violated any rules of BEDMAS. You still multiply first and add last from left to right.

Although the descending exponential format for writing expressions is generally preferred, for example, $3x^2 + 5x + 4$, which lists literal coefficients with higher exponents first, it does not matter if you do this or not. When checking your solutions against those given in this textbook, you only need to ensure that each of your terms matches the terms in the solution provided.

Example 2.4C: Multiplication in AlgebraSimplify the following algebraic expression: $(6x + 2 + 2)(3x - 2)$

Plan	You have been asked to simplify the expression.	
Understand	What You Already Know You are multiplying two expressions with each other. Each expression contains two or three terms.	How You Will Get There Applying the three steps for multiplication of algebraic expressions, you simplify, multiply each term, and combine.
Perform	$(6x + 2 + 2)(3x - 2)$	Step 1: Simplify the expression first.
	$(6x + 4)(3x - 2)$	Step 2: Multiply all terms in each expression with all terms in the other expression.
	$(6x)(3x) + (6x)(-2) + (4)(3x) + (4)(-2)$	Step 2: Resolve the multiplications.
	$18x^2 - 12x + 12x - 8$	Step 3: Perform the final simplifications. You can combine the middle two terms.
	$18x^2 - 8$	Final solution.
Present	The simplified algebraic expression is $18x^2 - 8$.	

Example 2.4D: More Challenging Algebraic MultiplicationSimplify the following algebraic expression: $-(3ab)(a^2 + 4b - 2a) - 4(3a + 6)$

Plan	You have been asked to simplify the expression.	
Understand	What You Already Know You have been provided with the expression. Notice that the first term consists of three expressions being multiplied together. The second term involves two expressions being multiplied together. Apply the rules of multiplication.	How You Will Get There Applying the three steps for multiplication of algebraic expressions, you will simplify, multiply each term, and combine.
Perform	$-(3ab)(a^2 + 4b - 2a) - 4(3a + 6)$	Step 1: You cannot simplify anything. Be careful with the negatives and write them out.
	$(-1)(3ab)(a^2 + 4b - 2a) + (-4)(3a + 6)$	Step 2: Work with the first pair of expressions in the first term and multiply.
	$(-3ab)(a^2 + 4b - 2a) + (-4)(3a + 6)$	Step 2: Multiply the resulting pair of expressions in the first term.
	$-3a^3b - 12ab^2 + 6a^2b + (-4)(3a + 6)$	Step 2: Work with the pair of expressions in the original second term and multiply.
	$-3a^3b - 12ab^2 + 6a^2b + [-12a - 24]$	Step 3: Drop the brackets to simplify.
	$-3a^3b - 12ab^2 + 6a^2b - 12a - 24$	Step 3: There are no like terms. You cannot simplify this expression any further.
Present	The simplified algebraic expression is $-3a^3b - 12ab^2 + 6a^2b - 12a - 24$.	

Division

You are often required to divide a monomial into either a monomial or a polynomial. In instances where the denominator consists of a polynomial, it is either not possible or extremely difficult to simplify the expression algebraically. Only division where denominators are monomials are discussed here.



How It Works

To simplify an expression when its denominator is a monomial, apply these rules:

Step 1: Just as in multiplication, determine if there is any way to combine like terms before completing the division. For example, with $\frac{3ab + 3ab - 3a^2b + 9ab^2}{3ab}$ you can simplify the numerator to $\frac{6ab - 3a^2b + 9ab^2}{3ab}$.

Step 2: Take *every* term in the numerator and divide it by the term in the denominator. This means that you must divide both the numerical and the literal coefficients. As with multiplication, it is usually best to work methodically from left to right so that you do not miss anything. So in our example we get:

$$\begin{aligned} \frac{6ab}{3ab} - \frac{3a^2b}{3ab} + \frac{9ab^2}{3ab} \\ (2)(1)(1) - (1)(a)(1) + (3)(1)(b) \\ 2 - a + 3b \end{aligned}$$

Step 3: Perform any final simplification by adding or subtracting the like terms as needed. As there are no more like terms, the final expression remains $2 - a + 3b$.



Things To Watch Out For

You may have heard of an outcome called "cancelling each other out." For example, in resolving the division $\frac{4a}{4a}$ many people would say that the terms cancel each other out. Many people will also mistakenly interpret this to mean that the quotient is zero and say that $\frac{4a}{4a} = 0$. In fact, when terms cancel each other out the quotient is one, not zero. The numerical coefficient is $\frac{4}{4} = 1$. The literal coefficient is $\frac{a}{a} = 1$. Thus, $\frac{4a}{4a} = (1)(1) = 1$. This also explains why a zero exponent equals one: $\frac{a^1}{a^1} = a^{1-1} = a^0 = 1$.



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A lot of people dislike fractions and find them hard to work with. Remember that when you are simplifying any algebraic expression you can transform any fraction into a decimal. For example, if your expression is $\frac{2x}{5} + \frac{3x}{4}$, you can convert the fraction into decimals: $0.4x + 0.75x$. In this format, it is easier to solve.

Example 2.4E: A Monomial Division

Simplify the following algebraic expression: $\frac{30x^6 + 5x^3 + 10x^3}{5x}$

Plan	You have been asked to simplify the expression.	
Understand	What You Already Know Notice that the provided expression is a polynomial divided by a monomial. Therefore, apply the division rules.	How You Will Get There Applying the three steps for division of an algebraic expression, you will simplify, divide each term, and combine.
Perform	$\frac{30x^6 + 5x^3 + 10x^3}{5x}$	Step 1: The numerator has two terms with the same literal coefficient (x^3). Combine these using the rules of addition.
	$\frac{30x^6 + 15x^3}{5x}$	Step 2: Now that the numerator is simplified, divide each of its terms by the denominator.
	$\frac{30x^6}{5x} + \frac{15x^3}{5x}$	Step 2: Resolve the divisions by dividing both numerical and literal coefficients.
	$6x^5 + 3x^2$	Step 3: There are no like terms, so this is the final solution.
Present	The simplified algebraic expression is $6x^5 + 3x^2$.	

Example 2.4F: A More Challenging Division

Simplify the following algebraic expression: $\frac{15x^2y^3 + 25xy^2 - xy + 10x^4y + 5xy^2}{5xy}$

Plan	You have been asked to simplify the expression.	
Understand	What You Already Know Notice that the provided expression is a polynomial divided by a monomial. Therefore, apply the division rules.	How You Will Get There Applying the three steps for division of algebraic expressions, you will simplify, divide each term, and combine.
Perform	$\frac{15x^2y^3 + 25xy^2 - xy + 10x^4y + 5xy^2}{5xy}$	Step 1: The numerator has two terms with the same literal coefficient (xy^2). Combine these by the rules of addition.
	$\frac{15x^2y^3 + 30xy^2 - xy + 10x^4y}{5xy}$	Step 2: Now that the numerator is simplified, divide each of its terms by the denominator.
	$\frac{15x^2y^3}{5xy} + \frac{30xy^2}{5xy} - \frac{xy}{5xy} + \frac{10x^4y}{5xy}$	Step 2: Resolve the divisions by dividing both numerical and literal coefficients.
	$3xy^2 + 6(1)(y) - 0.2(1)(1) + 2(x^3)(1)$	Step 3: Simplify and combine any like terms.
	$3xy^2 + 6y - 0.2 + 2x^3$	There are no like terms, so this is the final solution.
Present	The simplified algebraic expression is $3xy^2 + 6y - 0.2 + 2x^3$.	

Substitution

The ultimate goal of algebra is to represent a relationship between various variables. Although it is beneficial to simplify these relationships where possible and shorten the algebraic expressions, in the end you want to calculate a solution.

Substitution involves replacing the literal coefficients of an algebraic expression with known numerical values. Once the substitution has taken place, you solve the expression for a final value.



How It Works

Follow these steps to perform algebraic substitution:

Step 1: Identify the value of your variables. Assume the algebraic equation is $PV = \frac{FV}{1 + rt}$. You need to calculate the value of PV. It is known that $FV = \$5,443.84$, $r = 0.12$, and $t = \frac{270}{365}$.

Step 2: Take the known values and insert them into the equation where their respective variables are located, resulting in

$$PV = \frac{\$5,443.84}{1 + (0.12)\left(\frac{270}{365}\right)}$$

Step 3: Resolve the equation to solve for the variable. Calculate $PV = \frac{\$5,443.84}{1.088767} = \$5,000.00$.



Things To Watch Out For

It is common in algebra to represent a variable with more than just one letter. As you can see from the example above, FV is a variable, and it represents *future value*. This should not be interpreted to be two variables, F and V. Similarly, PMT represents *annuity payment*. When you learn new formulas and variables, take careful note of how a variable is represented.

As well, some literal coefficients have subscripts. For example, you could see d_1 and d_2 in the same formula. What sometimes happens is that there is more than one value for the same variable. As you will learn in Chapter 6 about merchandising, when you buy an item you may receive more than one discount rate (what d stands for). Therefore, the first discount gets a subscript of 1, or d_1 , and the second discount gets a subscript of 2, or d_2 . This allows you to distinguish between the two values in the equation and substitute the correct value in the correct place.



Paths To Success

If you are unsure whether you have simplified an expression appropriately, remember that you can make up your own values for any literal coefficient and substitute those values into both the original and the simplified expressions. If you have obeyed all the rules and simplified appropriately, both expressions will produce the same answer. For example, assume you simplified $2x + 5x$ into $7x$, but you are not sure if you are right. You decide to let $x = 2$. Substituting into $2x + 5x$, you get $2(2) + 5(2) = 14$. Substituting into your simplified expression you get $7(2) = 14$. Since both expressions produced the same answer, you have direct confirmation that you have simplified correctly.

Example 2.4G: Substitution

Substitute and solve the following equation:

$$N = L \times (1 - d_1) \times (1 - d_2) \times (1 - d_3)$$

Where $L = \$1,999.99$ $d_1 = 35\%$ $d_2 = 15\%$ $d_3 = 5\%$

Plan	You need to get a dollar value for the literal coefficient N.	
Understand	What You Already Know You are provided with the equation and the values of four literal coefficients.	How You Will Get There Step 1: The values of L, d_1 , d_2 , and d_3 are known. Step 2: Substitute those values into the equation. Step 3: Solve for N.
Perform	Step 1: $L = \$1,999.99$ $d_1 = 0.35$ $d_2 = 0.15$ $d_3 = 0.05$ Step 2: $N = \$1,999.99 \times (1 - 0.35) \times (1 - 0.15) \times (1 - 0.05)$ Step 3: $N = \$1,999.99 \times 0.65 \times 0.85 \times 0.95$ $N = \$1,049.74$	
Present	The value of N is \$1,049.74.	

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2.5: Linear Equations: Manipulating and Solving (Solving the Puzzle)

You are shopping at Old Navy for seven new outfits. The price points are \$10 and \$30. You really like the \$30 outfits; however, your total budget can't exceed \$110. How do you spend \$110 to acquire all the needed outfits without exceeding your budget while getting as many \$30 items as possible?

This is a problem of linear equations, and it illustrates how you can use them to make an optimal decision. Let L represent the quantity of clothing at the low price point of \$10, and H represent the quantity of clothing at the high price point of \$30. This results in the following algebraic equations:

$$L + H = 7 \text{ (the total number of outfits you need)}$$

$$\$10L + \$30H = \$110 \text{ (your total budget)}$$

By simultaneously solving these equations you can determine how many outfits at each price point you can purchase.

You will encounter many situations like this in your business career, for example, in making the best use of a manufacturer's production capacity. Assume your company makes two products on the same production line and sells all its output. Each product contributes differently to your profitability, and each product takes a different amount of time to manufacture. What combination of each of these products should you make such that you operate your production line at capacity while also maximizing the profits earned? This section explores how to solve linear equations for unknown variables.

Understanding Equations

To manipulate algebraic equations and solve for unknown variables, you must first become familiar with some important language, including linear versus nonlinear equations and sides of the equation.

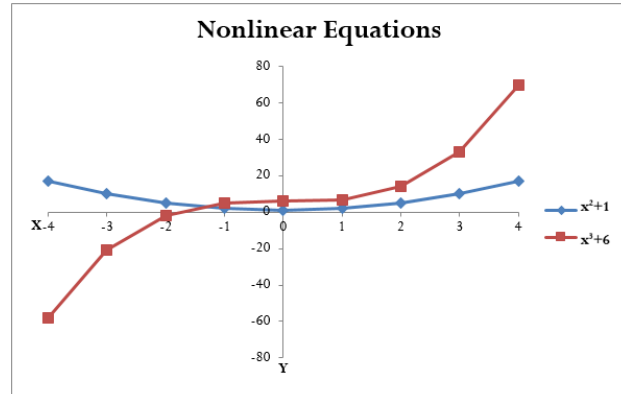
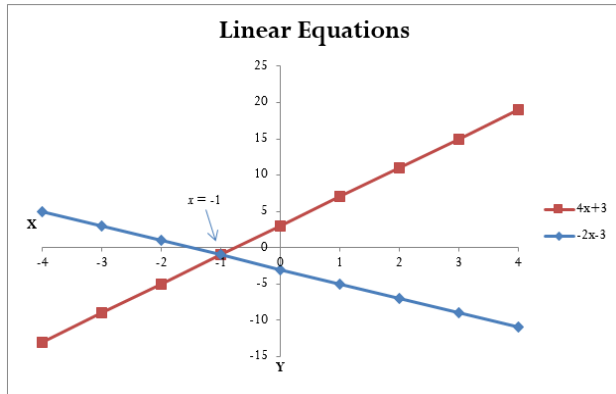
$4x+3$ is the Left Side: Every equation has two sides. Everything to the left side of the equal sign is known as the **left side of the equation**.

$-2x - 3$ is the Right Side: Everything to the right side of the equal sign is known as the **right side of the equation**.

$$4x + 3 = -2x - 3$$

x is the Variable x : Observe two important features of this variable:

1. Note that the variable has an exponent of 1. So if you were to plot each algebraic expression onto a Cartesian coordinate system, the expressions would form straight lines (see the left-hand figure next page). Such equations are called **linear equations**. Because each equation has only one variable, there is only one solution that makes the equation true. In business mathematics, this is the most common situation you will encounter.
2. If the exponent is anything but a one, the equation is a **nonlinear equation** since the graph of the expression does not form a straight line. The right-hand figure (next page) illustrates two algebraic expressions where the exponent is other than a 1.



The goal in manipulating and solving a linear equation is to find a value for the unknown variable that makes the equation true. If you substitute a value of $x = -1$ into the above example, the left-hand side of the equation equals the right-hand side of the equation (see Figure 2.17). The value of $x = -1$ is known as the **root**, or solution, to the linear equation.

Solving One Linear Equation with One Unknown Variable

In your study of solving linear equations, you need to start by manipulating a single equation to solve for a single unknown variable. Later in this section you will extend from this foundation to the solution of two linear equations with two unknowns.



How It Works

To determine the root of a linear equation with only one unknown variable, apply the following steps:

Step 1: Your first goal is to separate the terms containing the literal coefficient from the terms that only have numerical coefficients. Collect all of the terms with literal coefficients on only one side of the equation and collect all of the terms with only numerical coefficients on the other side of the equation. It does not matter which terms go on which side of the equation, so long as you separate them.

To move a term from one side of an equation to another, take the mathematical opposite of the term being moved and add it to both sides. For example, if you want to move the $+3$ in $4x + 3 = -2x - 3$ from the left-hand side to the right-hand side, the mathematical opposite of $+3$ is -3 . When you move a term, remember the cardinal rule: What you do to one side of an equation you must also do to the other side of the equation. Breaking this rule breaks the equality in the equation.

Step 2: Combine all like terms on each side and simplify the equation according to the rules of algebra.

Step 3: In the term containing the literal coefficient, reduce the numerical coefficient to a 1 by dividing both sides of the equation by the numerical coefficient.



Important Notes

When you are unsure whether your calculated root is accurate, an easy way to verify your answer is to take the original equation and substitute your root in place of the variable. If you have the correct root, the left-hand side of the equation equals the right-hand side of the equation. If you have an incorrect root, the two sides will be unequal. The inequality typically results from one of the three most common errors in algebraic manipulation:

1. The rules of BEDMAS have been broken.
2. The rules of algebra have been violated.
3. What was done to one side of the equation was not done to the other side of the equation.



Things To Watch Out For

When you move a term from one side of the equation to another using multiplication or division, remember that this affects *each and every term on both sides of the equation*. To remove the x from the denominator in the following equation, multiply both sides of the equation by x :

$$\frac{5}{x} + \frac{1}{x} = \frac{2}{x} + 2 \quad \text{becomes} \quad x\left(\frac{5}{x} + \frac{1}{x}\right) = \left(\frac{2}{x} + 2\right)x \quad \text{which then becomes} \quad 5 + 1 = 2 + 2x$$

Multiplying every term on both sides by x maintains the equality.



Paths To Success

Negative numbers can cause some people a lot of grief. In moving terms from a particular side of the equation, many people prefer to avoid negative numerical coefficients in front of literal coefficients. Revisiting $4x + 3 = -2x - 3$, you could move the $4x$ from the left side to the right side by subtracting $4x$ from both sides. However, on the right side this results in $-6x$. The negative is easily overlooked or accidentally dropped in future steps. Instead, move the variable to the left side of the equation, yielding a positive coefficient of $6x$.

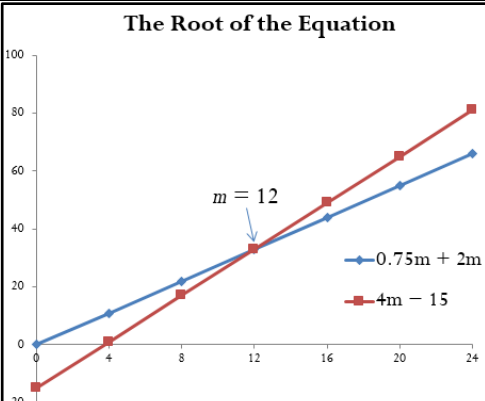
Example 2.5A: How to Solve the Opening Example		
Take the ongoing example in this section and solve it for x : $4x + 3 = -2x - 3$		
Plan	This is a linear equation since the exponent on the variable is 1. You are to solve the equation and find the root for x .	
Understand	What You Already Know The equation has already been provided.	How You Will Get There Apply the three steps for solving linear equations. To arrive at the root, you must follow the rules of algebra, BEDMAS, and equality.
Perform	$4x + 3 = -2x - 3$	Step 1: Move terms with literal coefficients to one side and terms with only numerical coefficients to the other side. Let's collect the literal coefficient on the left-hand side of the equation. Move $-2x$ to the left-hand side by placing $+2x$ on both sides.
	$4x + 3 + 2x = -2x - 3 + 2x$	On the right-hand side, the $-2x$ and $+2x$ cancel out to zero.
	$4x + 3 + 2x = -3$	Step 1: All of the terms with the literal coefficient are now on the left. Let's move all of the terms containing only numerical coefficients to the right-hand side. Move the $+3$ to the right-hand side by placing -3 on both sides.
	$4x + 3 + 2x - 3 = -3 - 3$	On the left-hand side, the $+3$ and -3 cancel out to zero.

	$4x + 2x = -3 - 3$	Step 2: The terms are now separated. Combine like terms according to the rules of algebra.
	$6x = -6$	Step 3: The term with the literal coefficient is being multiplied by the numerical coefficient of 6. Therefore, divide both sides by 6.
	$\frac{6x}{6} = \frac{-6}{6}$	The left-hand side numerical coefficients will divide to 1. Resolve the numerical coefficients on the right-hand side.
	$x = -1$	This is the root of the equation.
Present	The root of the equation is $x = -1$. To verify the accuracy of your manipulation, take the root of $x = -1$ and substitute it into the original equation: $4(-1) + 3 = -2(-1) - 3$ $-4 + 3 = 2 - 3$ $-1 = -1$ The left-hand side equals the right-hand side, so the root is correct.	

Example 2.5B: Solving a Linear Equation with One Unknown Variable

Solve the following equation for m : $\frac{3m}{4} + 2m = 4m - 15$

Plan	This is a linear equation since the exponent on the variable is a 1. You are to solve the equation and find the root for m .	
Understand	What You Already Know The equation has already been provided.	How You Will Get There Simplify the equations first and then apply the three steps for solving linear equations. To arrive at the root you must follow the rules of algebra, BEDMAS, and equality. You can use an approach that avoids negatives.
Perform	$\frac{3m}{4} + 2m = 4m - 15$	First, simplify all fractions to make the equation easier to work with.
	$0.75m + 2m = 4m - 15$	Still simplifying, collect like terms where possible.
	$2.75m = 4m - 15$	Step 1: Collect all terms with the literal coefficient on one side of the equation. Move all terms with literal coefficients to the right-hand side.
	$2.75m - 2.75m = 4m - 15 - 2.75m$	Step 1: Combine like terms and move all terms with only numerical coefficients to the left-hand side.
	$0 = 4m - 15 - 2.75m$	On the left-hand side, the $+2.75m$ and $-2.75m$ cancel each other out. Now move the numerical coefficients to the left-hand side.
	$0 + 15 = 4m - 15 - 2.75m + 15$	On the right-hand side, the -15 and $+15$ cancel each other out.
	$0 + 15 = 4m - 2.75m$	Step 2: Combine like terms on each side.
	$15 = 1.25m$	Step 3: Divide both sides by the numerical coefficient that accompanies the literal coefficient.
	$\frac{15}{1.25} = \frac{1.25m}{1.25}$	Simplify.
	$12 = m$	This is the root of the equation.

Present	The root of the equation is $m = 12$. This makes both sides of the equation, $\frac{3m}{4} + 2m \text{ and } 4m - 15, \text{ equal } 33.$	The Root of the Equation 

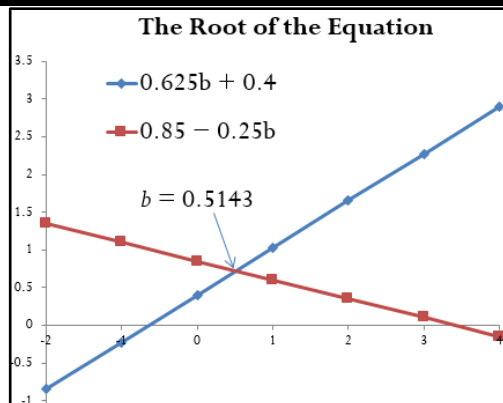
Example 2.5C: Solving a Linear Equation with One Unknown Variable Containing Fractions

Solve the following equation for b and round your answer to four decimals: $\frac{5}{8}b + \frac{2}{5} = \frac{17}{20} - \frac{b}{4}$

Plan	This is a linear equation since the exponent on the variable is a 1. You are to solve the equation and find the root for b .	
Understand	What You Already Know The equation has already been provided. Although you could attempt to clear each and every fraction or try to find a common denominator, recall that you can eliminate fractions by converting them to decimals.	How You Will Get There Simplify the fractions into decimal form. Then apply the three steps for solving linear equations. To arrive at the root, you must follow the rules of algebra, BEDMAS, and equality.
Perform	$\frac{5}{8}b + \frac{2}{5} = \frac{17}{20} - \frac{b}{4}$ $0.625b + 0.4 = 0.85 - 0.25b$ $0.625b + 0.4 + 0.25b = 0.85 - 0.25b + 0.25b$ $0.625b + 0.4 + 0.25b = 0.85$ $0.625b + 0.4 + 0.25b - 0.4 = 0.85 - 0.4$ $0.625b + 0.25b = 0.85 - 0.4$ $0.875b = 0.45$ $\frac{0.875b}{0.875} = \frac{0.45}{0.875}$ $b = 0.514285$ $b = 0.5143$	Simplify the fractions and convert to decimals. Step 1: Move the literal coefficient terms to the left-hand side. The literal coefficients on the right-hand side cancel each other out. Move the numerical coefficient terms to the right-hand side. The numerical coefficients on the left-hand side cancel each other out. Step 2: Combine like terms on each side. Step 3: Divide both sides by the numerical coefficient that accompanies the literal coefficient. Simplify. Round to four decimals as instructed. This is the root.

Present

The root of the equation, rounded to four decimals, is $b = 0.5143$.



Solving Two Linear Equations with Two Unknown Variables

The manipulation process you have just practiced works well for solving one linear equation with one variable. But what happens if you need to solve two linear equations with two variables simultaneously? Remember when you were at Old Navy purchasing seven outfits earlier in this chapter (equation 1)? You needed to stay within a pricing budget (equation 2). Each equation had two unknown variables representing the number of lower-priced and higher-priced outfits.

The goal is to reduce two equations with two unknowns into a single linear equation with one unknown. Once this transformation is complete, you then identify the unknown variable by applying the three-step procedure for solving one linear equation, as just discussed.

When you work with two linear equations with two unknowns, the rules of algebra permit the following two manipulations:

1. What you do to one side of the equation must be done to the other side of the equation to maintain the equality. Therefore, you can multiply or divide any equation by any number without changing the root of the equation. For example, if you multiply all terms of $x + y = 2$ by 2 on both sides, resulting in $2x + 2y = 4$, the equality of the equation remains unchanged and the same roots exist.
2. Terms that are on the same side of an equation can be added and subtracted between equations by combining like terms. Each of the two equations has a left side and right side. This rule permits taking the left side of the first equation and either adding or subtracting like terms on the left side of the second equation. When you perform this action, remember the first rule above. If you add the left sides of the equations together, you then must add the right side of both equations together to maintain equality.



How It Works

Follow these steps to solve two linear equations with two unknown variables:

Step 1: Write the two equations one above the other, vertically lining up terms that have the same literal coefficients and terms that have only the numerical coefficient. If necessary, the equations may need to be manipulated such that all of the literal coefficients are on one side with the numerical coefficients on the other side.

Step 2: Examine your two equations. Through multiplication or division, make the numerical coefficient on one of the terms containing a literal coefficient exactly equal to its counterpart in the other equation.

Step 3: Add or subtract the two equations as needed so as to eliminate the identical term from both equations.

Step 4: In the new equation, solve for the last literal coefficient.

Step 5: Substitute the root of the known literal coefficient into either of the two original equations. If one of the equations takes on a simpler structure, pick that equation.

Step 6: Solve your chosen equation for the other literal coefficient.

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Paths To Success

Sometimes it is unclear exactly how you need to multiply or divide the equations to make two of the terms identical. For example, assume the following two equations:

$$4.9x + 1.5y = 38.3$$

$$2.7x - 8.6y = 17.8$$

If the goal is to make the terms containing the literal coefficient x identical, there are two alternative solutions:

1. Take the larger numerical coefficient for x and divide it by the smaller numerical coefficient. The resulting number is the factor for multiplying the equation containing the smaller numerical coefficient. In this case, $4.9 \div 2.7 = 1.\overline{814}$. Multiply all terms in the second equation by $1.\overline{814}$ to make the numerical coefficients for x equal to each other, resulting in this pair of equations:

$$4.9x + 1.5y = 38.3$$

$$4.9x - 15.\overline{6074}y = 32.\overline{3037} \quad (\text{every term multiplied by } 1.\overline{814})$$

2. Take the first equation and multiply it by the numerical coefficient in the second equation. Then take the second equation and multiply it by the numerical coefficient in the first equation. In this case, multiply all terms in the first equation by 2.7. Then multiply all terms in the second equation by 4.9.

$$13.23x + 4.05y = 103.41 \quad (\text{every term multiplied by } 2.7)$$

$$13.23x - 42.14y = 87.22 \quad (\text{every term multiplied by } 4.9)$$

Note that both approaches successfully result in both equations having the same numerical coefficient in front of the literal coefficient x .

**TOP
SECRET**

Paths To Success

Ultimately, every pairing of linear equations with two unknowns can be converted into a single equation through substitution. To make the conversion, do the following:

1. Solve either equation for one of the unknown variables.
2. Take the resulting algebraic expression and substitute it into the other equation. This new equation is solvable for one of the unknown variables.
3. Substitute your newfound variable into one of the original equations to determine the value for the other unknown variable.

Take the following two equations:

$$a + b = 4$$

$$2a + b = 6$$

1. Solving the first equation for a results in $a = 4 - b$.
2. Substituting the expression for a into the second equation and solving for b results in $2(4 - b) + b = 6$, which solves as $b = 2$.
3. Finally, substituting the root of b into the first equation to calculate a gives $a + 2 = 4$ resulting in $a = 2$. Therefore, the roots of these two equations are $a = 2$ and $b = 2$.

Example 2.5D: Buying Those Outfits

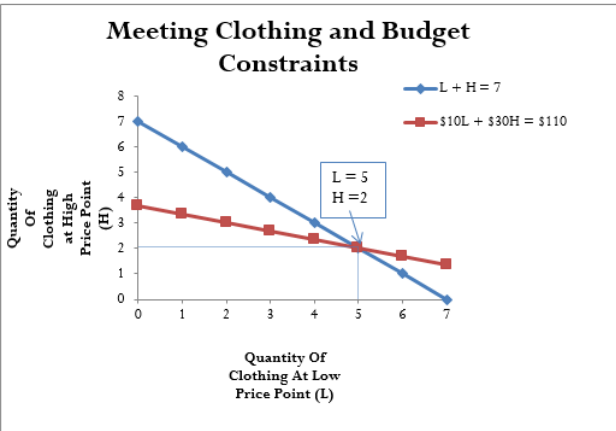
Recall from the section opener that in shopping for outfits there are two price points of \$10 and \$30, your budget is \$110, and that you need seven articles of clothing. The equations below represent these conditions. Identify how many low-priced outfits (L) and high-priced outfits (H) you can purchase.

$$L + H = 7$$

$$\$10L + \$30H = \$110$$

Plan

You need to determine the quantity of low-price-point items, or L , and high-price-point items, or H , that are within your limited budget. Note that the exponents on the variables are 1 and that there are two unknowns. So there are two linear equations with two unknowns.

Understand	What You Already Know You require seven articles of clothing and only have a budget of \$110. The equations express the relationships of quantity and budget.	How You Will Get There Apply the six-step procedure for solving two linear equations with two unknowns.
Perform	$L + H = 7$ $\$10L + \$30H = \$110$	Step 1: Write the equations one above the other and line them up.
	$10L + 10H = 70$ $\$10L + \$30H = \$110$	Step 2: Multiply all terms in the first equation by 10 so that L has the same numerical coefficient in both equations.
	$10L + 10H = 70$ subtract $\$10L + \$30H = \$110$ $-\$20H = -\$40$	Step 3: Subtract the equations by subtracting all terms on both sides.
	$\frac{-\$20H}{-\$20} = \frac{-\$40}{-\$20} \quad H = 2$	Step 4: Solve for H by dividing both sides by -20 .
	$L + H = 7$ $L + 2 = 7$	Step 5: Substitute the known value for H into one of the original equations. The first equation is simple, so choose that one.
	$L + 2 - 2 = 7 - 2$ $L = 5$	Step 6: Solve for L by subtracting 2 from both sides. You now have the roots for L and H .
Present	You can purchase five articles of clothing at the low price point and two articles of clothing at the high price point. This allows you to purchase seven articles of clothing and stay within your budget of \$110. 	

**TOP
SECRET**

Paths To Success

One of the most difficult areas of mathematics involves translating words into mathematical symbols and operations. To assist in this translation, the table below lists some common language and the mathematical symbol that is typically associated with the word or phrase.

Language			Math Symbol
Sum Addition	In addition to In excess	Increased by Plus	+
Divide Division	Divisible Quotient	Per	÷
Becomes Is/Was/Were	Will be	Results in Totals	=
Less than	Lower than		<
Less than or equal to			≤

Language			Math Symbol
Subtract Decreased by Diminished by	Less Minus	Difference Reduced by	−
Multiplied by Times	Percentage of	Product of Of	×
More than Greater than			>
Greater than or equal to			≥
Not equal to			≠

Example 2.5E: Solving Two Linear Equations with Two Unknowns for an Amusement Park

A Family Fun Park charges \$15 for a child wrist band and \$10.50 for an adult wrist band. On a warm summer day, the amusement park had total wrist band revenue of \$15,783 from sales of 1,279 wrist bands. How many adult and child wrist bands did the park sell that day?

Plan	You need the number of both adult and child wrist bands sold on the given day. Therefore, you must identify two unknowns.	
Understand	What You Already Know The price of the wrist bands, total quantity, and sales are known: Child wrist band price = \$15 Adult wrist band price = \$10.50 Total revenue = \$15,783 Total unit sales = 1,279 The quantity of adult wrist bands sold and the quantity of child wrist bands sold are unknown: Adult wrist bands quantity = a Child wrist bands quantity = c	How You Will Get There - Work with the quantities first. Calculate the total unit sales by adding the number of adult wrist bands to the number of child wrist bands: $\# \text{ of adult wrist bands} + \# \text{ of child wrist bands} = \text{total unit sales}$ $a + c = 1,279$ - Now consider the dollar figures. Total revenue for any company is calculated as unit price multiplied by units sold. In this case, you must sum the revenue from two products to get the total revenue. $\text{Total adult revenue} + \text{Total child revenue} = \text{Total revenue}$ $(\text{Adult price} \times \text{Adult quantity}) + (\text{Child price} \times \text{Child quantity}) = \text{Total revenue}$ $\\$10.50a + \\$15c = \\$15,783$ - Apply the six-step procedure for solving two linear equations with two unknowns.
Perform	$\begin{array}{rcl} a + c & = & 1,279 \\ \$10.50a + \$15c & = & \$15,783 \\ \hline 10.50a + 10.50c & = & 13,429.50 \\ \$10.50a + \$15c & = & \$15,783 \\ \hline 10.5a + 10.5c & = & 13,429.50 \\ \text{Subtract } \$10.5a + \$15c & = & \$15,783 \\ \hline -4.5c & = & -2,353.50 \\ \hline \frac{-4.5c}{-4.5} = \frac{-2,353.50}{-4.5} & & c = 523 \end{array}$ $\begin{array}{rcl} a + c & = & 1,279 \\ a + 523 & = & 1,279 \\ a + 523 - 523 & = & 1,279 - 523 \\ a & = & 756 \end{array}$	Step 1: Write the equations one above the other and line them up. Step 2: Multiply all terms in the first equation by 10.5, resulting in a having the same numerical coefficient in both equations. Step 3: Subtract the equations by subtracting all terms on both sides. Step 4: Solve for c by dividing both sides by -4.5. Step 5: Substitute the known value for c into one of the original equations. The first equation is simple, so choose that one. Step 6: Solve for a by subtracting 523 from both sides. You now have the roots for a and c.
Present	The Family Fun Park sold 523 child wrist bands and 756 adult wrist bands. Amusement Park Quantities and Dollars	



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2.6: Natural Logarithms

(How Can I Get That Variable Out of the Exponent?)

Your investment is earning compound interest of 6% per year. You wonder how long it will take to double in value. Answering this question requires you to solve an equation in which the unknown variable is located in the exponent instead of the base position. To find the answer, you need logarithms, which are reviewed in this section and will be applied in Chapters 9 and 11.

Logarithms

A **logarithm** is defined as the exponent to which a base must be raised to produce a particular power. Although the base can assume any value, two values for the base are most often used.

1. **A Base Value of 10.** This is referred to as a **common logarithm**. Thus, if you have $10^2 = 100$, then 2 is the common logarithm of 100 and this is written as $\log_{10}(100) = 2$, or just $\log(100) = 2$ (you can assume the 10 if the base is not written). The format for a common logarithm can then be summarized as

$$\log(\text{power}) = \text{exponent}$$

$$\text{If you have } 10^x = y, \text{ then } \log(y) = x.$$

2. **A Base Value of e .** This is referred to as a **natural logarithm**. In mathematics, there is a known constant e , which is a nonterminating decimal and has an approximate value of $e = 2.71828182845$. If you have $e^3 = 20.085537$, then 3 is the natural logarithm of 20.085537 and you write this as $\ln(20.085537) = 3$. The format for a natural logarithm is then summarized as:

$$\ln(\text{power}) = \text{exponent}$$

$$\text{If you have } e^x = 54.59815, \text{ then } \ln(54.59815) = x.$$

Common logarithms have been used in the past when calculators were not equipped with power functions. However, with the advent of computers and advanced calculators that have power functions, the natural logarithm is now the most commonly used format. From here on, this textbook focuses on natural logarithms only.

Properties of Natural Logarithms

Natural logarithms possess six properties:

1. **The natural logarithm of 1 is zero.** For example, if 1 is the power and 0 is the exponent, then you have $e^0 = 1$. This obeys the laws of exponents discussed in Section 2.4 of this chapter.
2. **The natural logarithm of any number greater than 1 is a positive number.** For example, the natural logarithm of 2 is 0.693147, or $e^{0.693147} = 2$.
3. **The natural logarithm of any number less than 1 is a negative number.** For example, the natural logarithm of 0.5 is -0.693147 , or $e^{-0.693147} = 0.5$. Similar to property 1, this obeys the laws of exponents discussed in Section 2.4, where $e^{-0.693147} = \frac{1}{e^{0.693147}}$, always producing a proper fraction.
4. **A natural logarithm cannot be less than or equal to zero.** Since e is a positive number with an exponent, there is no value of the exponent that can produce a power of zero. As well, it is impossible to produce a negative number when the base is positive.
5. **The natural logarithm of the quotient of two positive numbers is $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$.** For example,

$$\ln\left(\frac{\$20,000}{\$15,000}\right) = \ln(\$20,000) - \ln(\$15,000).$$

$$\ln(1.\bar{3}) = 9.903487 - 9.615805$$

$$0.287682 = 0.287682$$

6. **The natural logarithm of a power of a positive base is $\ln(x^y) = y(\ln x)$.** This property allows you to relocate the exponent down into the base. You will apply this property especially in Chapters 9 and 11. Demonstrating this principle,

$$\begin{aligned}\ln(x^y) &= y(\ln x) \\ \ln(1.05^6) &= 6 \times \ln(1.05) \\ \ln(1.340095) &= 6 \times 0.048790 \\ 0.292741 &= 0.292741\end{aligned}$$



Important Notes

Applying natural logarithms on your TI BAII Plus calculator requires two steps.

1. Input the power.
2. With the power still on the display, press the LN key located in the left-hand column of your keypad. The solution on the display is the value of the exponent. If you key in an impossible value for the natural logarithm, an "Error 2" message is displayed on the screen.

If you know the exponent and want to find out the power, remember that $e^x = \text{power}$. This is called the anti-log function. Thus, if you know the exponent is 2, then $e^2 = 7.389056$. On your calculator, the anti-log can be located on the second shelf directly above the LN button. To access this function, key in the exponent first and then press 2^{nd} e^x .

Natural
Logarithm



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Paths To Success

You do not have to memorize the mathematical constant value of e . If you need to recall this value, use an exponent of 1 and access the e^x function. Hence, $e^1 = 2.71828182845$.



Give It Some Thought

For each of the following powers, determine if the natural logarithm is positive, negative, zero, or impossible.

- a. 2.3 b. 1 c. 0.45 d. 0.97 e. -2 f. 4.83 g. 0

Solutions:
 1. a. positive (property 2)
 2. b. zero (Property 1)
 3. c. negative (property 3)
 4. e. impossible (Property 4)
 5. f. positive (property 2)
 6. g. impossible (property 4)

Example 2.6A: Applying Natural Logarithms and Properties

Solve the first two questions using your calculator. For the next two questions, demonstrate the applicable property.

- a. $\ln(2.035)$ b. $\ln(0.3987)$ c. $\ln\left(\frac{\$10,000}{\$6,250}\right)$ d. $\ln[(1.035)^{12}]$

Plan

You need to apply the properties of natural logarithms.

Understand

What You Already Know

The properties of natural logarithms are known.

How You Will Get There

- Apply property 2 and key this through your calculator.
- Apply property 3 and key this through your calculator.
- Apply property 5, $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- Apply property 6, $\ln(x^y) = y(\ln x)$

Perform	a. $\ln(2.035) = 0.710496$		Calculator Instructions	<table><tr><td colspan="3">Answer</td></tr><tr><td>a.</td><td>2.035</td><td>LN</td><td>0.710496</td></tr><tr><td>b.</td><td>0.3987</td><td>LN</td><td>-0.919546</td></tr><tr><td>c.</td><td>$10000 \div 6250 =$</td><td>LN</td><td>0.470004</td></tr><tr><td>d.</td><td>$1.035 \text{ y}^x 12 =$</td><td>LN</td><td>0.412817</td></tr></table>		Answer			a.	2.035	LN	0.710496	b.	0.3987	LN	-0.919546	c.	$10000 \div 6250 =$	LN	0.470004	d.	$1.035 \text{ y}^x 12 =$	LN	0.412817
	Answer																							
	a.	2.035				LN	0.710496																	
	b.	0.3987				LN	-0.919546																	
	c.	$10000 \div 6250 =$				LN	0.470004																	
d.	$1.035 \text{ y}^x 12 =$	LN	0.412817																					
b. $\ln(0.3987) = -0.919546$																								
c. $\ln\left(\frac{\$10,000}{\$6,250}\right) = \ln(\$10,000) - \ln(\$6,250)$ $\ln(1.6) = 9.210340 - 8.740336$ $0.470004 = 0.470004$																								
d. $\ln[(1.035)^{12}] = 12 \times \ln(1.035)$ $\ln(1.511068) = 12 \times 0.034401$ $0.412817 = 0.412817$																								
Present	Organizing your answers into a more common format:																							
	a. $e^{0.710496} = 2.035$ b. $e^{-0.919546} = 0.3987$		c. $\ln\left(\frac{\$10,000}{\$6,250}\right) = 0.470004$ (as proven by the property) d. $\ln[(1.035)^{12}] = 0.412817$ (as proven by the property)																					



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Chapter 2 Summary

Key Concepts Summary

Section 2.1: Order of Operations (Proceed in an Orderly Manner)

- A review of key mathematical operator symbols
- Rules for order of operations are known as BEDMAS

Section 2.2: Fractions, Decimals, and Rounding (Just One Slice of Pie, Please)

- The language and types of fractions
- Working with equivalent fractions by either solving for an unknown or increasing/reducing the fraction
- Converting any fraction to a decimal format
- Procedures for proper rounding
- The rounding rules that are used throughout this textbook

Section 2.3: Percentages (How Does It All Relate?)

- Converting decimal numbers to percentages
- Working with percentages in the form of rates, portions, and bases

Section 2.4: Algebraic Expressions (The Pieces of the Puzzle)

- Learning about the language of algebra
- The rules for manipulating exponents
- The rules of algebra for addition and subtraction
- The rules of algebra for multiplication
- The rules of algebra for division
- What is substitution and how it is performed?

Section 2.5: Linear Equations: Manipulating and Solving (Solving the Puzzle)

- A review of key concepts about equations
- The procedures required to solve one linear equation for one unknown variable
- The procedures required to solve two linear equations for two unknown variables

Section 2.6: Natural Logarithms (How Can I Get That Variable Out of the Exponent?)

- A review of the logarithm concept
- The rules and properties of natural logarithms

The Language of Business Mathematics

algebraic equation An equation that takes two algebraic expressions and makes them equal to each other.

algebraic expression Indicates the relationship between and mathematical operations that must be conducted on a series of numbers or variables.

base The entire amount or quantity of concern.

BEDMAS An order of operations acronym standing for **B**rackets, **E**xponents, **D**ivision, **M**ultiplication, **A**ddition, and **S**ubtraction.

common logarithm A logarithm with a base value of 10.

complex fraction A fraction that has fractions within fractions, combining elements of compound, proper, and/or improper fractions together.

compound fraction A fraction that combines an integer with either a proper or improper fraction.

denominator Any term by which some other term is divided; commonly the number on the bottom of a fraction.

divisor line The line that separates the numerator and the denominator in a fraction.

equivalent fractions Two or more fractions of any type that have the same numerical value upon completion of the division.

exponent A mathematical shorthand notation that indicates how many times a quantity is multiplied by itself

factor Components of terms that are separated from by multiplication or division signs.

fraction A part of a whole.

improper fraction A fraction in which the numerator is larger than the denominator.

left side of the equation The part of an equation that is to the left of the equal sign.

like terms Terms that have the same literal coefficient.

linear equation An algebraic expression in which the variable has an exponent of 1; when plotted, it will form a straight line.

literal coefficient A factor that is a variable.

logarithm The exponent to which a base must be raised to produce a particular power.

monomial An algebraic expression with only one term.

natural logarithm A logarithm with a base value of the mathematical constant e .

nomial The number of terms that appear in an algebraic expression.

nonlinear equation An algebraic expression in which the variable has an exponent other than 1; when plotted, it will not form a straight line.

numerator Any term into which some other term is being divided; commonly the number on the top in a fraction.

numerical coefficient A factor that is numerical.

percentage A part of a whole expressed in hundredths.

polynomial An algebraic expression with two or more terms.

portion Represents part of a whole or base.

proper fraction A fraction in which the numerator is smaller than the denominator.

rate Expresses a relationship between a portion and a base.

right side of the equation The part of an equation that is to the right of the equal sign.

root The value of the unknown variable that will make a linear equation true.

substitution Replacing the literal coefficients of an algebraic expression with their numerical values.

term In any algebraic expression, the components that are separated by addition and subtraction.

triangle technique A memorization technique that displays simple multiplication formulae in the form of a triangle. Anything on the same line is multiplied, and items above or below each other are divided to arrive at various solutions.

The Formulas You Need to Know

Symbols Used

% = percentage

dec = any number in decimal format

ln = natural logarithm

Rate = the relationship between the portion and base

Portion = the part of the whole

Base = the whole quantity

Formulas Introduced

Formula 2.1 Percentage Conversion: $\% = \text{dec} \times 100$ (Section 2.2)

Formula 2.2 Rate, Portion, Base: $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$ (Section 2.2)

Technology

Calculator

Formatting Instructions

Buttons Pushed	Calculator Display	What It Means
2nd Format	DEC=2.00	You have opened the Format window to its first setting. DEC tells your calculator how to round the calculations. In business math, it is important to be accurate. Therefore, we will set the calculator to what is called a floating display, which means your calculator will carry all of the decimals and display as many as possible on the screen.
9 Enter	DEC=9.	The floating decimal setting is now in place. Let us proceed.
↓	DEG	This setting has nothing to do with business math and is just left alone. If it does not read DEG, press 2nd Set to toggle it.
↓	US 12-31-1990	Dates can be entered into the calculator. North Americans and Europeans use slightly different formats for dates. Your display is indicating North American format and is acceptable for our purposes. If it does not read US, press 2nd Set to toggle it.
↓	US 1,000	In North America it is common to separate numbers into blocks of 3 using a comma. Europeans do it slightly differently. This setting is acceptable for our purposes. If your display does not read US, press 2nd Set to toggle it.
↓	Chn	There are two ways that calculators can solve equations. This is known as the Chain method, which means that your calculator will simply resolve equations as you punch it in without regard for the rules of BEDMAS. This is not acceptable and needs to be changed.
2nd Set	AOS	AOS stands for Algebraic Operating System. This means that the calculator is now programmed to use BEDMAS in solving equations.
2nd Quit	0.	Back to regular calculator usage.

Exponents and Signs

x^2 is used for exponents that are squares. 2^2 is keyed in as $2 x^2$.

y^x is used for exponents that are not squares. 2^3 is keyed in as $2 y^x 3 =$.

$\pm$ is used to change the sign of a number. To use, key in the number first and then press the $\pm$ key.

Memory

STO Store

RCL Recall

0–9 Memory cell numbers (10 in total)

To store a number on the display, press STO # (where # is a memory cell number).

To recall a number in the memory, press RCL # (where # is the memory cell where the number is stored).

Natural Logarithms

The natural logarithm function, LN, is located in the left-most column of your calculator. To use this function, input the power and then press the LN button. The solution displayed is the exponent. To calculate the power, input the exponent and then press 2nd e^x (called the anti-log).





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Chapter 3: General Business Management Applications

(Get Your Motor Running)

Basic calculations are so much a part of your everyday life that you could not escape them if you tried. While driving to school, you may hear on the radio that today's forecasted temperature of 27°C is 3°C warmer than the historical average. In the news, consumers complain that gas prices rose 11% in three months. A commercial tells you that the Toyota Prius you are driving uses almost 50% less gas than the Honda Civic Coupe Si. After class you head to your local gym to cancel a gym membership because you are too busy to work out. You have used only four months of an annual membership for which you paid \$449, and now you want to know how large a prorated refund you are eligible for.

On a daily basis you use basic calculations such as averages (the temperature), percent change (gas prices), ratios and proportions (energy efficiency), and prorating (the membership refund). To invest successfully, you must also apply these basic concepts. Or if you are an avid sports fan, you need basic calculations to understand your favourite players' statistics.

And it's not hard to see how these calculations would be used in the business world. Retail management examines historical average daily sales to predict future sales and to schedule the employees who will service those sales. Human resource managers continually calculate ratios between accounting and performance data to assess how efficiently labour is being used. Managers prorate a company's budget across its various departments.

This chapter covers universal business mathematics you will use whether your chosen business profession is marketing, accounting, production, human resources, economics, finance, or something else altogether. To be a successful manager you need to understand percent changes, averages, ratios, proportions, and prorating.

Outline of Chapter Topics

- 3.1: Percent Change (Are We Up or Down?)
- 3.2: Averages (What Is Typical?)
- 3.3: Ratios, Proportions, and Prorating (It Is Only Fair)

3.1: Percent Change

(Are We Up Or Down?)

On your way to work, you notice that the price of gasoline is about 10% higher than it was last month. At the office, reports indicate that input costs are down 5.4% and sales are up 3.6% over last year. Your boss asks you to analyze the year-over-year change in industry sales and submit a report. During your coffee break, you look through the day's e-flyers in your inbox. Home Depot is advertising that all garden furniture is 40% off this week; Safeway's ad says that Tuesday is 10% off day; and a *Globe and Mail* story informs you that the cost of living has risen by 3% since last year. You then log in to your financial services account, where you are happy to find that the mutual funds in your RRSP are up 6.7% from last year. What are you going to do with all this information?

Understanding how data changes from one period to the next is a critical business skill. It allows for quick assessment as to whether matters are improving or getting worse. It also allows for easy comparison of changes in different types of data over time. In this section, the concept of *percent change* is explored, which allows for the calculation of change between two points in time. Then a rate of change over time is introduced, which allows you to determine the change per period when multiple points in time are involved in the calculation. Finally, an application of these concepts to investment returns is discussed.

Percent Change

It can be difficult to understand a change when it is expressed in absolute terms. Can you tell at a glance how good a deal it is to buy a \$359 futon for \$215.40? It can also be difficult to understand a change when it is expressed as a percentage of its end result. Are you getting a good deal if that \$215.40 futon is 60% of the regular price? What most people do find easier to understand is a change expressed as a percentage of its starting amount. Are you getting a good deal if that \$359 futon is on sale at 40% off? A **percent change** expresses in percentage form how much any quantity changes from a starting period to an ending period.



The Formula

To calculate the percent change in a variable, you need to know the starting number and the ending number. Once you know these, Formula 3.1 (on the next page) represents how to express the change as a percentage.

Remember two critical concepts about percent change:

1. **Do Not Include the Original Quantity in the Change.** Percent change represents the changes in the quantities, not the values of the quantities themselves. The original quantity does not count toward the percent change. Therefore, if any quantity doubles, its percent change is 100%, not 200%. For example, if the old quantity was 25 and the new quantity is 50, note that the quantity has doubled. However, 25 of the final 50 comes from the original amount and therefore does not count toward the change. We subtract it out of the numerator through calculating $50 - 25 = 25$. Therefore, the change as a percentage is $\frac{50 - 25}{25} \times 100 = 100\%$. The same applies to a tripling of quantity. If our new quantity is 75 (which is triple the old quantity of 25), then $\frac{75 - 25}{25} \times 100 = 200\%$. The original value of 25 is once again subtracted out of the numerator. The original 100% is always removed from the formula.
2. **Negative Changes.** A negative change must be expressed with a negative sign or equivalent wording. For example, if the old quantity was 20 and the new quantity is 15, this is a decrease of 5, or a change of $15 - 20 = -5$. The percent change is $\frac{15 - 20}{20} \times 100 = -25\%$. Be careful in expressing a negative percent change. There are two correct ways to do this properly:
 - a. "The change is -25% ."
 - b. "It has decreased by 25% ."

Note in the second statement that the word "decreased" replaces the negative sign. To avoid confusion, do not combine the negative sign with the word "decreased"—recall that two negatives make a positive, so "It has decreased by -25% " would actually mean the quantity has *increased* by 25% .

$\Delta\%$ is Percent Change: The change in the quantity is always expressed in percent format. The Δ symbol is a commonly used symbol that represents the word *change*. Therefore, this algebraic symbol, which matches the BAII Plus calculator key, is read as "change percent."

New is New Quantity Value: This is the value that the quantity has become, or the number that you want to compare against a starting point.

Formula 3.1 – Percent Change:
$$\Delta\% = \frac{\text{New} - \text{Old}}{\text{Old}} \times 100$$

Old is Old Quantity Value: This is the value that the quantity used to be, or the number that you want all others to be compared against. Notice how the formula is structured:

1. First calculate "New – Old," the change in the quantity. This is the numerator.
2. Divide the change by "Old" to express the quantity change first as a fraction, then convert it into its decimal format by performing the division.

$\times 100$ is Percent Conversion: Recall from Section 2.3 that you convert a decimal to a percentage by multiplying it by 100. As the language suggests, percent changes are always percentages; therefore, you must include this component in the formula.



How It Works

To solve any question about percent change, follow these steps:

Step 1: Notice that there are three variables in the formula. Identify the two known variables and the one unknown variable.

Step 2: Solve for the unknown variable using Formula 3.1.

Assume last month your sales were \$10,000, and they have risen to \$15,000 this month. You want to express the percent change in sales.

Step 1: The known variables are Old = \$10,000 and New = \$15,000. The unknown variable is percent change, or $\Delta\%$.

Step 2: Using Formula 3.1,

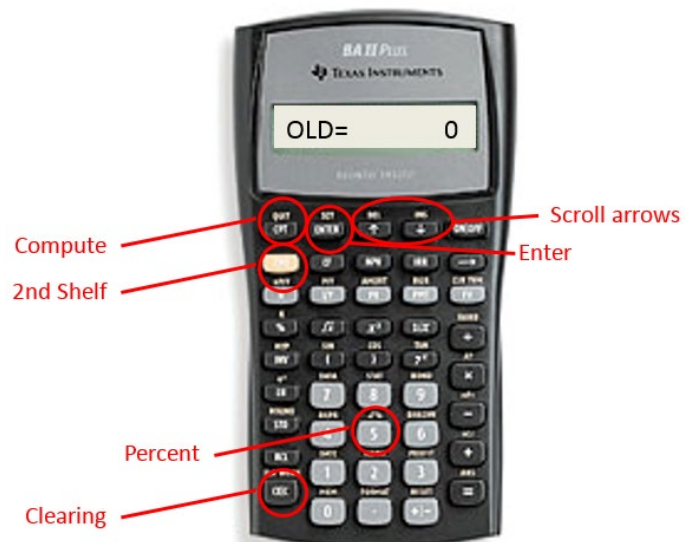
$$\begin{aligned}\Delta\% &= \frac{\$15,000 - \$10,000}{\$10,000} \times 100 = \frac{\$5,000}{\$10,000} \times 100 \\ &= 0.5 \times 100 = 50\%\end{aligned}$$

Observe that the change in sales is +\$5,000 month-over-month. Relative to sales of \$10,000 last month, this month's sales have risen by 50%.



Important Notes

To access the percent change function on your BAII Plus calculator, press 2nd $\Delta\%$ located above the number 5 on



your keypad. Always clear the memory of any previous question by pressing 2nd CLR Work once the function is open. Use the $\uparrow$ and $\downarrow$ arrows to scroll through the four lines of this function. To solve for an unknown variable, key in three of the four variables and then press Enter. With the unknown variable on your display, press CPT. Each variable in the calculator is as follows:

- OLD = The old or original quantity; the number that represents the starting point
- NEW = The new or current quantity; the number to compare against the starting point
- %CH = The percent change, or $\Delta\%$ in its percent format without the % sign
- #PD = Number of consecutive periods for change. By default, it is set to 1. For now, do not touch this variable. Later in this section, when we cover rate of change over time, this variable will be explained.



Things To Watch Out For

Watch out for two common difficulties involving percent changes.

1. **Rates versus Percent Changes.** Sometimes you may be confused about whether questions involve rates (Section 2.3) or percent changes. Recall that a *rate* expresses the relationship between a portion and a base. Look for some key identifiers, such as "is/are" (the portion) and "of" (the base). For percent change, key identifiers are "by" or "than." For example, "x has increased *by* y%" and "x is y% more *than* last year" both represent a percent change.
2. **Mathematical Operations.** You may imagine that you are supposed to add or subtract percent changes, but you cannot do this. Remember that percentages come from fractions. According to the rules of algebra, you can add or subtract fractions only if they share the same base (denominator). For example, if an investment increases in value in the first year by 10% and then declines in the second year by 6%, this is not an overall increase of $10\% - 6\% = 4\%$. Why? If you originally had \$100, an increase of 10% (which is $\$100 \times 10\% = \$10$) results in \$110 at the end of the first year. You must calculate the 6% decline in the second year using the \$110 balance, not the original \$100. This is a decline of $\$110 \times (-6\%) = -\$6.60$, resulting in a final balance of \$103.40. Overall, the percent change is 3.4%.



Paths To Success

A percent change extends the rate, portion, and base calculations introduced in Section 2.3. The primary difference lies in the portion. Instead of the portion being a part of a whole, the portion represents the change of the whole. Putting the two formulas side by side, you can calculate the rate using either approach.

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}} = \frac{\text{New} - \text{Old}}{\text{Old}}$$



Give It Some Thought

1. It has been five years since Juan went shopping for a new car. On his first visit to a car lot, he had sticker shock when he realized that new car prices had risen by about 20%. What does this situation involve?
 - a. Percent change
 - b. Rate, portion, base
2. Manuel had his home custom built in 2018 for \$300,000. In 2024 he had it professionally appraised at \$440,000. He wants to figure out how much his home has appreciated. How would he do so?
 - a. The 2018 price is the "New," and the 2024 price is the "Old."
 - b. The 2018 price is the "Old," and the 2024 price is the "New."

Solutions:

1. a (the question involves how car prices have changed; note the key word "by")
 2. b (the 2018 price is what the house used to be worth, which is the old quantity; the 2024 price represents the new value of the home)

Example 3.1A: Price of New Cars in Canada

In 1982, the average price of a new car sold in Canada was \$10,668. By 2009, the average price of a new car had increased to \$25,683. By what percentage has the price of a new car changed over these years?

Plan	You are trying to find the percent change in the price of the new car, or $\Delta\%$.										
Understand	What You Already Know Step 1: You know the old and new prices for the cars: Old = \$10,668 New = \$25,683	How You Will Get There Step 2: Apply Formula 3.1.									
Perform	Step 2: $\Delta\% = \frac{\$25,683 - \$10,668}{\$10,668} \times 100$ $= \frac{\$15,015}{\$10,668} \times 100 = 140.748\%$	Calculator Instructions <table border="1"> <thead> <tr> <th>OLD</th><th>NEW</th><th>%CH</th><th>#PD</th></tr> </thead> <tbody> <tr> <td>10668</td><td>25683</td><td>Answer: 140.748031</td><td>1</td></tr> </tbody> </table>		OLD	NEW	%CH	#PD	10668	25683	Answer: 140.748031	1
OLD	NEW	%CH	#PD								
10668	25683	Answer: 140.748031	1								
Present	From 1982 to 2009, average new car prices in Canada have increased by 140.748%.										

Example 3.1B: Figuring Out a Price after Taxes

When you purchase a retail item, the tax increases the price of the product. If you buy a \$799.00 Classic Home Gym machine in Ontario, it is subject to 13% HST. What amount do you pay for the machine including taxes?

Plan	You are looking for the price of the machine after increasing it by the sales tax. This is a "New" price for the machine.										
Understand	What You Already Know Step 1: You know the original price of the machine and how much to increase it by: Old = \$799.00 $\Delta\% = 13\%$	How You Will Get There Step 2: Apply Formula 3.1: $\Delta\% = \frac{\text{New} - \text{Old}}{\text{Old}} \times 100$									
Perform	Step 2: $13\% = \frac{\text{New} - \$799}{\$799} \times 100$ $0.13 = \frac{\text{New} - \$799}{\$799}$ $\$103.87 = \text{New} - \$799$ $\$902.87 = \text{New}$	Calculator Instructions <table border="1"> <thead> <tr> <th>OLD</th><th>NEW</th><th>%CH</th><th>#PD</th></tr> </thead> <tbody> <tr> <td>799</td><td>Answer: 902.87</td><td>13</td><td>1</td></tr> </tbody> </table>		OLD	NEW	%CH	#PD	799	Answer: 902.87	13	1
OLD	NEW	%CH	#PD								
799	Answer: 902.87	13	1								
Present	The price of a Classic Home Gym machine, after increasing the price by the taxes of 13%, is \$902.87.										

Example 3.1C: Handling Price Changes

Consumers often object to price changes in many daily products, even though inflation and other cost changes may justify these increases. To reduce the resistance to a price increase, many manufacturers adjust both prices and product sizes at the same time. For example, the regular selling price for a bottle of shampoo was \$5.99 for 240 mL. To account for cost changes, the manufacturer decided to change the price to \$5.79, but also reduce the bottle size to 220 mL. What was the percent change in the price per millilitre?

Plan	You need to find the percent change in the price per millilitre, or $\Delta\%$.										
Understand	What You Already Know Step 1: You know the old price and bottle size, as well as the planned price and bottle size: Old price = \$5.99 Old size = 240 mL New price = \$5.79 New size = 220 mL	How You Will Get There Step 1 (continued): Convert the price and size to a price per millilitre by taking the price and dividing by the size. Step 2: Apply Formula 3.1.									
Perform	Step 1: $\frac{\text{Old price}}{\text{Old size}} = \frac{\$5.99}{240 \text{ mL}} = \$0.024958 \text{ per mL}$ $\frac{\text{New price}}{\text{New size}} = \frac{\$5.79}{220 \text{ mL}} = \$0.026318 \text{ per mL}$ Step 2: $\Delta\% = \frac{\$0.026318 - \$0.024958}{\$0.024958} \times 100$ $= \frac{\$0.001359}{\$0.024958} \times 100 = 5.4485\%$	Calculator Instructions <table border="1"> <thead> <tr> <th>OLD</th><th>NEW</th><th>%CH</th><th>#PD</th></tr> </thead> <tbody> <tr> <td>$5.99 \div 240 =$</td><td>$5.79 \div 220 =$</td><td>Answer: 5.4484</td><td>1</td></tr> </tbody> </table>		OLD	NEW	%CH	#PD	$5.99 \div 240 =$	$5.79 \div 220 =$	Answer: 5.4484	1
OLD	NEW	%CH	#PD								
$5.99 \div 240 =$	$5.79 \div 220 =$	Answer: 5.4484	1								
Present	The price per mL has increased by 5.4485%. Note that to the consumer, it would appear as if the price has been lowered from \$5.99 to \$5.79, as most consumers would not notice the change in the bottle size.										

Rate of Change over Time

The percent change measures the change in a variable from start to end overall. It is based on the assumption that only a single change occurs. What happens when the ending number results from multiple changes and you want to know the typical value of each change? For example, the population of a large metropolitan area (CMA) has grown from 4,263,759 in 2008 to 5,113,149 in 2018. What annual percentage growth in population does this reflect? Notice that we are not interested in calculating the change in population over the 10 years; instead we want to determine the percentage change in *each* of the 10 years. The **rate of change over time** measures the percent change in a variable per time period.

**The Formula**

Calculating the rate of change over time is not as simple as dividing the percent change by the number of time periods involved, because you must consider the change for each time period relative to a different starting quantity. For example, in the metropolitan census example, the percent change from 2008 to 2009 is based on the original population size of 4,263,759. However, the percent change from 2009 to 2010 is based on the new population figure for 2009. Thus, even if the same number of people were added to the city in both years, the percent change in the second year is smaller because the population base became larger after the first year. As a result, when you need the percent change per time period, you must use Formula 3.2.

RoC is Rate of Change per time period:

This is a percentage that expresses how the quantity is changing per time period. It recognizes that any change in one period affects the change in the next period.

New is New Quantity Value:

What the quantity has become.

n is Total Number of Periods:

The total number of periods reflects the number of periods of change that have occurred between the Old and New quantities.

Formula 3.2 – Rate of Change Over Time:
$$\text{RoC} = \left(\left(\frac{\text{New}}{\text{Old}} \right)^{\frac{1}{n}} - 1 \right) \times 100$$

Old is Old Quantity Value: What the quantity used to be.

× 100 is Percent Conversion: Rates of change over time are always expressed as percentages.



How It Works

When you work with any rate of change over time, follow these steps:

Step 1: Identify the three known variables and the one unknown variable.

Step 2: Solve for the unknown variable using Formula 3.2.



Important Notes

On your calculator, calculate the rate of change over time using the percent change ($\Delta\%$) function. Previously, we had ignored the #PD variable in the function and it was always assigned a value of 1. In rate of change, this variable is the same as n in our equation. Therefore, if our question involved 10 changes, such as the annual population change of the metropolitan CMA from 2008 to 2018, then this variable is set to 10.



Paths To Success

You may find it difficult to choose which formula to use: percent change or rate of change over time. To distinguish between the two, consider the following:

1. If you are looking for the *overall* rate of change from beginning to end, you need to calculate the percent change.
2. If you are looking for the rate of change *per interval*, you need to calculate the rate of change over time.

Ultimately, the percent change formula is a simplified version of the rate of change over time formula where $n = 1$. Thus you can solve any percent change question using Formula 3.2 instead of Formula 3.1.



Give It Some Thought

For each of the following, distinguish whether you should solve the question by the percent change formula or the rate of change over time formula.

1. When Peewee started five-pin bowling with the Youth Bowling Canada (YBC) in 2009, his average was 53. In 2023, he finished his last year of the YBC with an average of 248. How did his average change from 2009 to 2023?

- A stock was priced at \$4.34 per share in 2018 and reached \$7.15 per share in 2024. What annual return did a shareholder realize?
- In 2016, total sales reached \$1.2 million. By 2022, sales had climbed to \$4.25 million. What is the growth in sales per year?

Solutions:

1. Percent change; looking for overall change
 2. Rate of change over time; looking for change per year
 3. Rate of change over time; looking for change per year

Example 3.1D: Population Growth

For a large city, where the population grew from 4,263,759 in 2008 to 5,113,149 in 2018, calculate the annual percent growth in the population.

Plan	We need to calculate the percent change per year, which is the rate of change over time, or RoC.										
Understand	What You Already Know Step 1: We know the starting and ending numbers for the population along with the time frame involved. Old = 4,263,759 New = 5,113,149 $n = 2018 - 2008 = 10$ years	How You Will Get There Step 2: Apply Formula 3.2.									
Perform	Step 2: $\text{RoC} = \left(\left(\frac{5,113,149}{4,263,759} \right)^{\frac{1}{10}} - 1 \right) \times 100$ $= (1.018332 - 1) \times 100$ $= 0.018332 \times 100 = 1.8332\%$	Calculator Instructions <table border="1" style="width: 100%; text-align: center;"> <tr> <th>OLD</th><th>NEW</th><th>%CH</th><th>#PD</th></tr> <tr> <td>4263759</td><td>5113149</td><td>Answer: 1.833244</td><td>10</td></tr> </table>		OLD	NEW	%CH	#PD	4263759	5113149	Answer: 1.833244	10
OLD	NEW	%CH	#PD								
4263759	5113149	Answer: 1.833244	10								
Present	Over the 10 year span from 2008 to 2018, the city's population grew by an average of 1.8332% per year.										

Example 3.1E: Percent Changes and Rate of Change Together

Kendra collects hockey cards. In her collection, she has a rookie year Wayne Gretzky card in mint condition. The book value of the card varies depending on demand for the card and its condition. If the estimated book value of the card fell by \$84 in the first year and then rose by \$113 in the second year, determine the following:

- What is the percent change in each year if the card is valued at \$1,003.33 at the end of the first year?
- Over the course of the two years, what was the overall percent change in the value of the card?
- What was the rate of change per year?



Plan	We need to provide four answers to the questions and find the percent change in year 1, or $\Delta\%_1$, then the percent change in year 2, or $\Delta\%_2$. Using these first two solutions, we calculate both the overall percent change across both years, or $\Delta\%_{\text{overall}}$, and the rate of change per year, or RoC.

Understand	What You Already Know Step 1: We know the price of the card at the end of the first year as well as how it has changed each year. Price at end of first year = \$1,003.33 Price change in first year = −\$84 Price change in second year = \$113	How You Will Get There Step 2: Calculate the price at the beginning of the first year. Step 3: For the percent change in Year 1, apply Formula 3.1: $\Delta\%_1$ Step 4: Calculate the price at the end of the second year. Step 5: For the percent change in Year 2, apply Formula 3.1: $\Delta\%_2$ Step 6: For the overall percent change, take the old price at the beginning of the first year and compare it to the new price at the end of the second year. Apply Formula 3.1: $\Delta\%_{\text{overall}}$ Step 7: Calculate the rate of change over the two years using the same two prices, but apply Formula 3.2.																									
Perform	Step 2: New = Old + Change; \$1,003.33 = Old − \$84.00; \$1,087.33 = Old Step 3: $\Delta\%_1 = \frac{\\$1,003.33 - \\$1,087.33}{\\$1,087.33} \times 100 = -7.7253\%$ Step 4: New = Old + Change = \$1,003.33 + \$113 = \$1,116.33 Step 5: $\Delta\%_2 = \frac{\\$1,116.33 - \\$1,003.33}{\\$1,003.33} \times 100 = 11.2625\%$ Step 6: $\Delta\%_{\text{overall}} = \frac{\\$1,116.33 - \\$1,087.33}{\\$1,087.33} \times 100 = 2.6671\%$ Step 7: $\text{RoC} = \left(\left(\frac{\\$1,116.33}{\\$1,087.33} \right)^{\frac{1}{2}} - 1 \right) \times 100 = 1.3248\%$	Calculator Instructions <table><tr><th>Calculation</th><th>OLD</th><th>NEW</th><th>%CH</th><th>#PD</th></tr><tr><td>Year 1</td><td>1087.33</td><td>1003.33</td><td>Answer: −7.725345</td><td>1</td></tr><tr><td>Year 2</td><td>1003.33</td><td>1116.33</td><td>Answer: 11.262495</td><td>1</td></tr><tr><td>Both Years</td><td>1087.33</td><td>1116.33</td><td>Answer: 2.667083</td><td>1</td></tr><tr><td><u>RoC</u></td><td>1087.33</td><td>1116.33</td><td>Answer: 1.324766</td><td>2</td></tr></table>	Calculation	OLD	NEW	%CH	#PD	Year 1	1087.33	1003.33	Answer: −7.725345	1	Year 2	1003.33	1116.33	Answer: 11.262495	1	Both Years	1087.33	1116.33	Answer: 2.667083	1	<u>RoC</u>	1087.33	1116.33	Answer: 1.324766	2
Calculation	OLD	NEW	%CH	#PD																							
Year 1	1087.33	1003.33	Answer: −7.725345	1																							
Year 2	1003.33	1116.33	Answer: 11.262495	1																							
Both Years	1087.33	1116.33	Answer: 2.667083	1																							
<u>RoC</u>	1087.33	1116.33	Answer: 1.324766	2																							
Present	The value of the hockey card dropped 7.7253% in the first year and rose 11.2625% in the second year. Overall, the card rose by 2.6671% across both years, which represents a growth of 1.3248% in each year.																										

Investment Returns

When investing, there are many ways to earn money. Very commonly, investors can benefit from their investment in two ways:

- 1) While holding the investment, you could receive **income** in the form of interest, distribution, or dividends. These amounts are paid out to the investor and represent actual cash inflows to be used in any manner the investor sees fit.
- 2) From the start to the end of the time period that the investor holds the investment, the value of investment may either increase or decrease. This difference in value is commonly referred to as **capital gains** or **capital losses**, respectively. These capital gains are usually taxed at a rate lower than investment income.



The Formulas

Working with investment returns is a combination of two previously introduced concepts: Rate/Portion/Base and Percent Change. No new formulas are required. These two formulas are adapted from their generic forms into “investment” language. There are three common investment return formulas:

- 1) Income Yield is adapted from Rate = Portion/Base (Formula 2.2) to become:

$$\text{Income Yield} = \text{Income} / \text{Old Investment Value} \times 100$$

- 2) Capital Gain Yield is adapted from Percent Change (Formula 3.1) to become:

$$\text{Capital Gain Yield} = \frac{\text{New Investment Value} - \text{Old Investment Value}}{\text{Old Investment Value}} \times 100$$

Note that (New Investment Value – Old Investment Value) represents a capital gain if it is a positive difference and a capital loss if it is a negative difference.

3) Investment Total Rate of Return is adapted from $\text{Rate} = \text{Portion} / \text{Base}$ (Formula 2.2) to become:

$$\text{Investment Total Rate of Return} = (\text{Income} + \text{Capital Gain}) / \text{Old Initial Investment} \times 100$$



How It Works

To solve any question about investment returns, follow these steps:

Step 1: Notice that amongst the three formulas, there are only three variables that are required including the income, old investment value, and new investment value. Identify these amounts.

Step 2: Using the three variables, calculate both yields and the total return.

To illustrate these concepts, give thought to buying a single share of a stock for \$10. Let's say you hold onto it for a period of 1 year and during that one year you receive \$1 in dividends. At the end of the year, you sell the single share of stock for \$11. From this scenario:

Step 1: Initial investment value = \$10, New investment value = \$11, and Income = \$1.

Step 2: Putting these values in the formulas:

$$\text{Income Yield} = \$1 / \$10 \times 100 = 10\%$$

$$\text{Capital Gain Yield} = \frac{\$11 - \$10}{\$10} \times 100 = \frac{\$1 \text{ Capital Gain}}{\$10} \times 100 = 10\%$$

$$\text{Investment Total Rate of Return} = (\$1 \text{ Income} + \$1 \text{ Capital Gain}) / \$10 \times 100 = 20\%$$



Important Notes

You may have noticed that in calculating the investment total rate of return, you can add together the income yield and the capital gain yield. Since both of these yields are calculated using the same denominator (the old investment value), the percentages can be added to arrive at the correct total rate of return.

Example 3.1F: Investment Returns

One year ago, \$20,000 was invested into a mutual fund. Over the course of the year, the investor received \$1,380 in distribution from the mutual fund. However, due to a global pandemic the mutual fund has declined and is now only worth \$17,600. Calculate the mutual fund's income yield, capital gain yield, and investment total rate of return.

Plan	Need to perform the three calculations requested using the adaptations of Formula 2.2 and Formula 3.1.	
Understand	What You Already Know Step 1: You know the following information: Old Investment Value = \$20,000 New Investment Value = \$17,600 Income = \$1,380	How You Will Get There Step 2: Substitute the values of the variables into the adapted versions of Formula 2.2 and Formula 3.1 in order to calculate the required information.
Perform	Step 2: $\text{Income Yield} = \$1,380 / \$20,000 \times 100 = 6.9\%$ $\text{Capital Gain Yield} = \frac{\$17,600 - \$20,000}{\$20,000} \times 100 = \frac{\$2,400 \text{ Capital Loss}}{\$20,000} \times 100 = -12\%$ $\text{Investment Total Rate of Return} = (\$1,380 \text{ Income} - \$2,400 \text{ Capital Loss}) / \$20,000 \times 100 = -5.1\%$	

Present

Despite earning 6.9% income from the mutual fund, the value of the fund decreased by 12% and the investment total rate of return is a loss of 5.1%.



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3.2: Averages

(What Is Typical?)

No matter where you go or what you do, averages are everywhere. Let's look at some examples:

- Three-quarters of your student loan is spent. Unfortunately, only half of the first semester has passed, so you resolve to squeeze the most value out of the money that remains. But have you noticed that many grocery products are difficult to compare in terms of value because they are packaged in different sized containers with different price points? For example, one tube of toothpaste sells in a 125 mL size for \$1.99 while a comparable brand sells for \$1.89 for 110 mL. Which is the better deal? A fair comparison requires you to calculate the average price per millilitre.
- Your local transit system charges \$2.25 for an adult fare, \$1.75 for students and seniors, and \$1.25 for children. Is this enough information for you to calculate the average fare, or do you need to know how many riders of each kind there are?
- Five years ago you invested \$8,000 in Roller Coasters Inc. The stock value has changed by 9%, -7%, 13%, 4%, and -2% over these years, and you wonder what the average annual change is and whether your investment kept up with inflation.
- If you participate in any sport, you have an average of some sort: bowlers have bowling averages; hockey or soccer goalies have a goals against average (GAA); and baseball pitchers have an earned run average (ERA).

Averages generally fall into three categories. This section explores simple, weighted, and geometric averages.

Simple Averages

An **average** is a single number that represents the middle of a data set. It is commonly interpreted to mean the “typical value.” Calculating averages facilitates easier comprehension of and comparison between different data sets, particularly if there is a large amount of data. For example, what if you want to compare year-over-year sales? One approach would involve taking company sales for each of the 52 weeks in the current year and comparing these with the sales of all 52 weeks from last year. This involves 104 weekly sales figures with 52 points of comparison. From this analysis, could you concisely and confidently determine whether sales are up or down? Probably not. An alternative approach involves comparing last year's average weekly sales against this year's average weekly sales. This involves the direct comparison of only two numbers, and the determination of whether sales are up or down is very clear.

In a **simple average**, all individual data share the same level of importance in determining the typical value. Each individual data point also has the same frequency, meaning that no one piece of data occurs more frequently than another. Also, the data do not represent a percent change. To calculate a simple average, you require two components:

1. The data itself—you need the value for each piece of data.
2. The quantity of data—you need to know how many pieces of data are involved (the count), or the total quantity used in the calculation.



The Formula

SAvg is Simple Average: A simple average for a data set in which all data has the same level of importance and the same frequency.

Σ is Summation: This symbol is known as the Greek capital letter *sigma*. In mathematics it denotes that all values written after it (to the right) are summed.

Formula 3.3 – Simple Average:
$$\text{SAvg} = \frac{\Sigma x}{n}$$

n is Total Quantity: This is the physical total count of the number of pieces of data or the total quantity being used in the average calculation. In business, the symbol *n* is a common standard for representing counts.

x is Any Piece of Data: In mathematics this symbol is used to represent an individual piece of data.

As expressed in Formula 3.3, you calculate a simple average by adding together all of the pieces of data then taking that total and dividing it by the quantity.



How It Works

The steps required to calculate a simple average are as follows:

Step 1: Sum every piece of data.

Step 2: Determine the total quantity involved.

Step 3: Calculate the simple average using Formula 3.3.

Assume you want to calculate an average on three pieces of data: 95, 108, and 97. Note that the data are equally important and each appears only once, thus having the same frequency. You require a simple average.

Step 1: $\Sigma x = 95 + 108 + 97 = 300$.

Step 2: There are three pieces of data, or $n = 3$.

Step 3: Apply Formula 3.3: $SAvg = \frac{300}{3} = 100$. The simple average of the data set is 100.



Important Notes

Although mentioned earlier, it is critical to stress that a simple average is calculated only when all of the following conditions are met:

1. All of the data shares the same level of importance toward the calculation.
2. All of the data appear the same number of times.
3. The data does not represent percent changes or a series of numbers intended to be multiplied with each other.

If any of these three conditions are not met, then either a weighted or geometric average is used depending on which of the above criteria failed. We discuss this later when each average is introduced.



Give It Some Thought

It is critical to recognize if you have potentially made any errors in calculating a simple average. Review the following situations and, without making any calculations, determine the best answer.

1. The simple average of 15, 30, 40, and 45 is:
 - a. lower than 20.
 - b. between 20 and 40, inclusive.
 - c. higher than 40.
2. If the simple average of three pieces of data is 20, which of the following data do not belong in the data set? Data set: 10, 20, 30, 40
 - a. 10
 - b. 20
 - c. 30
 - d. 40

Solutions:

1. b (a simple average should fall in the middle of the data set, which appears spread out between 15 and 45, so the middle would be around 30)
2. d (if the number 40 is included in any average calculation involving the other numbers, it is impossible to get a low average of 20)

Example 3.2A: Comparing Average Sales

First quarter sales for Buzz Electronics are as indicated in the table below.

Month	2023 Sales	2024 Sales
January	\$413,200	\$455,875
February	\$328,987	\$334,582
March	\$350,003	\$312,777

Martha needs to prepare a report for the board of directors comparing year-over-year quarterly performance. To do this, she needs you to do the following:

- Calculate the average sales in the quarter for each year.
- Express the 2024 sales as a percentage of the 2023 sales, rounding your answer to two decimals.

Plan	You need to calculate a simple average, or SAvg, for the first quarter in each of 2023 and 2024. Then convert the numbers into a percentage.																
Understand	What You Already Know You know the three quarters annually: 2023: $x_1 = \\$413,200$ $x_2 = \\$328,986$ $x_3 = \\$350,003$ 2024: $x_1 = \\$455,876$ $x_2 = \\$334,582$ $x_3 = \\$312,777$	How You Will Get There For each year, perform steps 1 to 3: Step 1: Sum the data. Step 2: Count the total quantity of data involved. Step 3: Calculate the simple average using Formula 3.3. Step 4: To calculate the rate expressed as a percentage, apply Formula 2.2 from Chapter 2, $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$, treating the 2023 average sales as the base (note the key word "of" in the question) and the 2024 sales as the portion. Round to two decimals.															
Perform	<table> <tr> <th>Step</th><th>Year 2023</th><th>Year 2024</th></tr> <tr> <td>1.</td><td>$\Sigma x = \\$413,200 + \\$328,986 + \\$350,003$ $= \\$1,092,189$</td><td>$\Sigma x = \\$455,876 + \\$334,582 + \\$312,777$ $= \\$1,103,235$</td></tr> <tr> <td>2.</td><td>$n = 3$</td><td>$n = 3$</td></tr> <tr> <td>3.</td><td>$\text{SAvg} = \frac{\\$1,092,189}{3} = \\$364,063$</td><td>$\text{SAvg} = \frac{\\$1,103,235}{3} = \\$367,745$</td></tr> <tr> <td>4.</td><td colspan="2">Step 4: $\% = \frac{\\$367,745}{\\$364,063} \times 100 = 101.01\%$</td></tr> </table>	Step	Year 2023	Year 2024	1.	$\Sigma x = \$413,200 + \$328,986 + \$350,003$ $= \$1,092,189$	$\Sigma x = \$455,876 + \$334,582 + \$312,777$ $= \$1,103,235$	2.	$n = 3$	$n = 3$	3.	$\text{SAvg} = \frac{\$1,092,189}{3} = \$364,063$	$\text{SAvg} = \frac{\$1,103,235}{3} = \$367,745$	4.	Step 4: $\% = \frac{\$367,745}{\$364,063} \times 100 = 101.01\%$		
Step	Year 2023	Year 2024															
1.	$\Sigma x = \$413,200 + \$328,986 + \$350,003$ $= \$1,092,189$	$\Sigma x = \$455,876 + \$334,582 + \$312,777$ $= \$1,103,235$															
2.	$n = 3$	$n = 3$															
3.	$\text{SAvg} = \frac{\$1,092,189}{3} = \$364,063$	$\text{SAvg} = \frac{\$1,103,235}{3} = \$367,745$															
4.	Step 4: $\% = \frac{\$367,745}{\$364,063} \times 100 = 101.01\%$																
Present	The average monthly sales in 2023 were \$364,063 compared to sales in 2024 of \$367,745. This means that 2024 sales are 101.01% of 2023 sales.																

Weighted Averages

Have you considered how your grade point average (GPA) is calculated? Your business program requires the successful completion of many courses. Your grades in each course combine to determine your GPA; however, not every course necessarily has the same level of importance as measured by your course credits.

Perhaps your math course takes one hour daily while your communications course is only delivered in one-hour sessions three times per week. Consequently, the college assigns the math course five credit hours and the communications course three credit hours. If you want an average, these different credit hours mean that the two courses do not share the same level of importance, and therefore a simple average cannot be calculated.

In a **weighted average**, not all pieces of data share the same level of importance or they do not occur with the same frequency. The data cannot represent a percent change or a series of numbers intended to be multiplied with each other. To calculate a weighted average, you require two components:

- The data itself—you need the value for each piece of data.

2. The weight of the data—you need to know how important each piece of data is to the average. This is either an assigned value or a reflection of the number of times each piece of data occurs (the frequency).



The Formula

As expressed in Formula 3.4, calculate a weighted average by adding the products of the weights and data for the entire data set and then dividing this total by the total of the weights.

WAvg is Weighted Average: An average for a data set where the data points may not all have the same level of importance or they may occur at different frequencies.

Σ is Summation: This symbol is known as the Greek capital letter *sigma*. In mathematics it denotes that all values written after it (to the right) are summed.

Formula 3.4 – Weighted Average:

$$\text{WAvg} = \frac{\sum wx}{\sum w}$$

w is Weighting Factor: A number that represents the level of importance for each piece of data in a particular data set. It is either predetermined or reflective of the frequency for the data.

x is Any Piece of Data: In mathematics this symbol is used to represent an individual piece of data.



How It Works

The steps required to calculate a weighted average are:

Step 1: Sum every piece of data multiplied by its associated weight.

Step 2: Sum the total weight.

Step 3: Calculate the weighted average using Formula 3.4.

Let's stay with the illustration of the math and communications courses and your GPA. Assume that these are the only two courses you are taking. You finish the math course with an A, translating into a grade point of 4.0. In the communications course, your C+ translates into a 2.5 grade point. These courses have five and three credit hours, respectively. Since they are not equally important, you use a weighted average.

Step 1: In the numerator, sum the products of each course's credit hours (the weight) and your grade point (the data). This means (math credit hours $\times$ math grade point) + (communications credit hours $\times$ communications grade point). Numerically, this is $\sum wx = (5 \times 4) + (3 \times 2.5) = 27.5$.

Step 2: In the denominator, sum the weights. These are the credit hours. You have $\sum w = 5 + 3 = 8$.

Step 3: Apply Formula 3.4 to calculate your GPA.

$$\text{WAvg} = \frac{27.5}{8} = 3.44 \text{ (GPAs have two decimals).}$$

Note that your GPA is higher than if you had just calculated a simple average of $\frac{4 + 2.5}{2} = 3.25$. This happens because your math course, in which you scored a higher grade, was more important in the calculation.



Things To Watch Out For

The most common error in weighted averages is to confuse the data with the weight. If you have the two backwards, your numerator is still correct; however, your denominator is incorrect. To distinguish the data from the weight, notice that the data forms a part of the question. In the above example, you were looking to calculate your *grade point* average; therefore, grade point is the data. The other information, the credit hours, must be the weight.

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The formula used for calculating a simple average is a simplification of the weighted average formula. In a simple average, every piece of data is equally important. Therefore, you assign a value of 1 to the weight for each piece of data. Since any number multiplied by 1 is the same number, the simple average formula omits the weighting in the numerator as it would have produced unnecessary calculations. In the denominator, the sum of the weights of 1 is no different from counting the total number of pieces of data. In essence, you can use a weighted average formula to solve simple averages.

**Give It Some Thought**

In each of the following, determine which information is the data and which is the weight.

1. Rafiki operates a lemonade stand during his garage sale today. He has sold 13 small drinks for \$0.50, 29 medium drinks for \$0.90, and 21 large drinks for \$1.25. What is the average price of the lemonade sold?
2. Natalie received the results of a market research study. In the study, respondents identified how many times per week they purchased a bottle of Coca-Cola. Calculate the average number of purchases made per week.

Purchases per Week	# of People
1	302
2	167
3	488
4	256

Solutions:
1. The price of the drinks is the data, and the number of drinks is the weight.
2. The purchases per week is the data, and the number of people is the weight.

Example 3.2B: Calculating Your Weighted Grade Point Average

A mark transcript received by a student at a local college. The chart shows how the grade translates into a grade point.

Course	Grade	Credit Hours	Grade	Grade Point	Grade	Grade Point
Economics 100	B	4	A+	4.5	C+	2.5
Math 100	A	5	A	4.0	C	2.0
Marketing 100	B+	3	B+	3.5	D	1.0
Communications 100	C	4	B	3.0	F	0.0
Computing 100	A+	3				
Accounting 100	D	4				

Calculate the student's grade point average (GPA). Round your final answer to two decimals.

Plan	The courses do not carry equal weights as they have different credit hours. Therefore, to calculate the GPA you must find a weighted average, or WAvg.	
Understand	What You Already Know Since the question asked for the grade point average, the grade points for each course are the data, or x. The corresponding credit hours are the weights, or w.	How You Will Get There You need to convert the grade for each course into a grade point using the secondary table. Then perform the following steps: Step 1: Sum every piece of data multiplied by its associated weight. Step 2: Sum the total weight. Step 3: Calculate the weighted average using Formula 3.4.

Perform	Convert the grades:				
	Course	Grade	Grade Point	Credit Hours	
	Economics 100	B	3.0	4	
	Math 100	A	4.0	5	
	Marketing 100	B+	3.5	3	
	Communications 100	C	2.0	4	
	Computing 100	A+	4.5	3	
	Accounting 100	D	1.0	4	
Step 1: $\sum wx = (4 \times 3.0) + (5 \times 4.0) + (3 \times 3.5) + (4 \times 2.0) + (3 \times 4.5) + (4 \times 1.0) = 68$					
Step 2: $\sum w = 4 + 5 + 3 + 4 + 3 + 4 = 23$					
Step 3: $W\text{Avg} = \frac{68}{23} = 2.96$					
Present					
	The student's GPA is 2.96. Note that math contributed substantially (almost one-third) to the student's grade point because this course was weighted heavily and the student performed well.				

Example 3.2C: Weighted Average Loan Balance				
Angelika started the month of March owing \$20,000 on her home equity line of credit (HELOC). She made a payment of \$5,000 on the fifth, borrowed \$15,000 on the nineteenth, and made another payment of \$5,000 on the twenty-sixth. Using each day's closing balance for your calculations, what was the average balance in the HELOC for the month of March?				
Plan	The balance owing in Angelika's HELOC is not equal across all days in March. Some balances were carried for more days than others. This means you will need to use the weighted average technique and find WAv _g .			
Understand	What You Already Know You know the following:			How You Will Get There Step 1: Sum every piece of data multiplied by its associated weight. Step 2: Sum the total weight. Step 3: Calculate the weighted average using Formula 3.4.
	Dates	Number of Days (<i>w</i>)	Balance in HELOC (<i>x</i>)	
	March 1–March 4	4	\$20,000	
	March 5–March 18	14	\$20,000 – \$5,000 = \$15,000	
	March 19–March 25	7	\$15,000 + \$15,000 = \$30,000	
	March 26–March 31	6	\$30,000 – \$5,000 = \$25,000	
Perform	Step 1: $\sum wx = (4 \times \$20,000) + (14 \times \$15,000) + (7 \times \$30,000) + (6 \times \$25,000) = \$650,000$ Step 2: $\sum w = 4 + 14 + 7 + 6 = 31$ Step 3: $W\text{Avg} = \frac{\$650,000}{31} = \$20,967.74$			
Present	Over the entire month of March, the average balance owing in the HELOC was \$20,967.74. Note that the balance with the largest weight (March 5 to March 18) and the largest balance owing (March 19 to March 25) account for almost two-thirds of the calculated average.			

Geometric Averages

How do you average a percent change? If sales increase 100% this year and decrease 50% next year, is the average change in sales an increase of $\frac{(100\% - 50\%)}{2} = 25\%$ per year? The answer is clearly “no.” If sales last year were \$100 and they increased by 100%, that results in a \$100 increase. The total sales are now \$200. If sales then decreased by 50%, you have a \$100 decrease. The total sales are now \$100 once again. In other words, you started with \$100 and finished with \$100. That is an average change of nothing, or 0% per year! Notice that the second percent change is, in fact, multiplied by the result of the

first percent change. A **geometric average** finds the typical value for a set of numbers that are meant to be multiplied together or are exponential in nature.



The Formula

In business mathematics, you most commonly use a geometric average to average a series of percent changes. Formula 3.5 is specifically written to address this situation.

GAvg is Geometric Average: The average of a series of percent changes expressed in percent format. Every percent change involved in the calculation requires an additional $(1 + \Delta\%)$ to be multiplied under the radical. The formula accommodates as many percent changes as needed.

n is Total Quantity: The physical total count of how many percent changes are involved in the calculation.

Formula 3.5 – Geometric Average:
$$\text{GAvg} = \left([(1 + \Delta\%_1) \times (1 + \Delta\%_2) \times \cdots \times (1 + \Delta\%_n)]^{\frac{1}{n}} - 1 \right) \times 100$$

$\Delta\%$ is Percent Change: The value of each percent change in the series from which the average is calculated. Express the percent changes in decimal format.

$\times 100$ is Percent Conversion: Because you are averaging percent changes, convert the final result from decimal form into a percentage.



How It Works

To calculate a geometric average follow these steps:

Step 1: Identify the series of percent changes to be multiplied.

Step 2: Count the total number of percent changes involved in the calculation.

Step 3: Calculate the geometric average using Formula 3.5.

Let's use the sales data presented above, according to which sales increase 100% in the first year and decrease 50% in the second year. What is the average percent change per year?

Step 1: The changes are $\Delta\%_1 = +100\%$ and $\Delta\%_2 = -50\%$.

Step 2: Two changes are involved, or $n = 2$.

Step 3: Apply Formula 3.5:

$$\text{GAvg} = \left([(1 + 100\%) \times (1 - 50\%)]^{\frac{1}{2}} - 1 \right) \times 100$$

$$\text{GAvg} = 0 \times 100 = 0\%$$

The average percent change per year is 0% because an increase of 100% and a decrease of 50% cancel each other out.



Important Notes

A critical requirement of the geometric average formula is that every $(1 + \Delta\%)$ expression must result in a number that is positive. This means that the $\Delta\%$ cannot be a value less than -100% else Formula 3.5 cannot be used.

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An interesting characteristic of the geometric average is that it will always produce a number that is either smaller than (closer to zero) or equal to the simple average. In the example, the simple average of +100% and -50% is 25%, and the geometric average is 0%. This characteristic can be used as an error check when you perform these types of calculations.



Give It Some Thought

For the first three questions, determine whether you should calculate a simple, weighted, or geometric average.

1. Randall bowled 213, 245, and 187 in his Thursday night bowling league and wants to know his average.
2. Cindy invested in a stock that increased in value annually by 5%, 6%, 3%, and 5%. She wants to know her average increase.
3. A retail store sold 150 bicycles at the regular price of \$300 and 50 bicycles at a sale price of \$200. The manager wants to know the average selling price.
4. Gonzalez has calculated a simple average of 50% and a geometric average of 60%. He believes his numbers are correct. What do you think?

Solutions:
 1. Simple; each item has equal importance and frequency.
 2. Geometric; these are a series of percent changes on the price of stock.
 3. Weighted; each item has a different frequency.
 4. At least one of the numbers is wrong since a geometric average is always smaller than or equal to the simple average.

Example 3.2D: How Are Sales Changing?

From 2006 to 2010, WestJet's year-over-year annual revenues changed by +21.47%, +19.89%, -10.55%, and +14.38%. This reflects growth from sales of \$1.751 billion in 2006 to \$2.609 billion in 2010.¹ What is the average percent growth in revenue for WestJet during this time frame?

Plan	Note that these numbers reflect percent changes in revenue. Year-over-year changes are multiplied together, so you would calculate a geometric average, or GAv.	
Understand	What You Already Know Step 1: You know the four percent changes: $\Delta\%_1 = 21.47\%$ $\Delta\%_2 = 19.89\%$ $\Delta\%_3 = -10.55\%$ $\Delta\%_4 = 14.38\%$ Step 2: Four changes are involved, or $n = 4$.	How You Will Get There Step 3: Express the percent changes in decimal format and substitute into Formula 3.5.
Perform	Step 3: $\text{GAv} = \left([(1 + 0.2147) \times (1 + 0.1989) \times (1 - 0.1055) \times (1 + 0.1438)]^{\frac{1}{4}} - 1 \right) \times 100 = 10.483\%$	
Present	On average, WestJet revenues have grown 10.483% each year from 2006 to 2010.	

Mechanics

¹ WestJet, *WestJet Fact Sheet*, www.westjet.com/pdf/investorMedia/investorFactSheet.pdf (accessed May 10, 2011).



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3.3: Ratios, Proportions, and Prorating

(It Is Only Fair)

You and your business partner have a good problem: Consumers are snapping up packets of your new eucalyptus loganberry facial scrub as fast as you can produce them. Each packet of the scrub contains 600 mg of loganberry extract and 80 mg of eucalyptus oil, as well as water and clay and other ingredients. Ratios are invaluable in understanding the relationship among different quantities, such as how much of each ingredient you need.

You have no trouble obtaining the water and clay, but the loganberry extract and eucalyptus oil are in short supply because of poor weather. In any time period, your suppliers can provide seven times as much loganberry extract as eucalyptus oil. To figure out which ingredient is limiting your production, you need proportions.

Sometimes you need to relate a proportion to the total of the quantities. This requires prorating skills. For example, once your business has grown, you start using a production line, which follows the common practice of producing more than one product. The papaya facial scrub you have introduced recently and the eucalyptus loganberry scrub require the same amount of production time. Your equipment capacity is 1,000 units per day. How many units of each type of scrub must you produce to meet market demand in the ratio of nine to two?

Years later, you are splitting the profits of your business partnership in proportion to each partner's total investment. You invested \$73,000, while your partner invested \$46,000. With total profits of \$47,500, what is your share?

Understanding the relationships among various quantities and how the components relate to an overall total emphasizes the need for understanding ratios, proportions, and prorating.

What Is a Ratio?

A **ratio** is a fixed relationship between two or more quantities, amounts, or sizes of a similar nature. For a ratio to exist, all terms involved in the ratio must be nonzero. Examine the criteria of this definition more closely:

1. **There Must Be Two or More Quantities.** A ratio does not exist if only one quantity is involved. For example, the fuel tank on your sports car takes 60 litres of gasoline. This is not a ratio, as there is no relationship to any other quantity, amount, or size. On the other hand, if you compare your fuel tank to the fuel tank of your friend's Hummer, you now have two quantities involved and could say that her fuel tank has twice the capacity of yours.
2. **All Terms Must Be of a Similar Nature.** In all of the examples provided note that all quantities, amounts, or sizes are based on the same unit. In the cosmetics formulation it was milligrams to milligrams; in the production line, it was units to units; in the investment scenario, it was dollars to dollars. For a ratio to have meaning and to be properly interpreted, all terms of the ratio must be expressed in a similar nature. When you place different units such as kilometres and metres into the same ratio, the result is confusing and will lead to misinterpretation of the relationship.
3. **All Terms Must Be Nonzero.** The numbers that appear in a ratio are called the **terms of the ratio**. If we have a recipe with four cups of flour to one cup of sugar, there are two terms: four and one. If any term is zero, then the quantity, amount, or size does not exist. For example, if the recipe called for four cups of flour to zero cups of sugar, there is no sugar! Therefore, every term must have some value other than zero.

Let's continue using the example of four cups of flour to one cup of sugar. Business ratios are expressed in five common formats, as illustrated in the table below.

Format	Ratio Example	Interpretation
To	4 to 1	Four cups of flour to one cup of sugar
: (colon)	4 : 1	Four cups of flour to one cup of sugar
Fraction	$\frac{4}{1}$	Four cups of flour per one cup of sugar
Decimal	4	Four times as much flour than sugar
Percent	400%	Flour is 400% of sugar (Hint: think rate, portion, base)

All of these formats work well when there are only two terms in the ratio. If there are three or more terms, ratios are best expressed in the colon format. For example, if the recipe called for four parts of flour to one part of sugar to two parts of chocolate chips, the ratio is 4 : 1 : 2. The fraction, decimal, and percent forms do not work with three or more terms.

Simplification and Reduction of Ratios

When a ratio is used to express a relationship between different variables, it must be easy to understand and interpret. Sometimes when you set up a ratio initially, the terms are difficult to comprehend. For example, what if the recipe called for $62\frac{1}{2}$ parts flour to 25 parts sugar? That is not very clear. Expressing the same ratio another way, you can say the recipe requires 5 parts flour to 2 parts sugar. Note how the relationship is clearer in the latter expression. Either way, though, both ratios mean the same thing; in decimal format this ratio is expressed as a value of 2.5. Recall that Section 2.2 discussed how fractions are expressed in higher and lower terms. We now apply the same knowledge to ratios to make the relationship as clear as possible.

When you reduce ratios to lower terms, remember two important characteristics involving the cardinal rule and integers:

1. **The Cardinal Rule.** Recall from Section 2.5 that the cardinal rule of algebra states, "What you do to one you must do to the other." In other words, whatever mathematical operation is performed on a term in a ratio must be equally performed on every other term in the ratio. If this rule is violated then the relationship between the terms is broken.
2. **Maintaining Integers.** Integers are easier to understand than decimals and fractions. In reducing a ratio to lower terms, aim to maintain every term as an integer (much as in Section 2.2).



How It Works

The steps involved in reducing ratios to lower terms are listed below. You may not need some steps, so skip them if the characteristic is not evident in the ratio.

Step 1: Clear any fractions that, when divided, produce a nonterminating decimal. Apply the rules of algebra and multiply each term by the denominator being cleared from the ratio. For example, if the ratio is $\frac{1}{3} : 2$, the first term when divided produces a nonterminating decimal. Clear the fraction by multiplying every term by the denominator of 3, resulting in a ratio of $1 : 6$.

Step 2: Perform division on all fractions that produce a terminating decimal. For example, if the ratio is $\frac{2}{5} : \frac{3}{10}$, both terms convert to terminating decimals, resulting in a ratio of $0.4 : 0.3$.

Step 3: Eliminate all decimals from the ratio through multiplication. In other words, express the ratio in higher terms by multiplying every term by a power of 10. The power of 10 you choose must be large enough to eliminate all decimals. For example, if the ratio is $0.2 : 0.25 : 0.125$, notice that the third term has the most decimal positions. A power of 1,000 (10^3) is required to move the decimal three positions to the right. Multiply every term by a power of 1,000, resulting in a ratio of $200 : 250 : 125$.

Step 4: Find a common factor that divides evenly into every term, thus producing integers. If you find no such factors, then the ratio is in its lowest terms. For example, if the ratio is $10 : 4 : 6$ it can be factored by dividing every term by 2, resulting in a ratio of $5 : 2 : 3$. There is no common factor that reduces this ratio further; therefore, it is in its lowest terms.



Important Notes

To factor any term, recall your multiplication tables. Assume the ratio is $36 : 24$. It is always best to pick the lowest term in the ratio when factoring. Looking at the 24 and recall what multiplies together to arrive at 24. In this case, your factors are 1×24 , 2×12 , 3×8 , and 4×6 . The goal is to find the largest value among these factors that also divides evenly into the other term of 36. The largest factor that makes this true is 12. Therefore, perform step 4 in our reduction process by dividing every term by a factor of 12, resulting in a ratio of $3 : 2$.



Things To Watch Out For

Always ensure that before you apply any of the reduction steps your relationship meets the requirements of being a ratio in the first place. For example, the expression of 10 km : 500 m is not a $1 : 50$ ratio since it violates the "similar nature"

characteristic of the ratio definition. You need to convert the metres into kilometres, producing $10 \text{ km} : 0.5 \text{ km}$. Now that you have a ratio, the reduction to lowest terms results in a $20 : 1$ ratio. This is very different than $1 : 50$! The lesson learned is to make sure you are working with a proper ratio before you manipulate it.

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In the fourth step of the procedure, do not feel compelled to find what is called the "magic factor." This is the single factor that reduces the ratio to its lowest terms in a single calculation. Although this is nice if it happens, you may as well find any factor that will make the ratio smaller and easier to work with.

- For example, assume a ratio of $144 : 72 : 96$. It may not be very apparent what factor goes into each of these terms on first glance. However, they are all even numbers, meaning they can all be divided by 2. This produces $72 : 36 : 48$.
- Once again, the "magic factor" may not be clear, but every term is still even. Divide by 2 again, producing $36 : 18 : 24$.
- If no magic factor is apparent, again note that all numbers are even. Divide by 2 yet again, producing $18 : 9 : 12$.
- The common factor for these terms is 3. Divided into every term you have $6 : 3 : 4$. There is no common factor for these terms, and the ratio is now in its lowest terms.

In the above example it took four steps to arrive at the answer—and that is all right. Did you notice the "magic factor" that could have solved this in one step? You can find it if you multiply all of the factors you have: $2 \times 2 \times 2 \times 3 = 24$. Dividing every term in the original ratio by 24 produces the solution in one step. If you did not notice this "magic factor," there is nothing wrong with taking four steps (or more!) to get the answer. In the end, both methods produce the same solution.

Example 3.3A: Reducing Ratios to Lowest Terms

Reduce the following ratios to their lowest terms:

- a. $49 : 21$ b. $0.33 : 0.066$ c. $\frac{9}{2} : \frac{3}{11}$ d. $5\frac{1}{8} : 6\frac{7}{8}$

Plan	You have been asked to reduce the ratios to their lowest terms.				
Understand	What You Already Know The four ratios to be reduced have been provided in colon format.		How You Will Get There Step 1: Clear all fractions that produce nonterminating decimals. Step 2: Perform division on all fractions that produce terminating decimals. Step 3: Eliminate all decimals through multiplication. Step 4: Divide every term by a common factor, reducing it to the lowest terms.		
Perform	Step	a. $49 : 21$	b. $0.33 : 0.066$	c. $\frac{9}{2} : \frac{3}{11}$	d. $5\frac{1}{8} : 6\frac{7}{8}$
	1.	No fractions	No fractions	Multiply both terms by 11: $\frac{99}{2} : 3$	No fractions that produce nonterminating decimals
	2.	No fractions	No fractions	Perform division: $49.5 : 3$	Perform division: $5.125 : 6.875$
	3.	No decimals	Clear decimals by multiplying by 1,000: $330 : 66$	Clear decimals by multiplying by 10: $495 : 30$	Clear decimals by multiplying by 1,000: $5,125 : 6,875$
	4.	Divide by 7: $7 : 3$	Divide by 66: $5 : 1$	Divide by 15: $33 : 2$	Divide by 125: $41 : 55$
Present	In lowest terms, the ratios are $7 : 3$, $5 : 1$, $33 : 2$, and $41 : 55$, respectively.				

Reducing a Ratio to the Smallest Term of One

Your goal in reducing a ratio is to make it easier to understand. Sometimes, even after you have applied the techniques for reducing the ratio, the end result is still hard to grasp. Look at part (d) of Example 3.3A. You arrived at a final solution of 41 : 55, with no further reduction possible. Think of this as 41 cups of flour to 55 cups of sugar. The relationship is not very clear.

In these circumstances, although integers are preferable in general, you must reduce the ratio further, reintroducing decimals to make the relationship more comprehensible. This means you apply a technique called “reducing the ratio to the smallest term of one.” In this technique, the smallest term in the ratio will have a value of 1 once you perform the ratio reduction and simplification.



How It Works

Follow these steps to reduce a ratio to the smallest term of one:

Step 1: Locate the smallest term in the ratio. (Do not just pick the first term.)

Step 2: Divide every term in the ratio by the selected smallest term. Every other term becomes a decimal number, for which either a clear rounding instruction is provided or you must obey the rounding rules used in this textbook. The smallest term by nature of the division equals one.

Let's continue with part (d) of Example 3.3A, in which the reduced ratio is 41 : 55.

Step 1: Locate the smallest term in the ratio. It is the first term and has a value of 41.

Step 2: Take every term and divide it by 41 to arrive at 1 : $1.\overline{34146}$. The decimal number allows you to roughly interpret the ratio as “one cup of flour to a touch over $1\frac{1}{3}$ cups of sugar.” Although not perfect, this is more understandable than 41 : 55.



Important Notes

You may have to make a judgment call when you decide whether to leave a reduced ratio alone or to reduce it to a ratio where the smallest term is one. There is no clear definitive rule; however, keep the following two thoughts in mind:

- For purposes of this textbook, you are provided with clear instructions on how to handle the ratio, allowing everyone to arrive at the same solution. Your instructions will read either “Reduce the ratio to its smallest terms” or “Reduce the ratio to the smallest term of one.”
- In the real world, no such instructions exist. Therefore you should always base your decision on which format is easier for your audience to understand. If the reduced fraction leaves your audience unable to understand the relationship, then use the reduction to a ratio with the smallest term of one instead.

Example 3.3B: Maintaining Inventory

The inventory management system at your company requires you to keep a ratio of 27 to 47 Nintendo consoles to Sony consoles on the shelf. There are 38 Nintendo consoles on the shelf.

- Express the ratio where the smallest term is one. Round your solution to two decimals.
- Using the ratio, determine how many Sony consoles should be in your inventory.

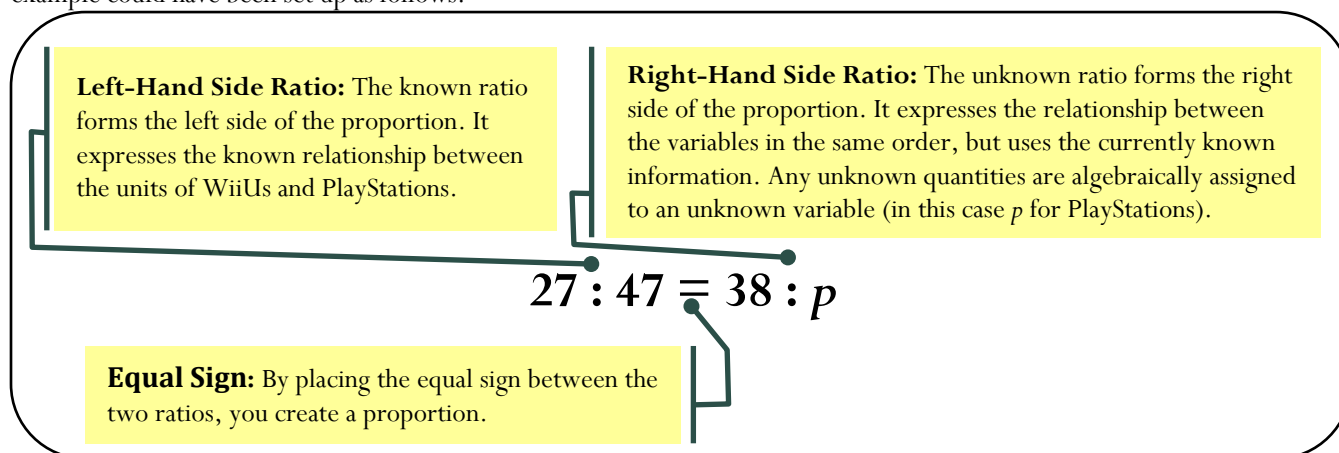
Plan	These terms have no common factor (note that 47 is a prime number). The terms in this ratio cannot be reduced in any way, which makes the relationship difficult to assess. Therefore, reduce the ratio to the smallest term of one. Once the relationship is more evident, determine how many Sony consoles need to be on the shelf.	
Understand	What You Already Know The ratio of Nintendo consoles to Sony consoles is 27 : 47. 38 Wiis are currently on the shelf.	How You Will Get There Step 1: Locate the smallest term. Step 2: Divide all terms by the smallest term to get the ratio. Step 3: Use the ratio to calculate the number of Sony consoles needed on the shelf.

Perform	Step 1: The smallest term is 27. Step 2: Divide every term by 27. $\frac{27}{27} : \frac{47}{27}$ becomes 1 : 1.74 The inventory management requires 1.74 units of Sony consoles to be on the shelf for every unit of Nintendo consoles. Step 3: $38 \times 1.74 = 66.12$ units. Since only whole units are possible, round to the nearest integer of 66.
Present	The inventory management system requires approximately 1.74 Sony consoles for every Nintendo console on the shelf. Therefore, with 38 Nintendo consoles on the shelf you require 66 Sony consoles to also be on the shelf.

What Are Proportions?

Knowing the relationship between specific quantities is helpful, but what if your quantity happens not to match the specific quantity expressed in the existing ratio? Example 3.3B illustrated that you need to learn how to apply the ratio to meet current conditions. The ratio in the inventory system was expressed in terms of 27 Nintendo WiiUs; however, the shelf displayed 38 units. How can we relate an existing ratio to a needed ratio?

A **proportion** is a statement of equality between two ratios. Just as we have both algebraic expressions and algebraic equations, there are ratios and proportions. With algebraic expressions, only simplification was possible. When the expression was incorporated into an algebraic equation, you solved for an unknown. The same is true for ratios and proportions. With ratios, only simplification is possible. Proportions allow you to solve for any unknown variable. The WiiU and PlayStation example could have been set up as follows:



A proportion must adhere to three characteristics, including ratio criteria, order of terms, and number of terms.

- **Characteristic #1: Ratio Criteria Must Be Met.** By definition, a proportion is the equality between two ratios. If either the left side or the right side of the proportion fails to meet the criteria for being a ratio, then a proportion cannot exist.
- **Characteristic #2: Same Order of Terms.** The order of the terms on the left side of the proportion must be in the exact same order of terms on the right side of the proportion. For example, if your ratio is the number of MP3s to CDs to DVDs, then your proportion is set up as follows:

$$\text{MP3} : \text{CD} : \text{DVD} = \text{MP3} : \text{CD} : \text{DVD}$$
- **Characteristic #3: Same Number of Terms.** The ratios on each side must have the same number of terms such that every term on the left side has a corresponding term on the right side. A proportion of $\text{MP3} : \text{CD} : \text{DVD} = \text{MP3} : \text{CD}$ is not valid since the DVD term on the left side does not have a corresponding term on the right side.

When you work with proportions, the mathematical goal is to solve for an unknown quantity or quantities. In order to solve any proportion, always obey the following four rules:

1. **Rule #1: At Least One Value for Any Term Is Known.** At least one of the left side or right side values for each term must be known. For example, $x : 5 = y : 10$ is not a solvable proportion since the corresponding first terms on both sides are unknown. However, $15 : 5 = y : 10$ is solvable since at least one of the first corresponding terms (the 15 and the y) is known.
2. **Rule #2: One Pair of Corresponding Terms Must Be Known.** At least one pair of corresponding terms on the left side and right side must have both quantities known. For example, $3 : x : 6 = y : 4 : z$ is not a solvable proportion since there is no pair of first terms (3 and y), second terms (x and 4), or third terms (6 and z) that produces a pair of known values. However, $3 : x : 6 = 9 : 4 : z$ is a solvable proportion since the first terms on both sides (the 3 and 9) are known.
3. **Rule #3: Obey BEDMAS and Perform Proper Algebraic Manipulation.** To manipulate a proportion, you must satisfy the rules of BEDMAS (Section 2.1) and all of the rules of algebra (Sections 2.4 and 2.5). Violating any of these rules breaks the equality of the ratios and produces an incorrect proportion.
4. **Rule #4: Use the Fractional Format.** The fractional format for ratios is recommended for solving a proportion. The other four formats generally make solving the proportion much more difficult, and the mathematical operations required become unclear.



How It Works

Follow these steps in solving any proportion for an unknown variable or variables:

Step 1: Set up the proportion with the known ratio on the left side. Place the ratio with any unknown variables on the right side.

Step 2: Work with only two terms at a time, and express the two terms in fractional format. This is not a problem if the proportion has only two terms on each side of the equation, for example, $27 : 47 = 38 : p$. This is expressed as $\frac{27}{47} = \frac{38}{p}$. If the proportion has three or more terms on each side, you can pick any two terms from each side of the proportion so long as you pick the same two terms on each side. In making your selection, aim to have a pair of terms on one side of the equation where both values are known while the other side of the equation is made up of one known term and one unknown term. For example, assume the proportion $6 : 5 : 4 = 18 : 15 : y$. The selection of the first and third terms only on each side of the equation produces $6 : 4 = 18 : y$. Notice that this is now a proportion with only two terms on each side, which you can express as $\frac{6}{4} = \frac{18}{y}$.

Step 3: Solve for the unknown variable. Obey the rules of BEDMAS and perform proper algebraic manipulation.

Step 4: If the proportion contained more than one unknown variable, go back to step 2 and select another pairing that isolates one of the unknown variables. Although one of the unknown variables is now known as a result of step 3, do not use this known value in making your selection. The danger in using a solved unknown variable is that if an error has occurred, the error will cascade through all other calculations. For example, assume the original proportion was $3 : 7 : 6 : 8 = x : y : z : 28$. From step 2 you may have selected the pairing of $3 : 8 = x : 28$, and in step 3 you may have erroneously calculated $x = 11.5$ (the correct answer is $x = 10.5$). In returning to step 2 to solve for another unknown variable, do not involve x in your next pairing. To isolate y , use $7 : 8 = y : 28$ and not $3 : 7 = 11.5 : y$, which will at least ensure that your calculation of y is not automatically wrong based on your previous error.



Things To Watch Out For

You must always pick terms from the same positions on both sides of the proportion. Otherwise, you will violate the equality of the proportion, since the terms are no longer in the same order on both sides and Characteristic #2 is not satisfied. For example, in working with the proportion $6 : 5 : 4 = 18 : 15 : y$ from above, you cannot select the first and third terms on the left side and also select the second and third terms on the right side. In other words, $6 : 4 \neq 15 : y$ since 1st term : 3rd term $\neq$ 2nd term : 3rd term.

**TOP
SECRET****Paths To Success**

It is always easiest to solve a proportion when the unknown variable is in the numerator. This characteristic requires minimal algebra and calculations to isolate the variable. If you find yourself with an unknown variable in the denominator, you can mathematically invert the fraction on both sides, since this obeys the cardinal rule of “what you do to one, you must do to the other.” When you invert, the numerator becomes the denominator and vice versa. For example, if the proportion is $\frac{3}{4} = \frac{12}{b}$, inversion then produces a proportion of $\frac{4}{3} = \frac{b}{12}$. Notice that isolating the unknown variable in the inverted proportion requires only a multiplication of 12 on both sides. This is a lot less work!

**Give It Some Thought**

- Some of the following proportions violate the characteristics or rules of proportions. Examine each and determine if all the rules and characteristics are met. If not, identify the problem.
 - $4 : 7 = 6 : y$
 - $5 : 3 = 6 : a : b$
 - $6 \text{ km} : 3 \text{ m} = 2 \text{ m} : 4 \text{ km}$
 - $6 : k = 18 : 12$
 - $4 : 0 = 8 : z$
 - $9 = p$
 - $4 : 7 : 10 = d : e : f$
 - $y : 10 : 15 = x : 30 : z$
- In the following solved proportions, which person properly executed step 2 of the proportion steps?

$6 : 5 : 4 : 3 = x : y : z : 9$

Person A: $6 : 5 = x : y$ Person B: $4 : 3 = y : 9$ Person C: $4 : 3 = z : 9$

Solutions:

1. a. OK
b. Not the same number of terms on each side (Characteristic #3)
c. Terms are not in same units; terms are not in the same order (Characteristics #1 and #2)
d. OK
e. Does not meet ratio criteria; there is a term of zero (Characteristic #1)
f. Does not meet ratio criteria; must have at least two terms on each side (Characteristic #1)
g. There is no corresponding known term on both sides; every pair of terms contains an unknown (Rule #2)
h. The first corresponding term is not known on both sides; at least one needs to be known (Rule #1)
2. Person C did it right. Person A failed to isolate a variable, and different terms were extracted by Person B (3rd term : 4th term $\neq$ 2nd term : 4th term).

Example 3.3C: Estimating Competitor Profits

A recent article reported that companies in a certain industry were averaging an operating profit of \$23,000 per 10 full-time employees. A marketing manager wants to estimate the operating profitability for one of her company's competitors, which employs 87 full-time workers. What is the estimated operating profit for that competitor?

Plan	There is a relationship between the number of full-time employees and the operating profit. This is a ratio. If you set up a similar ratio for the competitor, you create a proportion that you can solve for the competitor's estimated operating profit, or p .	
Understand	What You Already Know You know the industry and competitor information: Industry operating profit = \$23,000 Industry full-time employees = 10 Competitor operating profit = p Competitor full-time employees = 87	How You Will Get There Step 1: Set up the proportion. Step 2: Express in fractional format. Step 3: Isolate the unknown variable, p . Step 4: Since there is only one unknown, this step is not needed.
Perform	Step 1: industry profit : industry employees = competitor profit : competitor employees $\$23,000 : 10 = p : 87$ Step 2: $\frac{\$23,000}{10} = \frac{p}{87}$ Step 3: $\frac{\$23,000 \times 87}{10} = p$ $p = \$200,100$	

Present	If the industry is averaging \$23,000 in operating profits per 10 full-time employees, then the competitor with 87 full-time employees has an estimated \$200,100 in operating profits.

Example 3.3D: Making the Party Punch	
In preparing for a party, Brendan needs to mix a punch consisting of fruit juice, vodka, and 7UP in the ratio of 5 : 3 : 2, respectively. If Brendan has a 2 L bottle of vodka, how much fruit juice and 7UP must he mix with it?	
Plan	The relationship between the punch recipe ingredients of fruit juice, vodka, and 7UP forms a ratio. If you set up a similar ratio for what Brendan actually has and needs, you can calculate the exact amounts of fruit juice, vodka, and 7UP that are required.
Understand	What You Already Know You know the punch recipe ratio of fruit juice : vodka : 7UP is 5 : 3 : 2. Brendan has 2 L of vodka. How You Will Get There Step 1: Set up the proportion. Step 2: Pick two terms containing one unknown variable and express in fractional format. Step 3: Isolate the unknown variable. Step 4: Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable.
Perform	Step 1: $5 : 3 : 2 = f : 2 : s$ Step 2-3: $\frac{5}{3} = \frac{f}{2}$ $\frac{10}{3} = f$ $3.\bar{3} = f$ Step 4: $\frac{3}{2} = \frac{2}{s}$ $\frac{2}{3} = \frac{s}{2}$ $\frac{4}{3} = s$ $1.\bar{3} = s$
Present	Maintaining the ratio of 5 : 3 : 2 between the fruit punch, vodka, and 7UP, Brendan should mix $3\frac{1}{3}$ L of fruit juice and $1\frac{1}{3}$ L of 7UP with his 2 L of vodka.

What Is Prorating?

Ratios and proportions are commonly used in various business applications. But there will be numerous situations where your business must allocate limited funds across various divisions, departments, budgets, individuals, and so on. In the opener to this section, one example discussed the splitting of profits with your business partner, where you must distribute profits in proportion to each partner's total investment. You invested \$73,000 while your partner invested \$46,000. How much of the total profits of \$47,500 should you receive?

The process of **prorating** is the taking of a total quantity and allocating or distributing it proportionally. In the above example, you must take the total profits of \$47,500 and distribute it proportionally with your business partner based on the investment of each partner. The proportion is:

$$\text{your investment} : \text{your partner's investment} = \text{your profit share} : \text{your partner's profit share}$$

This proportion has two major concerns:

1. You don't know either of the terms on the right side. As per the rules of proportions, this makes the proportion unsolvable.
2. There is a piece of information from the situation that you didn't use at all! What happened to the total profits of \$47,500?

Every prorating situation involves a **hidden term**. This hidden term is usually the sum of all the other terms on the same side of the proportion and represents a total. In our case, it is the \$47,500 of total profits. This quantity must be placed as an extra term on both sides of the proportion to create a proportion that can actually be solved.



How It Works

Prorating represents a complex proportion. As such, the steps involved in prorating are similar to the steps for solving any proportion:

Step 1: Set up the proportion with the known ratio on the left side. Place the ratio with any unknown variables on the right side.

Step 2: Insert the hidden term on both sides of the proportion. Usually this term represents the total of all other terms on the same side of the proportion.

Step 3: Working with only two terms at a time, express the two terms in fractional format. Ensure that only one unknown variable appears in the resulting proportion.

Step 4: Solve for the unknown variable. Ensure that the rules of BEDMAS are obeyed and proper algebraic manipulation is performed.

Step 5: If the prorating contains more than one unknown variable, go back to step 3 and select a pairing that isolates another one of the unknown variables.

To solve your profit-splitting scenario, let y represent your profit share and p represent your partner's share:

Step 1: your investment : your partner's investment = your profit share : your partner's profit share

$$\$73,000 : \$46,000 = y : p$$

Step 2: Insert the hidden total term on both sides:

$$\text{your investment} : \text{your partner's investment} : \text{total investment} = \text{your profit share} : \text{your partner's profit share} : \text{total profits}$$

$$\$73,000 : \$46,000 : \$119,000 = y : p : \$47,500$$

Step 3: Set up one proportion: $\frac{\$73,000}{\$119,000} = \frac{y}{\$47,500}$

Step 4: Solve for y . Calculate $y = \$29,138.66$.

Step 5: Set up the other proportion: $\frac{\$46,000}{\$119,000} = \frac{p}{\$47,500}$

Solve for p . Calculate $p = \$18,361.34$.

Your final proportion is:

$$\$73,000 : \$46,000 : \$119,000 = \$29,138.66 : \$18,361.34 : \$47,500.00$$

So you will receive \$29,138.66 of the total profits and your partner will receive \$18,361.34.



Important Notes

To make prorating situations easier to solve, it is always best to insert the hidden term as the *last term* on both sides of the equation. This forces the unknown variables into the numerator when you select pairs of terms. It then takes less algebra for you to isolate and solve for the unknown variable.



Things To Watch Out For

Ensure that when you insert the hidden term you put it in the same position on both sides of the proportion. A common error is to put the total on the "outsides" of the proportion:

$$A : B = C : D \text{ when prorated becomes}$$

$$\text{Total} : A : B = C : D : \text{Total}$$

This violates the proportion characteristics, since the terms are not in the same order on both sides. The correct insertion of the hidden total term makes the proportion look like this:

$$A : B : \text{Total} = C : D : \text{Total}$$

Paths To Success

$$3 : 5 : 2 : 10 = \$30 : \$50 : \$20 : \$100.$$

Plan	You need to calculate the sum of the direct and overhead costs for each product. The direct costs are known. The individual overhead costs are unknown but can be calculated using prorating of the total overhead costs. You have three unknowns, namely, chocolate overhead (Co), wafers overhead (Wo), and nougat overhead (No).	
Understand	What You Already Know You have some information on direct costs and overhead costs: Chocolate direct (Cd) = \$743,682 Wafers direct (Wd) = \$2,413,795 Nougat direct (Nd) = \$347,130 Total overhead costs = \$721,150	How You Will Get There Step 1: Set up the proportion. Step 2: Insert the hidden term. Step 3: Pick two terms containing one unknown variable and express in fractional format. You can start with chocolate overhead, or Co. Step 4: Isolate the unknown variable. Step 5: Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable. Proceed with the wafers overhead, or Wo. Step 6: Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable. Proceed with the nougat overhead, or No. Step 7: Sum the direct and overhead costs for each product.
Perform	Step 1: $Cd : Wd : Nd = Co : Wo : No$ $\$743,682 : \$2,413,795 : \$347,130 = Co : Wo : No$ Step 2: $\$743,682 : \$2,413,795 : \$347,130 : \text{Total direct costs} = Co : Wo : No : \$721,150$ Calculate the total direct costs as a sum of all three direct costs: $\text{Total direct costs} = \$743,682 + \$2,413,795 + \$347,130 = \$3,504,607$ The proportion is now: $\$743,682 : \$2,413,795 : \$347,130 : \$3,504,607 = Co : Wo : No : \$721,150$ Step 3: $\frac{\$743,682}{\$3,504,607} = \frac{Co}{\$721,150}$ Step 4: $\$153,028.93 = Co$ Step 5: $\frac{\$2,413,795}{\$3,504,607} = \frac{Wo}{\$721,150}$ $\$496,691.43 = Wo$ Step 6: $\frac{\$347,130}{\$3,504,607} = \frac{No}{\$721,150}$ $\$71,429.64 = No$ Step 7: Chocolate total cost = $\$743,682.00 + \$153,028.93 = \$896,710.93$ Wafer total cost = $\$2,413,795.00 + \$496,691.43 = \$2,910,486.43$ Nougat total cost = $\$347,130.00 + \$71,429.64 = \$418,559.64$	
Present	Putting together the direct costs with the allocated overhead costs, the total cost for chocolate is \$896,710.93, wafers is \$2,910,486.43, and nougat is \$418,559.64.	



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Chapter 3 Summary

Key Concepts Summary

Section 3.1: Percent Change (Are We Up or Down?)

- Measuring the percent change in a quantity from one value to another
- Measuring the constant rate of change over time in a quantity

Section 3.2: Averages (What Is Typical?)

- The calculation of simple averages when everything is equal
- The calculation of weighted averages when not everything is equal
- The calculation of geometric averages when everything is multiplied together

Section 3.3: Ratios, Proportions, and Prorating (It Is Only Fair)

- The characteristics of a ratio
- How to simplify and reduce a ratio to its lowest terms
- How to simplify a ratio by reducing its smallest term to a value of one
- How to equate two ratios into the form of a proportion
- How to use proportions to prorate

The Language of Business Mathematics

average A single number that represents the middle of a data set.

capital gains or losses The increase (or decrease) in the value of an investment over the holding period of an investment.

geometric average A typical value for a set of numbers that are meant to be multiplied together or are exponential in nature. This may take the form of a series of percent changes.

hidden term In prorating, this is the sum of all the other terms on the same side of the proportion and represents a total.

investment income While holding onto an investment, this is any amount received by the investor from sources including interest, dividends, or distributions.

percent change An expression in percent form of how much any quantity changes from a starting period to an ending period.

proportion A statement of equality between two ratios.

prorating The process of taking a total quantity and allocating or distributing it proportionally.

rate of change over time A measure of the percent change in a variable per time period.

ratio A fixed relationship between two or more quantities, amounts, or sizes of a similar nature where all terms are nonzero.

simple average An average where each piece of data shares the same level of importance and frequency but does not represent percent changes or numbers that are intended to be multiplied with each other.

terms of the ratio The numbers appearing in a ratio.

weighted average An average where not all pieces of data share the same level of importance or they do not occur with the same frequency; the data cannot represent percent changes or numbers that are intended to be multiplied with each other.

The Formulas You Need to Know

Symbols Used

$\Delta\%$ = percent change

Σ = summation symbol, which means you add everything up

GAvg = geometric average

n = a count of something, whether a total number of periods or a total quantity

New = the value that a quantity has become; the number that is being compared

Old = the value that a quantity used to be; the number to compare to

RoC = rate of change per time period

SAvg = simple average

x = any individual piece of data

w = weighting factor

WAvg = weighted average

Formulas Introduced

Formula 3.1 Percent Change: $\Delta\% = \frac{\text{New}-\text{Old}}{\text{Old}} \times 100$ (Section 3.1)

Formula 3.2 Rate of Change over Time: $\text{RoC} = \left(\sqrt[n]{\frac{\text{New}}{\text{Old}}} - 1 \right) \times 100$ (Section 3.1)

Formula 3.3 Simple Average: $\text{SAvg} = \frac{\sum x}{n}$ (Section 3.2)

Formula 3.4 Weighted Average: $\text{WAvg} = \frac{\sum wx}{\sum w}$ (Section 3.2)

Formula 3.5 Geometric Average: $\text{GAvg} = \left[\sqrt[n]{(1 + \Delta\%_1) \times (1 + \Delta\%_2) \times \cdots \times (1 + \Delta\%_n)} - 1 \right] \times 100$ (Section 3.2)

Technology

Calculator

Percent Change

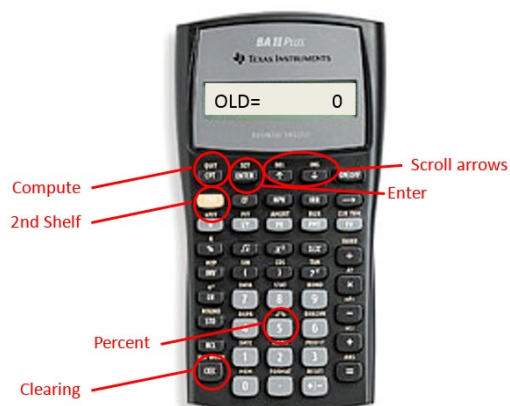
2nd $\Delta\%$ (located above the 5 key) to access and enter three of the following. Press CPT on the unknown to compute. It is recommended to clear a previous question from memory by then pressing 2nd CLR Work.

OLD = The old or original quantity; the number to compare to

NEW = The new or current quantity; the number wanting to be compared

%CH = The percent change; in percent format

#PD = Number of consecutive periods for the change. By default, it is set to 1. However, if you want to increase a number by the same percentage (a constant rate of change) consecutively, enter the number of times the change occurs. For example, if increasing a number by 10% three times in a row, set this variable to 3.





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Chapter 4: Human Resources and Economic Applications

(It's All about the People)

Has your employer paid you all that you have earned? While this problem is difficult to track, the Canadian Payroll Association estimates the frequency of payroll errors to be 5.7%.¹ Studies in the United States have found that 33% of employers have made some form of payroll mistake, whether in the calculation of pay or in deductions.² This means that you must always check your paycheque to ensure that you receive the correct amount of pay. If you are overpaid, you are not entitled to keep the overage and must reimburse your employer. If you are underpaid, your employer owes you money.

We all work to earn enough income to cover our expenses, such as mortgages, car payments, and recreational activities. There are many ways to earn that income, ranging from straight salaries and hourly wages to commission rates and piecework wages. The amount of money you generate by working is called your *gross earnings*. If you are an employee, this amount is then subject to automatic deductions, including federal taxes, provincial taxes, employment insurance, Canada Pension Plan, along with any other deductions required by your employer. All of these government deductions are submitted to the government on your behalf. What is left over after all of these deductions is your net pay, which is your take-home amount.

Payroll forms one of the largest and most contentious expenses in most organizations. It is critical to get it right. Workers often go on strike if they believe their take-home pay is unfair. And payroll errors are costly and time consuming to fix, aside from their impact on employee satisfaction.

Payroll decisions are affected by inflation. Ask your parents how much things cost when they were young. In the 1950s, my parents attended movie theatres for 25¢ admission. A chocolate bar was 5¢. Today, attending a first-run movie costs \$8.99 or more, and chocolate bars are typically around one dollar. To keep up with these rising prices, incomes have constantly increased. In 1959, family income was approximately \$4,500 compared to the 2009 median family income of \$68,410.³ Can you still afford the same things that people could afford back then? Business indexes are regularly used to understand the change over time in various quantities or how a quantity at one time point compares to the corresponding value at a certain reference time point.

For example, the next time you are offered a raise at work, an index can tell you whether you are truly receiving a raise or not! Assume that product prices have risen by 2% year-over-year. If you are offered a raise less than 2%, you have taken a wage cut and have less purchasing power than before. A raise of 2% allows you to at least break even, and anything above 2% increases your purchasing power.

Whether you are pursuing a career in human resources, economics, or some other field, the issues of gross earnings, income taxes, and indexes remain important both personally and professionally. This chapter examines some of the basic mathematical concepts involved in these calculations.

Outline of Chapter Topics

- 4.1: Gross Earnings (Off to Work You Go)
- 4.2: Personal Income Tax (The Taxman Taketh)
- 4.3: Indexes (The Times Are Changing)

¹ Steven Van Alstine, Vice-President of Education, the Canadian Payroll Association.

² Mie-Yun Lee, "Outsource Your Payroll," *Entrepreneur*. www.entrepreneur.com/humanresources/article47340.html (accessed November 29, 2009).

³ Statistics Canada, "Median Total Income, by Family Type, by Province and Territory," CANSIM table 111-0009, <http://www40.statcan.ca/l01/cst01/famil108a-eng.htm> (accessed October 19, 2010).

4.1: Gross Earnings

(Off to Work You Go)

You work hard at your job, and you want to be compensated properly for all the hours you put in. Assume you work full time with an hourly rate of pay of \$10. Last week you worked eight hours on Sunday and eight hours on Monday, which was a statutory holiday. Then you took Tuesday off, worked eight hours on each of Wednesday and Thursday, took Friday off, and worked 10 hours on Saturday. That's a total of 42 hours of work for the week. What is your gross pay? Give or take a small amount depending on provincial employment standards, it should be about \$570. But if you don't understand how to calculate gross earnings, you could be underpaid without ever realizing it.

Here are some notes about the content in this chapter: About 10% of Canadian workers fall under federal employment standards, which are not discussed here. This textbook generalizes the most common provincial employment standards; however, to calculate your earnings accurately requires you to apply your own provincial employment standards legislation. Part-time employment laws are extremely complex, so this textbook assumes in all examples that the employee is full time.

This section addresses the calculation of **gross earnings**, which is the amount of money earned before any deductions from your paycheque. The four most common methods of employee remuneration include salaries, hourly wages, commissions, and piecework wages.

Salary and Hourly Wages

One ad in the employment classifieds indicates compensation of \$1,270 biweekly, while a similar competing ad promotes wages of \$1,400 semi-monthly. If both job ads are similar in every other way, which job has the higher annual gross earnings? To make this assessment, you must understand how salaries work. A **salary** is a fixed compensation paid to a person on a regular basis for services rendered. Most employers pay employees by salary in occupations where the employee's work schedule generally remains constant.

In contrast, an **hourly wage** is a variable compensation based on the time an employee has worked. In contrast to a salary, this form of compensation generally appears in occupations where the number of hours is unpredictable or continually varies from period to period.

Employment Contract Characteristics⁴

Salaried and hourly full-time employees are similar with regard to their gross earnings. The major earnings issues in an employment contract involve regular earnings structure, overtime, and holidays.

Regular Earnings Structure

An agreement with your employer outlines the terms of your employment, including the time frame and frequency of pay.

1. **Time Frame.** For salaried employees, the time frame that the salary covers must be clearly stated. For example, you could receive a salary of \$2,000 monthly or \$50,000 annually. Notice that each of these salaries is followed by the specific time frame for the compensation. For hourly employees, the time frame requires identification of the wage earned per hour.
2. **Frequency.** How often the gross earnings are paid out to the employee must be defined.
 - a. **Monthly:** Earnings are paid once per month. By law, employees must receive compensation from their employer at least once per month, which equals 12 times per year.
 - b. **Daily:** Earnings are paid at the end of every day. This results in about 260 paydays per year (5 days per week multiplied by 52 weeks per year). In a leap year, there might be one additional payday.

⁴ Special thanks to Steven Van Alstine (CPM, CAE), Vice-President of Education, the Canadian Payroll Association, for assistance in summarizing Canadian payroll legislation and jurisdictions.

- c. **Weekly:** Earnings are paid once every week. This results in 52 paydays in any given year since there are 52 weeks per year.
- d. **Biweekly:** Earnings are paid once every two weeks. This results in 26 paydays in any given year since there are $52 \div 2 = 26$ biweekly periods per year.
- e. **Semi-monthly:** Earnings are paid twice a month, usually every half month (meaning on the 15th and last day of the month). This results in 24 paydays per year.

Thus, the earnings structure specifies both the time frame and the frequency of earnings. For a salaried employee, this may appear as “\$2,000 monthly paid semi-monthly” or “\$50,000 annually paid biweekly.” For an hourly employee, this may appear as “\$10 per hour paid weekly.” No matter whether you are salaried or hourly, earnings determined by your regular rate of pay are called your **regular earnings**.

Overtime

Overtime is work time in excess of your regular workday, regular workweek, or both. In most jurisdictions it is paid at 1.5 times your regular hourly rate (called *time-and-a-half*), though your company may voluntarily pay more or a union may have negotiated a more favourable rate such as two times your regular hourly rate (called *double time*). A contract with an employer will specify your regular workday and workweek, and some occupations are exempt from overtime. Due to the diversity of occupations, there is no set rule on what constitutes a regular workday or workweek. In most jurisdictions, a regular workweek is eight hours per day and 40 hours per week. Once you exceed these regular hours, you are eligible to receive **overtime or premium earnings**, which are based on your overtime rate of pay.

Holidays

A **statutory holiday** is a legislated day of rest with pay. Five statutory holidays are recognized throughout Canada, namely, New Year's Day, Good Friday (or Easter Monday in Quebec), Canada Day, Labour Day, and Christmas Day. Each province or territory has an additional four to six **public holidays** (or general holidays), which may include Family Day (known as Louis Riel Day in Manitoba and Islander Day in PEI) in February, Victoria Day in May, the Civic Holiday in August, Thanksgiving Day in October, Remembrance Day in November, and Boxing Day in December. These public holidays may or may not be treated the same as statutory holidays, depending on provincial laws.

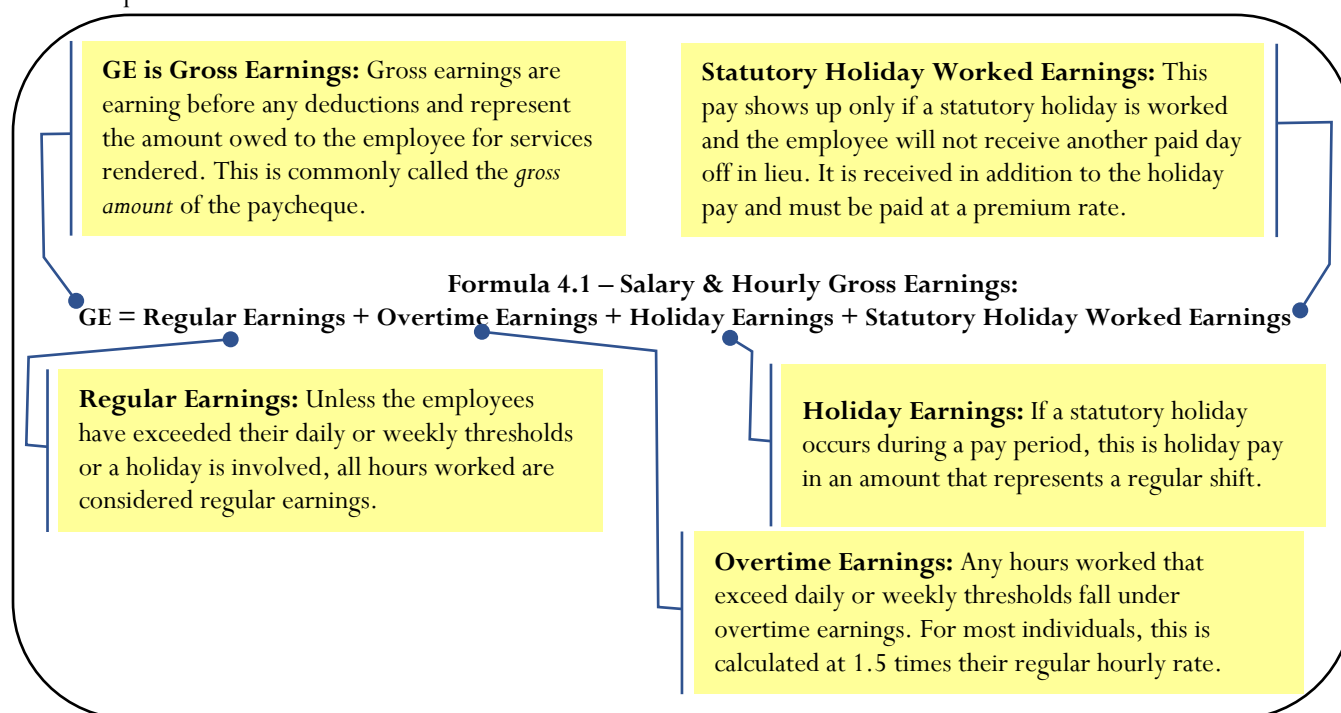
Statutory and public holidays generally require employees to receive the day off with pay. If the holiday falls on a nonworking day, it is usually the next working day that is given off instead. For example, if Christmas Day falls on a Saturday, typically the following Monday is given off with pay. Here's how holidays generally work (though you should always consult legislation for your specific jurisdiction):

- You should be given the day off with pay, called **holiday earnings**. The holiday earnings are in the amount of a regular day's earnings, and the hours involved count toward your weekly hourly totals for overtime purposes (preventing employers from shifting your work schedule that week).
- If you are required to work, the employer must offer another day off in lieu with pay. Your work on the statutory holiday is then paid at *regular earnings* and the hours involved contribute toward your weekly hourly totals for overtime purposes. You are paid holiday earnings on your future day off.
- If you are required to work and no day or rest is offered in lieu, this poses the most complex situation. Under these conditions:
 - You are entitled to the holiday earnings you normally would have received for the day off. The hours that make up your holiday earnings contribute toward your weekly hourly totals for overtime purposes (again, consult your local jurisdiction).
 - In addition, for the hours you worked on the statutory holiday you are entitled to overtime earnings known as **statutory holiday worked earnings**. These hours do not contribute toward your weekly hourly totals for overtime purposes since you are already compensated at a premium rate of pay. For example, assume you work eight hours on Labour Day, your normal day is eight hours, and you won't get another day off in lieu. Your employer owes you the eight hours of holiday earnings you should have received for getting the day off *plus* the eight hours of statutory holiday worked earnings for working on Labour Day.



The Formula

The four forms of compensation consist of regular earnings, overtime earnings, holiday earnings, and statutory holiday worked earnings. Add these four elements together to determine total gross earnings. Formula 4.1 shows the relationship.



How It Works

Salaried Employees

To calculate the total gross earnings for a salaried employee, follow these steps:

Step 1: Analyze the employee's work performed and assign hours as needed into each of the four categories of pay. If the employee has only regular hours of pay, skip to step 6.

Step 2: Calculate the employee's equivalent hourly rate of pay. This means converting the salary into an equivalent hourly rate:

$$\text{Equivalent Hourly Rate} = \frac{\text{Annual Salary}}{\text{Annual Hours Worked}}$$

For example, use a \$2,000 monthly salary requiring 40 hours of work per week. Express the salary annually by multiplying it by 12 months, yielding \$24,000. Express the 40 hours per week annually by multiplying by 52 weeks per year, yielding 2,080 hours worked. The equivalent hourly rate is $\$24,000 \div 2,080 = \$11.538461$.

Step 3: Calculate any holiday earnings. Take the unrounded hourly rate and multiply it by the number of hours in a regular shift, or

$$\text{Holiday Earnings} = \text{Unrounded Hourly Rate} \times \text{Hours in a Regular Shift}$$

A salaried employee earning \$11.538461 per hour having a daily eight-hour shift receives $\$11.538461 \times 8 = \$92.31$ in holiday earnings.

Step 4: Calculate any overtime earnings.

- Determine the overtime hourly rate of pay by multiplying the unrounded hourly rate by the minimum standard overtime factor of 1.5 (this could be higher if the company pays a better overtime rate than this):

$$\text{Overtime Hourly Rate} = \text{Unrounded Hourly Rate} \times 1.5$$

- b. Round the final result to two decimals. For the salaried worker, $\$11.538461 \times 1.5 = \$17.31$ per overtime hour.
- c. Multiply the overtime hourly rate by the overtime hours worked.

Step 5: Calculate any statutory holiday worked earnings which is similar to calculating overtime earnings:

$$\text{Statutory Holiday Worked Earnings} = \text{Statutory Hourly Rate} \times \text{Statutory Hours Worked}$$

The statutory hourly rate is at minimum 1.5 times the unrounded hourly rate of pay. The salaried employee working eight hours on a statutory holiday receives $\$17.31 \times 8 = \$138.48$.

Step 6: Calculate the gross earnings paid at the regular rate of pay. Take the amount of the salary and divide it by the number of pay periods involved, then subtract any holiday earnings:

$$\text{Regular Earnings} = \frac{\text{Salary}}{\text{Pay Periods}} - \text{Holiday Earnings}$$

You need to calculate the number of pay periods based on the regular earnings structure. For example, an annual \$52,000 salary paid biweekly would have 26 pay periods annually. Therefore, a regular paycheck is $\$52,000 \div 26 = \$2,000$ per paycheck. As another example, a \$2,000 monthly salary paid semi-monthly has two pay periods in a single month, resulting in regular earnings of $\$2,000 \div 2 = \$1,000$ per paycheck. If a holiday is involved in the pay period, you must deduct the holiday earnings from these amounts.

Step 7: Calculate the total gross earnings by applying Formula 4.1.

Hourly Employees

To calculate the total gross earnings for an hourly employee, follow steps similar to those for the salaried employee:

Step 1: Analyze the employee's work performed and assign hours as needed into each of the four categories of pay. It is usually best to set up a table similar to the one below. This table allows you to visualize the employee's week at a glance along with totals, enabling proper assessment of their hours worked.

This table separates the four types of earnings into different rows. Enter the information into the table about the employee's workweek, placing it in the correct day and on the correct row. If any daily thresholds are exceeded, place the appropriate hours into the overtime row. Once you have completed this, total the regular hours and holiday hours for the week and check to see if they exceed any regular weekly threshold. If so, starting with the last workday of the week and working backwards, convert regular hours into overtime hours until you have reduced the regular hours to the regular weekly total.

Type of Pay	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Total Hours	Rate of Pay	Earnings
Regular										
Holiday										
Overtime										
Holiday Worked										
									TOTAL EARNINGS	

Once you have completed the table, if the employee has only regular hours of pay, skip to step 5. Otherwise, proceed with the next step.

Step 2: Calculate any holiday earnings. Take the hourly rate and multiply it by the number of hours in a regular shift:

$$\text{Holiday Earnings} = \text{Hourly Rate} \times \text{Hours in a Regular Shift}$$

Step 3: Calculate any overtime earnings.

- a. Determine the overtime hourly rate of pay rounded to two decimals by multiplying the hourly rate by the minimum standard overtime factor of 1.5 (or higher).

$$\text{Overtime Hourly Rate} = \text{Hourly Rate} \times 1.5$$

- b. Multiply the overtime hourly rate by the overtime hours worked.

Step 4: Calculate any statutory holiday worked earnings. This is the same procedure as for a salaried employee.

Step 5: Calculate the gross earnings paid at the regular rate of pay. Take the number of hours worked and multiply it by the hourly rate of pay:

$$\text{Regular Earnings} = \text{Hours Worked} \times \text{Hourly Rate}$$

For example, 20 hours worked at \$10 per hour with no holiday earnings is $20 \times \$10 = \$200$.

Step 6: Calculate the total gross earnings by applying Formula 4.1.



Things To Watch Out For

Be careful about the language of the payment frequency. It is very common to confuse *semi* and *bi*, and sometimes businesses use the terms incorrectly. The term *semi* generally means half. Therefore, to be paid semi-monthly means to be paid every half month. The term *bi* means two. Therefore, to be paid biweekly means to be paid every two weeks. Some companies that pay semi-monthly mistakenly state that they pay bimonthly, which in fact would mean they paid every two months.



Paths To Success

In calculating the pay for a salaried employee, this textbook assumes for simplicity that a year has exactly 52 weeks. In reality, there are 52 weeks plus one day in any given year. In a leap year, there are 52 weeks plus two days. This extra day or two has no impact on semi-monthly or monthly pay, since there are always 24 semi-months and 12 months in every year. However, weekly and biweekly earners are impacted as follows:

- If employees are paid weekly, approximately once every six years there are 53 pay periods in a single year. This would "reduce" the employees' weekly paycheck in that year. For example, assume they earn \$52,000 per year paid weekly. Normally, they are paid $\$52,000 \div 52 = \$1,000$ per week. However, since there are 53 pay periods approximately every sixth year, this results in $\$52,000 \div 53 = \$981.13$ per week for that year.
- If employees are paid biweekly, approximately once every 12 years there are 27 pay periods in a single year. This has the same effect as the extra pay period above. For example, if they are paid \$52,000 per year biweekly they normally receive $\$52,000 \div 26 = \$2,000$ per biweekly cheque. Approximately every twelfth year, they are paid $\$52,000 \div 27 = \$1,925.93$ per biweekly cheque for that year.

Many employers ignore these technical nuances in pay structure since the extra costs incurred to modify payroll combined with the effort required to calm down employees who don't understand the smaller paycheck are not worth the savings in labour. Therefore, most employers treat every year as if it has 52 weeks (26 biweeks) regardless of the reality. In essence, employees receive a bonus paycheck approximately once every six or twelve years!



Give It Some Thought

A salaried employee whose normal workweek is 8 hours per day and 40 hours per week works 8 hours each day from Monday to Saturday inclusive, where Monday was a statutory holiday. Which of the following statements is correct (assuming she will not get another day off in lieu of the holiday)?

- The employee receives only her regular weekly earnings for 40 hours.
- The employee receives 32 hours of regular earnings, 8 hours of holiday earnings, 8 hours of overtime earnings, and 8 hours of statutory holiday worked earnings.
- The employee receives 40 hours of regular earnings, 8 hours of overtime earnings, and 8 hours of statutory holiday worked earnings.
- The employee receives 40 hours of regular earnings and 8 hours of overtime earnings.

Solutions: The correct answer is b. When working on the statutory holiday and not getting another day off in lieu, the salaried employee is eligible for eight hours of holiday earnings plus eight hours of statutory holiday worked earnings. The holiday earnings count toward the weekly total, but not the statutory holiday worked earnings. Thus the employee from Tuesday to Friday inclusive worked an additional 32 regular hours, bringing her weekly total to 40 hours. The work on Saturday exceeds her 40 hour workweek, and therefore all eight hours are paid as overtime earnings.

Example 4.1A: Salary with Overtime and a Holiday

Tristan is compensated with an annual salary of \$65,000 paid biweekly. His regular workweek consists of four 10-hour days, and he is eligible for overtime at 1.5 times pay for any work in excess of his regular requirements. Tristan worked regular hours for the first two weeks. Over the next two weeks, Tristan worked his regular hours and became eligible for 11 hours of overtime. During these two weeks, he worked his regular shift on Good Friday but his employer has agreed to give him another day off with pay in the future.

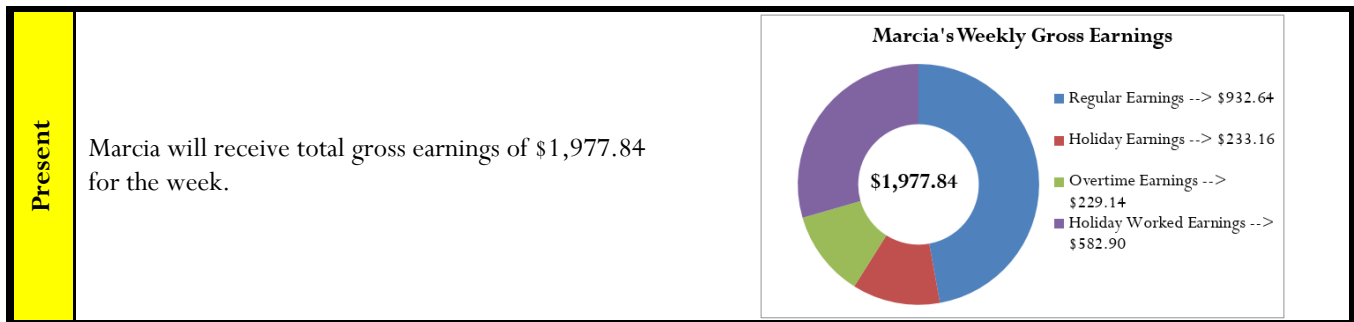
- Determine Tristan's gross earnings for the first two-week pay period.
- Determine Tristan's gross earnings for the second two-week pay period.

Plan	You have been asked to calculate Tristan's gross earnings, or GE, for two consecutive pay periods.																						
Understand	What You Already Know Step 1: You know Tristan's compensation: Annual Salary = \$65,000 Pay Periods = biweekly = 26 times per year Annual Hours = $10/\text{day} \times 4 \text{ days/week} \times 52 \text{ weeks} = 2,080$ You also know his work schedule: First Two Weeks = regular pay Overtime in First Two Weeks = 0 Second Two Weeks = regular pay Overtime in Second Two Weeks = 11 hours There is a holiday in the second two weeks, but he will receive another day off in lieu.	How You Will Get There For each biweekly pay period, apply the following steps: Step 2: Calculate Tristan's equivalent hourly rate of pay. Step 3: Calculate holiday earnings using the equivalent hourly rate. Step 4: Calculate overtime earnings by taking the overtime hourly pay rate multiplied by hours worked. Step 5: Calculate statutory holiday worked earnings at the premium rate of pay. Step 6: Calculate regular earnings: Step 7: Determine total gross earnings using Formula 4.1.																					
Perform	<table> <tr> <th>Step</th><th>First Two Weeks</th><th>Second Two Weeks</th></tr> <tr> <td>2.</td><td>There are only regular earnings; skip to step 6.</td><td>Equivalent Hourly Rate = $\frac{\\$65,000}{2,080} = \\$31.25$</td></tr> <tr> <td>3.</td><td>Holiday earnings = \$0</td><td>Tristan will take his holiday pay another day. Therefore, he is not eligible for this pay, and his work counts as regular hours; holiday earnings = \$0</td></tr> <tr> <td>4.</td><td>Overtime earnings = \$0</td><td>Overtime hourly rate = $\\$31.25 \times 1.5 = \\$46.88$ Overtime Earnings = $\\$46.88 \times 11 = \\$515.68$</td></tr> <tr> <td>5.</td><td>Statutory Worked = \$0</td><td>Since Tristan is receiving another day off in lieu, he is not eligible for this pay; statutory Worked = \$0</td></tr> <tr> <td>6.</td><td>Regular Earnings = $\frac{\\$65,000}{26} - \\$0 = \\$2,500$</td><td>Regular Earnings = $\frac{\\$65,000}{26} - \\$0 = \\$2,500$</td></tr> <tr> <td>7.</td><td>GE = $\\$2,500 + \\$0 + \\$0 + \\$0 = \\$2,500$</td><td>GE = $\\$2,500 + \\$515.68 + \\$0 + \\$0 = \\$3,015.68$</td></tr> </table>	Step	First Two Weeks	Second Two Weeks	2.	There are only regular earnings; skip to step 6.	Equivalent Hourly Rate = $\frac{\$65,000}{2,080} = \$31.25$	3.	Holiday earnings = \$0	Tristan will take his holiday pay another day. Therefore, he is not eligible for this pay, and his work counts as regular hours; holiday earnings = \$0	4.	Overtime earnings = \$0	Overtime hourly rate = $\$31.25 \times 1.5 = \$46.88$ Overtime Earnings = $\$46.88 \times 11 = \$515.68$	5.	Statutory Worked = \$0	Since Tristan is receiving another day off in lieu, he is not eligible for this pay; statutory Worked = \$0	6.	Regular Earnings = $\frac{\$65,000}{26} - \$0 = \$2,500$	Regular Earnings = $\frac{\$65,000}{26} - \$0 = \$2,500$	7.	GE = $\$2,500 + \$0 + \$0 + \$0 = \$2,500$	GE = $\$2,500 + \$515.68 + \$0 + \$0 = \$3,015.68$	
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Present	For the first two-week pay period, Tristan worked only his regular hours and therefore is compensated \$2,500 as per his salary. For the second two-week pay period, Tristan is eligible to receive his regular hours plus his overtime, but he receives no additional pay for the worked holiday since he will receive another day off in lieu. His total gross earnings are \$3,015.68.																						

Example 4.1B: A Week in the Life of an Hourly Wage Earner

Marcia receives an hourly wage of \$32.16 working on an automotive production line. Her union has negotiated a regular work day of 7.25 hours for five days totalling 36.25 hours for the week. Overtime is paid at 1.5 times her regular rate for any work that exceeds the daily or weekly limits. If work is required on a statutory holiday, her company does not give a day off in lieu and pays a premium rate of 2.5 times her regular rate. Last week, Marcia worked nine hours on Monday, her regular hours on Tuesday through Friday inclusive, and three hours on Saturday. Friday was a statutory holiday. Calculate Marcia's gross earnings for the week.

Plan	You need to calculate Marcia's gross earnings, or GE, for the week.																								
Understand	What You Already Know Step 1: You know Marcia's pay structure and workweek: Regular Hourly Rate = \$32.16 Overtime Hourly Pay = 1.5 times Statutory Holiday Worked Rate = 2.5 times Exceeding 7.25 hours daily or 36.25 hours weekly is overtime.							How You Will Get There Step 1 (continued): Take Marcia's hours and place them into the table. Assess whether any daily or weekly totals are considered overtime and make any necessary adjustments. Step 2: Calculate holiday earnings using the hourly rate. Step 3: Calculate overtime earnings by determining the overtime hourly rate of pay multiplied by hours worked. Step 4: Calculate statutory holiday worked earnings at the premium rate of pay. Step 5: Calculate regular earnings. Step 6: Determine total gross earnings.																	
	<table><tr><td>Sunday</td><td>0</td></tr><tr><td>Monday</td><td>9</td></tr><tr><td>Tuesday</td><td>7.25</td></tr><tr><td>Wednesday</td><td>7.25</td></tr><tr><td>Thursday</td><td>7.25</td></tr><tr><td>Friday (statutory holiday)</td><td>7.25</td></tr><tr><td>Saturday</td><td>3</td></tr></table>							Sunday	0	Monday	9	Tuesday	7.25	Wednesday	7.25	Thursday	7.25	Friday (statutory holiday)	7.25	Saturday	3				
	Sunday	0																							
	Monday	9																							
	Tuesday	7.25																							
	Wednesday	7.25																							
	Thursday	7.25																							
Friday (statutory holiday)	7.25																								
Saturday	3																								
Perform	Step 1:																								
	Type	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total	Rate	Earnings														
	Regular	0	①7.25	7.25	7.25	7.25		3	③39.25	\$32.16															
	Holiday						②7.25																		
	Overtime		①1.75						1.75																
	Holiday Worked						②7.25		7.25																
									TOTAL EARNINGS																
	① She worked nine hours. Therefore, the first 7.25 hours are regular pay and the last 1.75 are overtime pay.																								
	② Friday was a statutory holiday, and she will not receive another day off in lieu. She must receive statutory holiday worked pay in addition to her hours worked.																								
	③ Note the weekly total of 36.25 has been exceeded by three hours. Move Saturday's hours into overtime.																								
The following table is the final layout of her workweek:																									
Type	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total	Rate	Earnings															
Regular	0	7.25	7.25	7.25	7.25			36.25	\$32.16																
Holiday						7.25																			
Overtime		1.75					3	4.75																	
Holiday Worked						7.25		7.25																	
								TOTAL EARNINGS																	
Steps 2–6: Explanations are noted in the table below.																									
Type	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total	Rate	Earnings															
Regular	0	7.25	7.25	7.25	7.25			36.25	\$32.16	④\$932.64															
Holiday						7.25				① \$233.16															
Overtime		1.75					3	4.75	②\$48.24	② \$229.14															
Holiday Worked						7.25		7.25	③\$80.40	③ \$582.90															
								TOTAL EARNINGS		⑤ \$1,977.84															
① Holiday Earnings = 7.25 × \$32.16 = \$233.16																									
② Overtime Hourly Rate = \$32.16 × 1.5 = \$48.24; Overtime Earnings = 4.75 × \$48.24 = \$229.14																									
③ Statutory Holiday Worked Rate = \$32.16 × 2.5 = \$80.40; Statutory Worked Earnings = 7.25 × \$80.40 = \$582.90																									
④ Regular Earnings = 29 × \$32.16 = \$932.64																									
⑤ GE = \$932.64 + \$229.14 + \$233.16 + \$582.90 = \$1,977.84																									



Commission

Over the last two weeks you sold \$50,000 worth of machinery as a sales representative for an office solution company. The company's compensation plan involves a straight commission rate of 3.5%. What are your gross earnings? If you sold an additional \$12,000 in machinery, how much more would you earn?

Particularly in the fields of marketing and customer service, many workers are paid on a commission basis. A **commission** is an amount or a fee paid to an employee for performing or completing some form of transaction. The commission typically takes the form of a percentage of the dollar amount of the transaction. Marketing and customer service industries use this form of compensation as an incentive to perform: If the representative doesn't sell anything then the representative does not get paid. Issues to be discussed about commission include what constitutes regular earnings, how to handle holidays and overtime, and the three different types of commission structures.

- **Regular Earnings.** All commissions are considered to be regular earnings. To calculate the gross earnings for an employee, take the total amount of the transactions and multiply it by the commission rate:

$$\text{Gross Earnings} = \text{Total Transaction Amount} \times \text{Commission Rate}$$

This is not a new formula. It is a specific application of Formula 2.2: Rate, Portion, Base. In this case, the Base is the total amount of the transactions, the Rate is the commission rate, and the Portion is the gross earnings for the employee.

- **Holidays and Overtime.** Commission earners are eligible to receive overtime earnings, holiday earnings, and statutory holiday worked earnings. However, the provincial standards on these matters vary widely and the mathematics involved do not necessarily follow any one procedure or calculation. As such, this textbook leaves these issues to be covered in a payroll administration course.
- **Types of Commission.** Commission earnings typically follow one of the following three structures:
 - **Straight Commission.** If your entire earnings are based on your dollar transactions and calculated strictly as a percentage of the total, you are on **straight commission**. An application of Formula 2.2 (Rate, Portion, Base) calculates your gross earnings under this structure.
 - **Graduated Commission.** Within a **graduated commission** structure, you are offered increasing rates of commission for higher levels of performance. The theory behind this method of compensation is that the higher rewards motivate employees to perform better. An example of a graduated commission scale is found in the table below.

Transaction Level	Commission Rate
\$0–\$100,000.00	3%
\$100,000.01–\$250,000.00	4.5%
\$250,000.01–\$500,000.00	6%
Over \$500,000.00	7.5%

Recognize that the commission rates are applied against the portion of the sales falling strictly into the particular category, not the entire balance. Thus, if the total sales equal \$150,000, then the first \$100,000 is paid at 3% while the next \$50,000 is paid at 4.5%.

- **Salary Plus Commission.** If your earnings combine a basic salary together with commissions on your dollar transactions, you have a **salary plus commission** structure. No new mathematics are required for this commission type. You must combine salary calculations, discussed earlier in this section, with either a straight commission or

graduated commission, as discussed above. Usually this form of compensation pays the lowest commission rate since a basic salary is already provided.



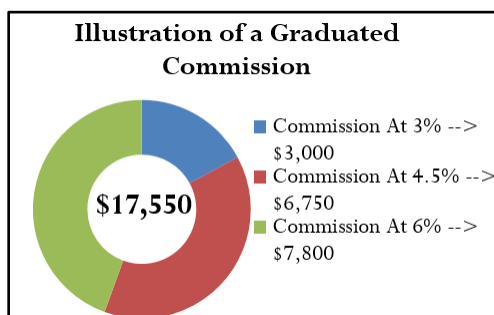
How It Works

Follow these steps to calculate commission earnings:

Step 1: Determine which commission structure is used to pay the employee. Identify information on commission rates, graduated scales, and any salary.

Step 2: Determine the dollar amounts that are eligible for any particular commission rate and calculate commissions.

Step 3: Sum all earnings from every eligible commission rate plus any salary.



Let's assume \$380,000 of merchandise is sold. Using the previous table as our graduated commission scale, calculate commission earnings.

Step 1: Sales total \$380,000 and all commission rates and scales are found in the table.

Step 2: The first \$100,000 is compensated at 3%, equalling \$3,000. The next \$150,000 is compensated at 4.5%, equalling \$6,750. The last \$130,000 is compensated at 6%, equalling \$7,800. There is no compensation at the 7.5% level since sales did not exceed \$500,000.

Step 3: The total commission on sales of \$380,000 is $\$3,000 + \$6,750 + \$7,800 = \$17,550$.

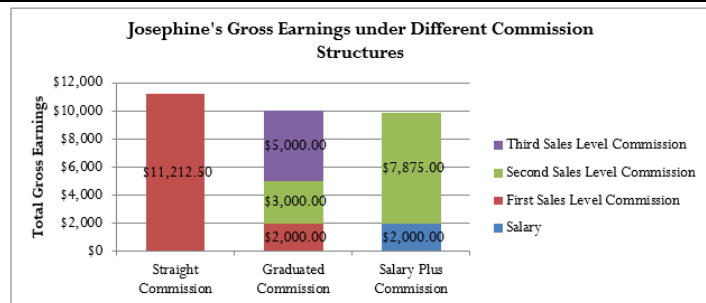
Example 4.1C: Different Types of Commissions

Josephine is a sales representative for a food company. Over the past two weeks, she closed \$325,000 in retail distribution contracts. Calculate the total gross earnings that Josephine earns if

- She is paid a straight commission of 3.45%.
- She is paid 2% for sales on the first \$100,000, 3% on the next \$100,000, and 4% on all remaining sales.
- She is paid a base salary of \$2,000 plus a commission of 3.5% on all sales above \$100,000.

Plan	You need to calculate the gross earnings, or GE, for Josephine under various commission structures.															
Understand	What You Already Know Step 1: For all three options, total sales = \$325,000 a. This is a straight commission where the commission rate = 3.45%. b. This is a graduated commission, structured as follows: <table><tr><th>Sales Level</th><th>Rate</th></tr><tr><td>\$0–\$100,000.00</td><td>2%</td></tr><tr><td>\$100,000.01–\$200,000.00</td><td>3%</td></tr><tr><td>\$200,000.01 and above</td><td>4%</td></tr></table> c. This is a salary plus graduated commission, with salary = \$2,000 and a graduated commission structure as follows: <table><tr><th>Sales Level</th><th>Rate</th></tr><tr><td>\$0–\$100,000.00</td><td>0%</td></tr><tr><td>\$100,000.01 and above</td><td>3.5%</td></tr></table>	Sales Level	Rate	\$0–\$100,000.00	2%	\$100,000.01–\$200,000.00	3%	\$200,000.01 and above	4%	Sales Level	Rate	\$0–\$100,000.00	0%	\$100,000.01 and above	3.5%	How You Will Get There Step 2: Determine the sales eligible for each commission and calculate total commissions. Note that all calculations are $\text{Portion} = \text{Rate} \times \text{Base}$. Step 3: Sum all commissions plus any salary to calculate total gross earnings.
	Sales Level	Rate														
	\$0–\$100,000.00	2%														
\$100,000.01–\$200,000.00	3%															
\$200,000.01 and above	4%															
Sales Level	Rate															
\$0–\$100,000.00	0%															
\$100,000.01 and above	3.5%															

Perform	Steps 2 & 3:																									
	a. $3.45\% \times \$325,000 = 0.0345 \times \$325,000 = \$11,212.50 = \text{Total Gross Earnings}$																									
	b.																									
	<table border="1"> <tr> <th>Sales Level</th><th>Rate</th><th>Eligible Sales at Each Rate</th><th>Commission Earned</th></tr> <tr> <td>\$0–\$100,000.00</td><td>2%</td><td>$\\$100,000 - \\$0 = \\$100,000$</td><td>$0.02 \times \\$100,000 = \\$2,000$</td></tr> <tr> <td>\$100,000.01–\$200,000.00</td><td>3%</td><td>$\\$200,000 - \\$100,000 = \\$100,000$</td><td>$0.03 \times \\$100,000 = \\$3,000$</td></tr> <tr> <td>\$200,000.01 and above</td><td>4%</td><td>$\\$325,000 - \\$200,000 = \\$125,000$</td><td>$0.04 \times \\$125,000 = \\$5,000$</td></tr> </table>	Sales Level	Rate	Eligible Sales at Each Rate	Commission Earned	\$0–\$100,000.00	2%	$\$100,000 - \$0 = \$100,000$	$0.02 \times \$100,000 = \$2,000$	\$100,000.01–\$200,000.00	3%	$\$200,000 - \$100,000 = \$100,000$	$0.03 \times \$100,000 = \$3,000$	\$200,000.01 and above	4%	$\$325,000 - \$200,000 = \$125,000$	$0.04 \times \$125,000 = \$5,000$									
Sales Level	Rate	Eligible Sales at Each Rate	Commission Earned																							
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\$200,000.01 and above	4%	$\$325,000 - \$200,000 = \$125,000$	$0.04 \times \$125,000 = \$5,000$																							
	Total Gross Earnings = $\$2,000 + \$3,000 + \$5,000 = \$10,000$																									
	c. Recall Base Salary = \$2,000																									
	<table border="1"> <tr> <th>Sales Level</th><th>Rate</th><th>Base</th><th>Commission Earned</th></tr> <tr> <td>\$0–\$100,000.00</td><td>0%</td><td>$\\$100,000 - \\$0 = \\$100,000$</td><td>$0 \times \\$100,000 = \\$0$</td></tr> <tr> <td>\$100,000.01 and above</td><td>3.5%</td><td>$\\$325,000 - \\$100,000 = \\$225,000$</td><td>$0.035 \times \\$225,000 = \\$7,875$</td></tr> </table>	Sales Level	Rate	Base	Commission Earned	\$0–\$100,000.00	0%	$\$100,000 - \$0 = \$100,000$	$0 \times \$100,000 = \$0$	\$100,000.01 and above	3.5%	$\$325,000 - \$100,000 = \$225,000$	$0.035 \times \$225,000 = \$7,875$													
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	Total Gross Earnings = $\$0 + \$7,875 + \$2,000 = \$9,875$																									
Present	If Josephine is paid under straight commission, her total gross earnings will be \$11,212.50. Under the graduated commission, she will receive \$10,000 in total gross earnings. For the salary plus commission, she will receive \$9,875 in total gross earnings.	Josephine's Gross Earnings under Different Commission Structures <table border="1"> <caption>Josephine's Gross Earnings Breakdown</caption> <thead> <tr> <th>Commission Structure</th> <th>Salary</th> <th>First Sales Level Commission</th> <th>Second Sales Level Commission</th> <th>Third Sales Level Commission</th> <th>Total Gross Earnings</th> </tr> </thead> <tbody> <tr> <td>Straight Commission</td> <td>\$0</td> <td>\$11,212.50</td> <td>\$0</td> <td>\$0</td> <td>\$11,212.50</td> </tr> <tr> <td>Graduated Commission</td> <td>\$0</td> <td>\$2,000.00</td> <td>\$3,000.00</td> <td>\$5,000.00</td> <td>\$10,000.00</td> </tr> <tr> <td>Salary Plus Commission</td> <td>\$2,000.00</td> <td>\$0</td> <td>\$7,875.00</td> <td>\$0</td> <td>\$9,875.00</td> </tr> </tbody> </table>	Commission Structure	Salary	First Sales Level Commission	Second Sales Level Commission	Third Sales Level Commission	Total Gross Earnings	Straight Commission	\$0	\$11,212.50	\$0	\$0	\$11,212.50	Graduated Commission	\$0	\$2,000.00	\$3,000.00	\$5,000.00	\$10,000.00	Salary Plus Commission	\$2,000.00	\$0	\$7,875.00	\$0	\$9,875.00
	Commission Structure	Salary	First Sales Level Commission	Second Sales Level Commission	Third Sales Level Commission	Total Gross Earnings																				
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Salary Plus Commission	\$2,000.00	\$0	\$7,875.00	\$0	\$9,875.00																					



Piecework

Have you ever heard the phrase "pay-for-performance"? Although this phrase has many interpretations in different industries, for some people this phrase means that they get paid based on the quantity of work that they do. For example, many workers in clothing manufacturing are paid a flat rate for each article of clothing they produce. As another example, employees in fruit orchards may get paid by the number of pieces of fruit that they harvest, or simply by the kilogram. As you can see, these workers are neither salaried nor paid hourly, nor are they on commission. They earn their paycheck for performing a specific task. Therefore, a **piecework** wage compensates such employees on a per-unit basis.

This section focuses on the regular earnings only for piecework wage earners. Similar to workers on commission, piecework earners are eligible to receive overtime earnings, holiday earnings, and statutory holiday worked earnings. However, the standards vary widely from province to province, and there is not necessarily any one formula to calculate these earnings. As with commissions, this textbook leaves those calculations for a payroll administration course.



The Formula

To calculate the regular gross earnings for a worker paid on a piecework wage, you require the piecework rate and how many units they are to be paid for:

$$\text{Gross Earnings} = \text{Piecework Rate} \times \text{Eligible Units}$$

This is not a new formula but another application of Formula 2.2 (Rate, Portion, Base). The Piecework Rate is the Rate, the Eligible Units are the Base, and the Gross Earnings are the Portion.



How It Works

To calculate an employee's gross earnings for piecework, follow these steps:

Step 1: Identify the piecework rate and the level of production or units.

Step 2: Perform any necessary modifications on the production or units to match how the piecework is paid.

Step 3: Calculate the commission gross earnings by multiplying the rate by the eligible units.

Assume that Juanita is a piecework earner at a blue jean manufacturer. She is paid daily and earns \$1.25 for every pair of jeans that she sews. On a given day, Juanita sewed 93 pairs of jeans. Her gross earnings are calculated as follows:

Step 1: Her piecework rate = \$1.25 per pair with production of 93 units.

Step 2: The rate and production are both expressed by the pair of jeans. No modification is necessary.

Step 3: Her gross piecework earnings are the product of the rate and units produced, or $\$1.25 \times 93 = \$116.25$.



Things To Watch Out For

Pay careful attention to step 2 in the procedure. In some industries, the piecework rate and the units of production do not match. For example, a company could pay a piecework rate per kilogram, but a single unit may not represent a kilogram. This is typical in some canning industries, where workers are paid per kilogram for canning the products, but the cans may only be 200 grams in size. Therefore, if workers produce five cans, they are not paid for five units produced. Rather, they are paid for only one unit produced since $5 \text{ cans} \times 200 \text{ g} = 1,000 \text{ g} = 1 \text{ kg}$. Before calculating piecework earnings, ensure that both the piecework rate and the eligible units are in the same terms, whether it be metric tonnes, kilograms, or otherwise.

Example 4.1D: A Telemarketer Earning Piecework Wages

In outbound telemarketing, some telemarketers are paid on the basis of "completed calls." This is not commission since their pay is not based on actually selling anything. Rather, a completed call is defined as simply any phone call for which the agent speaks with the customer and a decision is reached, regardless of whether the decision was to accept, reject, or request further information. If a telemarketer produces five completed calls per hour and works 7½-hour shifts five times per week, what are the total gross earnings she can earn over a biweekly pay period if her piecework wage is \$3.25 per completed call?

Plan	We are looking for the total gross earnings, or GE, for the telemarketer over the biweekly pay period.	
Understand	What You Already Know Step 1: The frequency of the telemarketer's pay, along with her hours of work, piecework wage, and unit of production are known: Piecework Rate = \$3.25 per completed call Hourly Units Produced = 5 Hours of Work = 7½ hours per day, five days per week Frequency of Pay = biweekly	How You Will Get There Step 2: You must determine the telemarketer's production level. Calculate how many completed calls she achieves per biweekly pay period: $\text{Eligible Units} = \text{Units Produced per Hour} \times \text{Hours per Day} \times \text{Days per Week} \times \text{Weeks}$ Step 3: Apply Formula 2.2 (adapted for piecework wages) to get the portion owing.
Perform	Step 2: Eligible Units = $1.5 \times 7.5 \times 5 \times 2 = 375$ Step 3: GE = $\$3.25 \times 375 = \$1,218.75$	
Present	Over a biweekly period, the telemarketer completes 375 calls. At her piecework wage, this results in total gross earnings of \$1,218.75.	



Section 4.1 Exercises

Mechanics

- Laars earns an annual salary of \$60,000. Determine his gross earnings per pay period under each of the following payment frequencies:
a. Monthly b. Semi-monthly c. Biweekly d. Weekly
- A worker earning \$13.66 per hour works 47 hours in the first week and 42 hours in the second week. What are his total biweekly earnings if his regular workweek is 40 hours and all overtime is paid at 1.5 times his regular hourly rate?
- Marley is an independent sales agent. He receives a straight commission of 15% on all sales from his suppliers. If Marley averages semi-monthly sales of \$16,000, what are his total annual gross earnings?
- Sheila is a life insurance agent. Her company pays her based on the annual premiums of the customers that purchase life insurance policies. In the last month, Sheila's new customers purchased policies worth \$35,550 annually. If she receives 10% commission on the first \$10,000 of premiums and 20% on the rest, what are her total gross earnings for the month?
- Tuan is a telemarketer who earns \$9.00 per hour plus 3.25% on any sales above \$1,000 in any given week. If Tuan works 35 regular hours and sells \$5,715, what are his gross earnings for the week?
- Adolfo packs fruit in cans on a production line. He is paid a minimum wage of \$9.10 per hour and earns \$0.09 for every can packed. If Adolfo manages to average 160 cans per hour, what are his total gross earnings daily for an eight-hour shift?

Applications

- Charles earns an annual salary of \$72,100 paid biweekly based on a regular workweek of 36.25 hours. His company generously pays all overtime at twice his regular wage. If Charles worked 85.5 hours over the course of two weeks, what are his gross earnings?
- Armin is the payroll administrator for his company. In looking over the payroll, he notices the following workweek (from Sunday to Saturday) for one of the company's employees: 0, 6, 8, 10, 9, 8, and 9 hours, respectively. Monday was a statutory holiday, and with business booming the employee will not be given another day off in lieu. Company policy pays all overtime at time-and-a-half, and all hours worked on a statutory holiday are paid at twice the regular rate. A normal workweek consists of five, eight-hour days. If the employee receives \$22.20 per hour, what are her total weekly gross earnings?
- In order to motivate a manufacturer's agent to increase his sales, a manufacturer offers monthly commissions of 1.2% on the first \$125,000, 1.6% on the next \$150,000, 2.25% on the next \$125,000, and 3.75% on anything above. If the agent managed to sell \$732,000 in a single month, what commission is he owed?
- Humphrey and Charlotte are both sales representatives for a pharmaceutical company. In a single month, Humphrey received \$5,545 in total gross earnings while Charlotte received \$6,388 in total gross earnings. In sales dollars, how much more did Charlotte sell if they both received 5% straight commission on their sales?
- Mayabel is a cherry picker working in the Okanagan Valley. She can pick 17 kg of cherries every hour. The cherries are placed in pails that can hold 13.6 kg of cherries. If she works 40 hours in a single week, what are her total gross earnings if her piecework rate is \$17.00 per pail?
- Miranda is considering three relatively equal job offers and wants to pick the one with the highest gross earnings. The first job is offering a base salary of \$1,200 semi-monthly plus 2% commission on monthly sales. The second job offer consists of a 9.75% straight commission. Her final job offer consists of monthly salary of \$1,620 plus 2.25% commission on her first \$10,000 in monthly sales and 6% on any monthly sales above that amount. From industry publications, she knows that a typical worker can sell \$35,000 per month. Which job offer should she choose, and how much better is it than the other job offers?

13. A Canadian travel agent is paid a flat rate of \$37.50 for every vacation booked through a certain airline. If the vacation is in North America, the agent also receives a commission of 2.45%. If the vacation is international, the commission is 4.68%. What are the total monthly gross earnings for the agent if she booked 29 North American vacations worth \$53,125 and 17 international vacations worth \$61,460?
14. Vladimir's employer has just been purchased by another organization. In the past, he has earned \$17.90 per hour and had a normal workweek of 37.5 hours. However, his new company only pays its employees a salary semi-monthly. How much does Vladimir need to earn each paycheck to be in the same financial position?

Challenge, Critical Thinking, & Other Applications

15. An employee on salary just received his biweekly paycheck in the amount of \$1,832.05, which included pay for five hours of overtime at time-and-a-half. If a normal workweek is 40 hours, what is the employee's annual salary?
16. A graduated commission scale pays 1.5% on the first \$50,000, 2.5% on the next \$75,000, and 3.5% on anything above. What level of sales would it take for an employee to receive total gross earnings of \$4,130?
17. A sales organization pays a base commission on the first \$75,000 in sales, base + 2% on the next \$75,000 in sales, and base + 4% on anything above. What is the base commission if an employee received total gross earnings of \$7,500 on \$200,000 in sales?
18. A typical sales agent for a company has annual sales of \$4,560,000, equally spread throughout the year, and receives a straight commission of 2%. As the new human resource specialist, to improve employee morale you have been assigned the task of developing different pay options of equivalent value to offer to the employees. Your first option is to pay them a base salary of \$2,000 per month plus commission. Your second option is to pay a base commission monthly on their first \$200,000 in sales, and a base + 2.01% on anything over \$200,000 per month. In order to equate all the plans, determine the required commission rates, rounded to two decimals in percent format, in both options.
19. Shaquille earns an annual salary of \$28,840.50 paid biweekly. His normal workweek is 36.25 hours and overtime is paid at twice the regular rate. In addition, he is paid a commission of 3% of sales on the first \$25,000 and 4% on sales above that amount. What are his total gross earnings during a pay period if he worked 86 hours and had sales of \$51,750?
20. Mandy is paid \$9.50 per hour and also receives a piecework wage of \$0.30 per kilogram, or portion thereof. A regular workday is 7.5 hours and 37.5 hours per week. Overtime is paid at time-and-a-half, and any work on a statutory holiday is paid at twice the regular rate. There is no premium piecework wage. Mandy's work record for a two-week period is listed below. Determine her total gross earnings.

Week		Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
1	Hours worked	7.5	7.5	9	8	7.5	3
	250 g items produced	1,100	1,075	1,225	1,150	1,025	450
2	Hours worked	4 Statutory holiday, no day off in lieu	7.5	10	7.5	8	
	250 g items produced	575	1,060	1,415	1,115	1,180	

4.2: Personal Income Tax

(The Taxman Taketh)

You have just filled out your income tax return, where you see that your total payable for the year is more than \$5,000. You are glad you do not have to pay this lump sum all at once! Instead, almost all of it has actually been paid already: Your employer has made regular deductions from each paycheck to cover your federal and provincial taxes.

In Canada, businesses are taxed differently than individuals and there are many nuances and complexities to address. For these reasons, this textbook does not cover business taxes, which require a course in corporate taxes.

For individuals, gross personal income taxes are relatively straightforward. This section of the book uses the 2021 tax year federal and provincial/territorial personal income tax brackets to calculate the amount of gross taxes that are ultimately deducted from your gross earnings. You can find current-year tax brackets at <https://www.canada.ca/en/revenue-agency/services/tax/individuals/frequently-asked-questions-individuals/canadian-income-tax-rates-individuals-current-previous-years.html>. Tax credits and other various deductions are too complex and not discussed or calculated.

Federal and Provincial/Territorial Tax Brackets and Tax Rates

Personal income tax is a tax on gross earnings levied by both the federal and provincial/territorial governments. All of Canada uses a **progressive tax system** in which the tax rate increases as the amount of income increases; however, the increased tax rates apply only to income amounts above minimum thresholds. Thus, higher income earners pay higher marginal tax rates than lower income earners. The format is similar to a graduated commission structure.

The federal and provincial/territorial governments offer a **basic personal amount**, which is the amount of income for which the wage earner is granted a tax exemption. In other words, it is tax-free income. This is designed to help low-income earners in Canada. Tax brackets are adjusted annually federally and in most provinces and territories to reflect increases in the cost of living, as measured by the consumer price index (CPI).



The Formula

To calculate the gross income taxes for any individual before tax credits and deductions, you must total all taxes from all eligible gross earnings in every tax bracket. Formula 4.2 shows the calculation.

Income Tax: The total of all tax portions in all tax brackets represents the amount of taxes that must be deducted from annual gross earnings and remitted to the appropriate government(s). You must complete these calculations both federally and provincially to determine the total annual income taxes. The formula is an adaptation of Formula 2.2: Rate, Portion, and Base, where:

- Each eligible income in any tax bracket is the base.
- Each associated tax bracket rate is the rate.

Multiplying each base by each rate produces the portion of eligible gross earnings owed in taxes.

Formula 4.2 – Annual Income Tax:

$$\text{Income Tax} = \sum (\text{Eligible Income in Tax Bracket} \times \text{Tax Bracket Rate})$$

Eligible Income in Tax Bracket: All earnings fall into tax brackets, as listed in the two tables above. Therefore, you must separate gross earnings into the different tax brackets. The amount of gross earnings in any tax bracket is then taxed at the associated tax rate.

Tax Bracket Rate: Every tax bracket has an associated tax rate as listed in the two tables above. This rate increases as the gross earnings increase. Multiply the gross earnings in any tax bracket by this rate to calculate the portion of the earnings that are owed in taxes.

2021 Provincial & Territory Income Taxes

Prov. Or Terr.	Taxable Income	Tax Rate	Prov. Or Terr.	Taxable Income	Tax Rate
British Columbia	\$0.00–\$11,070.00*	0%	Saskatchewan	\$0.00–\$16,225.00*	0%
	\$11,070.01–\$42,184.00	5.06%		\$16,225.01–\$45,677.00	10.5%
	\$42,184.01–\$84,369.00	7.7%		\$45,677.01–\$130,506.00	12.5%
	\$84,369.01–\$96,686.00	10.5%		\$130,506.01 and up	14.5%
	\$96,686.01–\$117,623.00	12.29%	Nova Scotia	\$0.00–\$8,481.00*	0%
	\$117,623.01–\$159,483.00	14.7%		\$8,481.01–\$29,590.00	8.79%
	\$159,483.01–\$222,420.00	16.8%		\$29,590.01–\$59,180.00	14.95%
Alberta	\$222,420.01 and up	20.5%		\$59,180.01–\$93,000.00	16.67%
	\$0.00–\$19,369.00*	0%		\$93,000.01–\$150,000.00	17.5%
	\$19,369.01–\$131,220.00	10%		\$150,000.01 and up	21%
	\$131,220.01–\$157,464.00	12%	Prince Edward Island**	\$0.00–\$10,500.00*	0%
	\$157,464.01–\$209,952.00	13%		\$10,500.01–\$31,984.00	9.8%
Manitoba	\$209,952.01–\$314,928.00	14%		\$31,984.01–\$63,969.00	13.8%
	\$314,928.01 and up	15%		\$63,969.01 and up	16.7%
	\$0.00–\$9,936.00*	0%	Newfoundland and Labrador	\$0.00–\$9,536.00*	0%
	\$9,936.01–\$33,723.00	10.8%		\$9,536.01–\$38,081.00	8.7%
Ontario**	\$33,723.01–\$72,885.00	12.75%		\$38,081.01–\$76,161.00	14.5%
	\$72,885.01 and up	17.4%		\$76,161.01–\$135,973.00	15.8%
	\$0.00–\$10,880.00*	0%		\$135,973.01–\$190,363.00	17.3%
	\$10,880.01–\$45,142.00	5.05%		\$190,363.01 and up	18.3%
	\$45,142.01–\$90,287.00	9.15%	Yukon**	\$0.00–\$11,038.00*	0%
Quebec	\$90,287.01–\$150,000.00	11.16%		\$11,038.01–\$43,561.00	7.04%
	\$150,000.01–\$220,000.00	12.16%		\$43,561.01–\$87,123.00	9.68%
	\$220,000.01 and up	13.16%		\$87,123.01–\$135,054.00	11.44%
	\$0.00–\$15,728.00*	0%		\$135,054.01 and up	12.76%
	\$15,728.01–\$45,105.00	15%	Northwest Territories	\$0.00–\$15,423.00*	0%
New Brunswick	\$45,105.01–\$90,200.00	20%		\$15,423.01–\$44,396.00	5.9%
	\$90,200.01–\$109,755.00	24%		\$44,396.01–\$88,796.00	8.6%
	\$109,755.01 and up	25.75%		\$88,796.01–\$144,362.00	12.2%
	\$0.00–\$10,564.00*	0%	Nunavut	\$144,362.01 and up	14.05%
	\$10,564.01–\$43,835.00	9.4%		\$0.00–\$16,467.00*	0%
	\$43,835.01–\$87,761.00	14.82%		\$16,467.01–\$46,740.00	4%
	\$87,761.01–\$142,534.00	16.52%		\$46,740.01–\$93,480.00	7%
	\$142,534.01–\$162,383.00	17.84%		\$93,480.01–\$151,978.00	9%
	\$162,383.01 and up	20.3%		\$151,978.01 and up	11.5%

2021 Federal Income Tax⁵

Taxable Income	Tax Rate
\$0.00–\$13,808.00*	0%
\$13,808.01–\$49,020.00	15%
\$49,020.01–\$98,040.00	20.5%
\$98,040.01–\$151,978.00	26%
\$151,978.01–\$216,511.00	29%
\$216,511.01 and up	33%

* These amounts represent the basic personal amount.

**There are additional income surtaxes in these provinces and Yukon. These surtaxes are not covered in this textbook.

⁵ Note that the focus of this textbook is on the mathematics of income taxes – the mathematics do not change. The tax brackets and tax rates do change every year both provincially and federally. For current income tax information, visit: <https://www.canada.ca/en/revenue-agency/services/tax/individuals/frequently-asked-questions-individuals/canadian-income-tax-rates-individuals-current-previous-years.html>.



How It Works

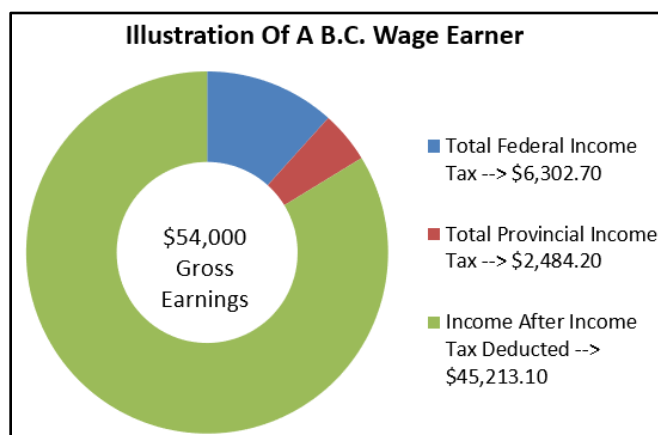
To calculate the total annual income tax owing in any tax year, follow these steps:

Step 1: Determine the total amount of gross earnings that are taxable for the individual.

Step 2: Calculate the federal income taxes owing. Using the federal tax table, split the gross earnings into each of the listed tax brackets, then calculate the taxes owing in each tax bracket by multiplying the income in the bracket and the tax rate. Round any tax amounts to two decimals.

Step 3: Calculate the provincial/territorial income taxes owing. Repeat the same process as in step 2, but use the appropriate tax brackets from the provincial tax table instead. Round any tax amounts to two decimals.

Step 4: Calculate the total annual income taxes owing by applying Formula 4.2. This means adding all calculated tax amounts from both steps 2 and 3 together.



Assume you live in British Columbia and your taxable gross income is \$54,000. Here is how you would calculate your annual total federal and provincial taxes:

Step 1: The gross earnings are \$54,000.

Step 2: Federally, your income falls into the first three brackets. Your first \$13,808 is taxed at 0%; therefore, there are no income taxes on this amount. The next \$35,212 (from \$13,808.01 to \$49,020.00) is taxed at 15%, thus $35,212 \times 15\% = \$5,281.80$. The last \$4,980 (from \$49,020.01 to \$54,000.00) is taxed at 20.5%, thus $4,980 \times 20.5\% = \$1,020.90$.

Step 3: Provincially, your income falls into the first three brackets for British Columbia. Your first \$11,070 is taxed at

0%; therefore, there are no income taxes on this amount. The next \$31,114 (from \$11,070.01 to \$42,184.00) is taxed at 5.06%, thus $31,114 \times 5.06\% = \$1,574.37$. The last \$11,816 (from \$42,184.01 to \$54,000) is taxed at 7.7%, thus $11,816 \times 7.7\% = \$909.83$.

Step 4: Your total federal income tax is $0.00 + \$5,281.80 + \$1,020.90 = \$6,302.70$. Your total provincial income tax is $0.00 + \$1,574.37 + \$909.83 = \$2,484.20$. Applying Formula 4.1, income tax = $\$6,302.70 + \$2,484.20 = \$8,786.90$, which is the amount that will be deducted from total gross earnings.



Things To Watch Out For

Have you ever heard someone say, "I earn more income and moved to a higher tax bracket, so now I am earning less money and my paycheck is lower"? The progressive tax system used in Canada makes this statement untrue.

Remember that tax rates apply only to the portion of the gross earnings in the tax bracket and are not retroactive to lower levels of income. For example, if your taxable gross income increased from \$96,000 to \$106,000, your highest federal tax bracket is now 26% instead of 20.5%. However, your federal income tax is not calculated at 26% for your entire income. Rather, the first \$13,808 is tax free, the next \$35,212 is taxed at 15%, the next \$49,020 is taxed at 20.5%, and the final \$7,960 is taxed at 26%.



Paths To Success

It is sometimes beneficial to pre-calculate the total income taxes in any tax bracket under the assumption that the individual's income comprises the entire tax bracket. This technique is particularly useful when repetitive calculations are required, such as determining the federal income taxes for each employee in an entire company. For example, the second federal tax bracket extends from \$13,808.01 to \$49,020, representing \$35,212 of employee income. Since this bracket is taxed at 15%, someone who earns a higher income always owes the full amount of tax in this category, which is $35,212 \times$

15% = \$5,281.80. The next tax bracket covers \$49,020.01 to \$98,040, representing \$49,020 of income. Someone earning a higher income always owes the full amount of tax in this category, which is $\$49,020 \times 20.5\% = \$10,049.10$.

Assume taxable gross earnings of \$106,000. With your pre-calculated income taxes in each bracket, you know the income taxes in the first three tax brackets are \$0, \$5,281.80, and \$10,049.10, or \$15,330.90 in total. You only need to calculate the tax on the portion of income in the final tax bracket of $\$7,960 \times 26\% = \$2,069.60$. The grand total is $\$15,330.90 + \$2,069.60 = \$17,400.50$.

Example 4.2A: Federal and Provincial Income Taxes

A Canadian wage earner has taxable gross income of \$107,250. Calculate the federal and provincial annual income taxes individually and then sum the amounts to calculate the total annual income taxes if the wage earner lives in

a. Ontario b. New Brunswick c. Alberta

Plan	For each of the provinces, you need to calculate the individual's total federal income tax and total provincial income tax. Then sum these amounts to arrive at the annual income tax.				
Understand	What You Already Know Step 1: The employee's gross taxable earnings are \$97,250. From the tax tables, you also know the federal and provincial income tax brackets and corresponding tax rates.	How You Will Get There Step 2: Calculate the federal income tax. This amount is the same in all three provinces and needs to be calculated only once. Looking at the federal tax table, you see that this wage earner falls into the first four tax brackets, which requires the sum of the income tax calculations for each bracket. Step 3: Calculate the provincial income tax. Using the provincial tax table, the income spans the first four of Ontario's tax brackets. In New Brunswick, the individual falls into the first four tax brackets. In Alberta, the individual falls into the first two of Alberta's tax brackets. In all three provinces, the total provincial income is the sum of the income tax calculations for each bracket. Step 4: Calculate total annual income taxes using Formula 4.2.			
Perform	Step 2: $(\$13,808 - \$0) \times 0 = \$0.00$ $(\$49,020 - \$13,808) \times 0.15 = \$5,281.80$ $(\$98,040 - \$49,020) \times 0.205 = \$10,049.10$ $(\$107,250 - \$98,040) \times 0.26 = \$2,394.60$ Total federal income tax = $\\$0.00 + \\$5,281.80 + \\$10,049.10 + \\$2,394.60 = \\$17,725.50$ Step 3: <table border="1"> <tbody> <tr> <td> Ontario $(\\$10,880 - \\$0) \times 0 = \\$0.00$ $(\\$45,142 - \\$10,880) \times 0.0505 = \\$1,730.23$ $(\\$90,287 - \\$45,142) \times 0.0915 = \\$4,130.77$ $(\\$107,250 - \\$90,287) \times 0.1116 = \\$1,893.07$ Total Ontario income tax = $\\$0.00 + \\$1,730.23 + \\$4,130.77 + \\$1,893.07 = \\$7,754.07$ </td><td> New Brunswick $(\\$10,564 - \\$0) \times 0 = \\$0.00$ $(\\$43,835 - \\$10,564) \times 0.094 = \\$3,127.47$ $(\\$87,761 - \\$43,835) \times 0.1482 = \\$6,509.83$ $(\\$107,250 - \\$87,761) \times 0.1652 = \\$3,219.58$ Total New Brunswick income tax = $\\$0.00 + \\$3,127.47 + \\$6,509.83 + \\$3,219.58 = \\$12,856.88$ </td><td> Alberta $(\\$19,369 - \\$0) \times 0 = \\$0.00$ $(\\$107,250 - \\$19,369) \times 0.10 = \\$8,788.10$ Total Alberta income tax = $\\$0.00 + \\$8,788.10 = \\$8,788.10$ </td></tr> </tbody> </table> Step 4: Total income tax if living in Ontario = $\\$17,725.50 + \\$7,754.07 = \\$25,479.57$ Total income tax if living in New Brunswick = $\\$17,725.50 + \\$12,856.88 = \\$30,582.38$ Total income tax if living in Alberta = $\\$17,725.50 + \\$8,788.10 = \\$26,513.60$		Ontario $(\$10,880 - \$0) \times 0 = \$0.00$ $(\$45,142 - \$10,880) \times 0.0505 = \$1,730.23$ $(\$90,287 - \$45,142) \times 0.0915 = \$4,130.77$ $(\$107,250 - \$90,287) \times 0.1116 = \$1,893.07$ Total Ontario income tax = $\$0.00 + \$1,730.23 + \$4,130.77 + \$1,893.07 = \$7,754.07$	New Brunswick $(\$10,564 - \$0) \times 0 = \$0.00$ $(\$43,835 - \$10,564) \times 0.094 = \$3,127.47$ $(\$87,761 - \$43,835) \times 0.1482 = \$6,509.83$ $(\$107,250 - \$87,761) \times 0.1652 = \$3,219.58$ Total New Brunswick income tax = $\$0.00 + \$3,127.47 + \$6,509.83 + \$3,219.58 = \$12,856.88$	Alberta $(\$19,369 - \$0) \times 0 = \$0.00$ $(\$107,250 - \$19,369) \times 0.10 = \$8,788.10$ Total Alberta income tax = $\$0.00 + \$8,788.10 = \$8,788.10$
Ontario $(\$10,880 - \$0) \times 0 = \$0.00$ $(\$45,142 - \$10,880) \times 0.0505 = \$1,730.23$ $(\$90,287 - \$45,142) \times 0.0915 = \$4,130.77$ $(\$107,250 - \$90,287) \times 0.1116 = \$1,893.07$ Total Ontario income tax = $\$0.00 + \$1,730.23 + \$4,130.77 + \$1,893.07 = \$7,754.07$	New Brunswick $(\$10,564 - \$0) \times 0 = \$0.00$ $(\$43,835 - \$10,564) \times 0.094 = \$3,127.47$ $(\$87,761 - \$43,835) \times 0.1482 = \$6,509.83$ $(\$107,250 - \$87,761) \times 0.1652 = \$3,219.58$ Total New Brunswick income tax = $\$0.00 + \$3,127.47 + \$6,509.83 + \$3,219.58 = \$12,856.88$	Alberta $(\$19,369 - \$0) \times 0 = \$0.00$ $(\$107,250 - \$19,369) \times 0.10 = \$8,788.10$ Total Alberta income tax = $\$0.00 + \$8,788.10 = \$8,788.10$			
Present	For an individual earning \$107,250 in taxable gross income, the person will pay \$25,479.57 in total gross income tax if living in Ontario, \$30,582.38 total gross income tax if living in New Brunswick, and \$26,513.60 if living in Alberta.				



Section 4.2 Exercises

Use the provincial and federal tax tables introduced earlier in this chapter when you determine income taxes in these exercises.

Mechanics

For questions 1–3, determine only the federal income taxes on the following gross taxable incomes.

1. \$22,375
2. \$158,914
3. \$102,100

For questions 4–6, determine only the provincial income taxes on the following gross taxable incomes.

4. \$61,000 in Newfoundland and Labrador
5. \$83,345 in Saskatchewan
6. \$88,775 in British Columbia

For questions 7 and 8, determine the total federal and provincial/territorial taxes on the following gross taxable incomes.

7. \$58,910 in the Northwest Territories
8. \$65,525 in Prince Edward Island

Applications

9. Nadia lives in Manitoba and has three part-time jobs. Her gross annual taxable income from each job was \$5,300, \$21,450, and \$25,390. How much federal and provincial income tax does Nadia owe?
10. If Delaney's gross taxable income increases from \$71,000 to \$79,000, by what dollar amount will her after-tax pay change once federal and provincial income taxes are deducted in the province of Quebec?
11. Helen just moved from Saskatchewan to her new job in Alberta at the same rate of pay of \$51,225. By what dollar amount will her provincial income taxes change?
12. Jane has received two job offers that she thinks are relatively equal. The only difference lies in the salary. The first job offer is for a position in Nova Scotia earning gross taxable income of \$63,375. The second job offer is for a position in British Columbia earning gross taxable income of \$61,990. If Jane will select the job that has the highest after-tax income, which one should she choose? How much better is this option in dollars?
13. Suppose an employee earns \$111,300 in gross taxable income. Calculate the federal and provincial income taxes that need to be deducted if he lives in either Saskatchewan, Ontario, or the Northwest Territories. In which province or territory will he earn the most income after income tax deductions? The least? What is the dollar amount difference among the three alternatives?
14. After both federal and provincial income taxes are deducted, who would earn more money, an individual earning \$85,000 in New Brunswick or an individual earning \$79,000 in Nunavut?

Challenge, Critical Thinking, & Other Applications

15. Rawatha is the human resources manager for her firm located in Yukon. The salespeople for her organization are paid an annual base salary of \$30,000 plus 8% straight commission on their annual sales. If the average salesperson sells \$310,000 annually, what is the total annual federal and provincial income tax owing on the salesperson's gross taxable income?
16. Esmerelda works on the production line and is paid a piecework rate of \$2.25 per unit produced. She is capable of producing an average of five units per hour, and works eight hours every weekday. Assuming a 52-week year, what is the annual federal income tax and provincial income tax owing if she lives in Prince Edward Island?

17. Mary Jane is paid biweekly at a rate of \$34.68 per hour and works 36.25 hours every week. Assuming a 52-week year, what are the amounts of the federal income tax deduction and provincial income tax deduction that must be removed from each paycheck if Mary Jane lives in Manitoba? (Hint: income tax would be deducted equally across all paycheques for the year.)
18. Felix was transferred from his head office in Ontario to Alberta. In accepting the transfer, his employer agreed to increase his salary of \$88,000 by \$7,000. What is the percent change in Felix's after-tax income?
19. If a wage earner paid \$2,277.98 in total annual Ontario provincial income taxes, determine the annual gross taxable income.
20. Perform a comparison across all territories and provinces of the income earned after both federal and provincial income taxes are deducted for a Canadian who earns \$56,738 in gross taxable income. Which province or territory has the highest income after deductions? Which has the lowest? What is the percent difference between the two?

4.3: Indexes

(The Times Are Changing)

You read in the *Financial Post* that the S&P/TSX Composite Index is up 12.74 points while the NASDAQ is down 45.98, and you wonder what this means for your investment portfolio. The consumer price index rose 3.1% since last year, and you wonder if the raise your employer offered you keeps pace. You need to understand indexes to make sense of these numbers.

What Is An Index?

An **index** is a number used to compare two quantities sharing the same characteristic as measured under different circumstances. Usually, the comparison involves two different points of time. The two quantities are expressed as a rate of one another and are then converted to an index number using a base value (usually 100 or 1,000). The result is called an **index number**, which expresses the relationship between the two quantities.



The Formula

Chosen Quantity is The Portion: This variable is the quantity to be compared against some other number. It is the quantity of interest in your calculation.

Base Quantity is The Base: This variable is the quantity that you want to compare the chosen value against. The result of taking the Chosen Quantity divided by the Base Quantity is a rate, since it involves taking a portion and dividing it by a base.

Base Value is The Conversion: Most indexes are expressed in terms of percentages; therefore, the base value is commonly 100. This is the default value to use unless otherwise stated.

Formula 4.3 – Index Numbers:

$$\text{Index Number} = \frac{\text{Chosen quantity}}{\text{Base quantity}} \times \text{Base value}$$

Index Number is a Modified Rate: The expression of the relationship between the chosen quantity and the base quantity in the unit of the base value. There are two common interpretations to this number.

1. Most index numbers represent a percentage (where the base value is 100); however, the percent sign is never written after the index number. Therefore, a calculation resulting in an index number of 117 is interpreted as the chosen quantity being 117% of the base quantity. This is also interpreted as a percent change in which the chosen quantity is 17% higher than the base quantity. Index numbers of this nature (i.e., with a base of 100) are rounded to one decimal in most cases, which is the rounding standard for this textbook.
2. If the base value is other than 100, then the index number is not a percentage and only expresses the ratio between the two numbers. For example, an index of 2,000 using a base value of 1,000 concludes that the chosen quantity is twice as large as the base quantity. Ultimately, this type of index number is convertible into a percentage using Formula 2.2 (Rate, Portion, Base), and the chosen quantity is expressed as 200% of the base quantity, or a 100% change or increase from the base quantity. If the value being represented is dollars, round this index to two decimals. If the value is in other units, a suitable rounding standard must be



How It Works

Follow these steps to calculate an index number:

Step 1: Identify the base and the base value for the index.

Step 2: Identify the value of the chosen quantity to be compared against the base value.

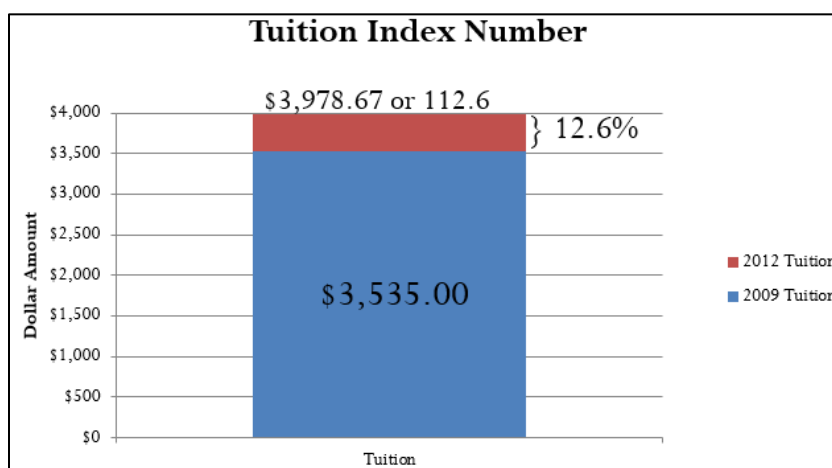
Step 3: Calculate the index number by applying Formula 4.3.

In the above steps, note that in the event that the index number is known and you are solving for a different unknown, identify the known variables in steps 1 and 2. Assume the price of tuition was \$3,535.00 in 2009 and \$3,978.67 in 2012. The goal is to express the 2012 tuition indexed against the 2009 tuition.

Step 1: The tuition in 2009 is the amount to be compared against; thus, Base Quantity = \$3,535. No base value is specified; thus, use the default of Base Value = 100.

Step 2: The 2012 tuition is the quantity for which you wish to compute the index, resulting in Chosen Quantity = \$3,978.67.

Step 3: The Index Number = $\frac{\$3,978.67}{\$3,535.00} \times 100 = 112.6$. You can conclude that tuition in 2012 is 112.6% higher than 2009, or that tuition has increased by 12.6% over the three years.



Example 4.3A: Indexing the Price of Gold

In January 2000, the average price of gold per troy ounce was \$284.32 (in US dollars). In May 2011, the average price of gold was \$1,494.89. Express the gold price in 2011 indexed against the gold price in 2000.

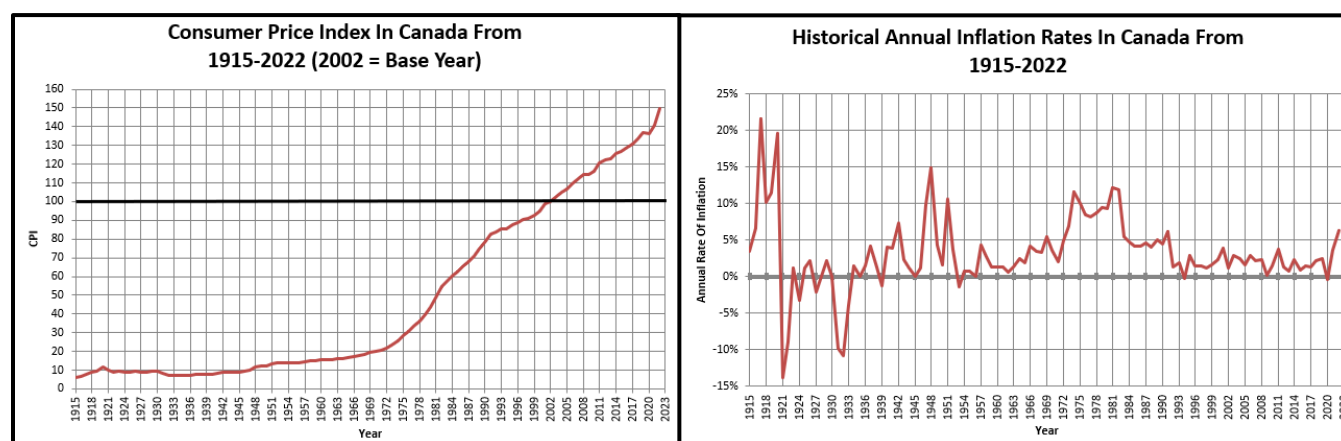
Plan	You need to calculate an index number that compares the 2011 gold price against the 2000 gold price.	
Understand	What You Already Know Step 1: Base Quantity = 2000 price = \$284.32; assume the standard Base Value = 100 Step 2: Chosen Quantity = 2011 price = \$1,494.89	How You Will Get There Step 3: Apply Formula 4.3.
Perform	Step 3: Index Number = $\frac{\$1,494.89}{\$284.32} \times 100 = 525.8$	
Present	The May 2011 gold price has an index number of 525.8 when compared to the January 2000 price. This means that the price of gold has increased by 425.8% over the approximate 11-year period.	

Specialty Indexes

While indexes can be created for any comparison of two quantities, two specialized indexes are widely used across the business and financial sectors of Canada: the consumer price index and the S&P/TSX Composite Index.

Consumer Price Index

One of the most closely watched price indexes in Canada is the **consumer price index**, or CPI. Historical values for this index (base year = 2002) are illustrated the graph on the left while the historical rates of inflation (1915-2022) are depicted in the graph on the right.⁶

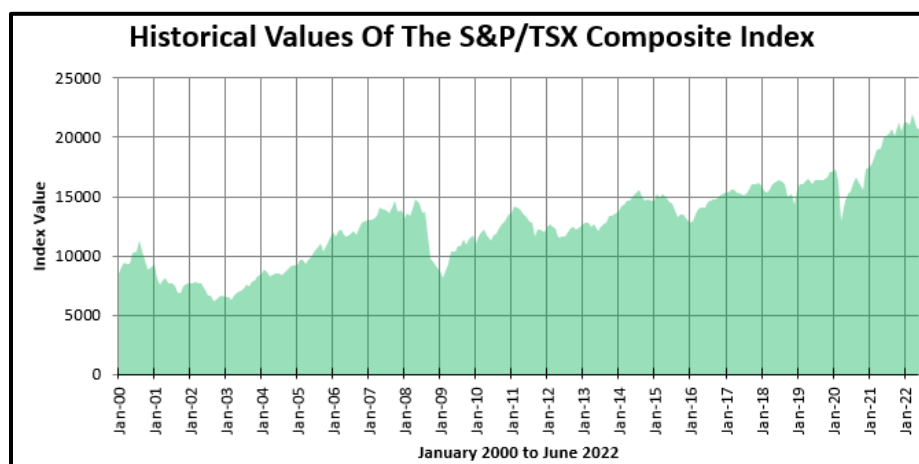


The CPI, calculated by Statistics Canada on a monthly basis, is used as a measure for estimating the rate of inflation or the cost of living (notice the slope of the CPI line increases or decreases in accordance with the change in the rate of inflation). The CPI measures the average price of the goods and services that a typical Canadian household commonly purchases, which is called the **market basket**. This basket contains about 600 items in categories such as food, housing, transportation, furniture, recreation, and clothing. The CPI is used for many purposes, such as increases in wages, pensions, salaries, prices, the Canada Pension Plan (CPP), and Old Age Security (OAS).

In the section opener, you need to know the CPI in discussing your raise with your employer. Since the CPI reflects the changing cost of living, you must ensure that any raise your employer offers covers, at the barest minimum, the increased costs of living. For example, if the CPI increased from 114.3 to 116.5 over the year, reflecting a 1.78% rate of inflation, then your raise needs to be at least 1.78% to keep you in the same financial position as last year.

S&P/TSX Composite Index

Another index reported daily and nationally across Canada is the Standard & Poor's/Toronto Stock Exchange Composite Index, or **S&P/TSX Composite Index**. This index captures the equity prices of approximately 200 of Canada's



⁶ Values determined from Bank of Canada "Inflation Calculator" (all index values measured in May of each year). www.bankofcanada.ca/rates/related/inflation-calculator/?page_moved=1 (accessed June 13, 2022).

largest companies on the Toronto Stock Exchange as measured by their market capitalization, representing approximately 70% of the Canadian market capitalization listed on the TSX. The historical values of the index are illustrated on the previous page. The exact number of companies in the index continually changes because of strict requirements on market capitalization, liquidity, and domicile that must be met. The index is a measure of the strength and direction of the economy of Canada. For example, during the 2008 subprime mortgage crisis in the United States, the index retreated from a value of 14,714 to just over 8,123 in a very short time frame (demonstrating the close interconnectedness of Canada's economy with that of the United States). At this time, the Canadian economy entered its first recession in years.

In the section opener, knowing the S&P/TSX Composite Index is invaluable for the meeting with your financial adviser to evaluate the health of your investment portfolio. For example, from May 2010 to May 2011 the index rose approximately 15.7%. If your investment portfolio rose by 9% over the same time frame (which on its own sounds impressive), you know from comparison to the index that your portfolio in fact had a subpar performance.



The Formula

The base year used in CPI calculations is mid-2002, and the base value is 100. The CPI is always rounded to one decimal. The base year used in S&P/TSX calculations is 1975, and the base value is 1,000. The S&P/TSX is always rounded to two decimals. Regardless of which index is being calculated, Formula 4.3 remains unchanged.

Often when you are working with these indexes, you want to compare how quantities have increased or decreased. This requires you to use Formula 3.1 (Percent Change) to calculate percent changes.



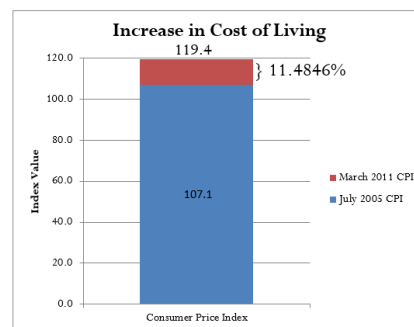
How It Works

No new steps or procedures are required when you work with the CPI or the S&P/TSX Composite Index. To calculate the value of an index, apply the index number steps. To calculate the percent change between two index values, apply the percent change steps.

For example, in July 2005 the CPI was 107.1. In March 2011 the CPI was 119.4. How much did the cost of living rise from July 2005 to March 2011? To solve this problem, apply the percent change steps:

Step 1: New = 119.4 and Old = 107.1. You are looking for the $\Delta\%$.

Step 2: Substituting into Formula 3.1: $\Delta\% = \frac{119.4 - 107.1}{107.1} \times 100 = 11.4846\%$. You can conclude that the cost of living has risen by 11.4846% over the approximate six-year time frame.



Important Notes

From time to time, Statistics Canada updates the base period for the CPI and adjusts all CPI numbers accordingly. Previously, the base period was 1992 before being adjusted to 2002. Another update to the base period can probably be expected soon; however, this occurs retroactively since it takes time to recalculate all indexes and gather all of the latest data.

Example 4.3B: How Much Money Do You Have?

If you had invested \$15,000 into an S&P/TSX portfolio in 1975, how much money would you have in 2011 when the S&P/TSX Composite Index indicated a value of 13,607.25?

Plan	You need to determine the amount of money in the portfolio at the current time period of 2011. This amount represents the Chosen Quantity.	
Understand	What You Already Know Step 1: Since the money was invested in the base year (1975), the amount of the investment is the base quantity. Base Quantity = \$15,000 Base Value = 1,000 Step 2: In this case, seek the Chosen Quantity with a known Index Number = 13,607.25	How You Will Get There Step 3: Apply Formula 4.3 and rearrange for Chosen Quantity.
Perform	Step 3: $13,607.25 = \frac{\text{Chosen Quantity}}{\$15,000} \times 1,000$ $204,108,750 = \text{Chosen Quantity} \times 1,000$ $\$204,108.75 = \text{Chosen Quantity}$	
Present	If you had invested \$15,000 in the S&P/TSX Composite Index back in 1975, your investment portfolio is now valued at \$204,108.75, representing an increase of 1,260.725%.	

Purchasing Power of a Dollar

The consumer price index is applied in determining the **purchasing power of a dollar**, which is the amount of goods and services that can be exchanged for a dollar. The result of the calculation is a percentage showing how much more or less product is received in return for every dollar spent. For example, recall that my parents could attend a movie for 25¢ in the 1950s. If my parents had lost that quarter under their mattress and found it today, the quarter no longer can purchase admission to the movie (nor much else!). Therefore, the purchasing power of that quarter has drastically declined.



The Formula

PPD is Purchasing Power of a Dollar: This percentage gives you a true understanding of how the ability of your money to purchase products has changed over time. The change is based on the base year for the consumer price index used in the calculation. Therefore, if the CPI is based in 2002, then the result of this formula expresses the change in your purchasing power since that year.

× 100 is Percent Conversion: Express the purchasing power of a dollar in percent format.

Formula 4.4 – Purchasing Power Of A Dollar:
$$\text{PPD} = \frac{\$1}{\text{CPI} / 100} \times 100$$

CPI is Consumer Price Index:
The index value used is the CPI at the chosen point in time.

/ 100 is Decimal Conversion: Convert the CPI to decimal format before completing the division.



How It Works

Follow these steps to calculate the purchasing power of a dollar:

Step 1: Choose the point in time to perform the calculation. Identify the value of the CPI at that point in time as well as noting the base year upon which the CPI is determined (so you can properly interpret the calculated amount). Note that in the event that the purchasing power is known and you are solving for a different unknown, you must identify the known variables in this step.

Step 2: Calculate the purchasing power of a dollar by applying Formula 4.4.

Assume the CPI is 111.1 today. How has your purchasing power changed since 2002 (the base year for the CPI)?

Step 1: The CPI = 111.1 and the base year is 2002.

Step 2: Substitute into the formula: $PPD = \frac{\$1}{111.1 / 100} \times 100 = 90.009\%$. Therefore, your purchasing power is 90.009% what it was in 2002. In 2002, if you could buy 10 loaves of bread with your dollar, today you could buy approximately nine loaves of bread with the same amount of money. In other words, your dollar purchases about 10% less product with the same amount of money.

Example 4.3C: How Has Your Purchasing Power Changed?

In March 2006, the CPI was 108.6. By March 2011 it had risen to 119.4. As a percent change, how did your purchasing power change over that time frame?

Plan	You need to calculate the percent change, or $\Delta\%$, between the purchasing power of a dollar (PPD) in 2006 and 2011.										
Understand	What You Already Know Step 1: 2006 represents your purchasing power of a dollar in the past, and 2011 represents the present. Old = PPD in 2006 New = PPD in 2011 CPI 2006 = 108.6 CPI 2011 = 119.4 The current CPI is based in 2002.	How You Will Get There Step 2: Apply Formula 4.4 to calculate the PPD for both 2006 and 2011. Step 3: Apply Formula 3.1 to calculate the percent change.									
Perform	Step 2: $PPD\ 2006 = \frac{\$1}{108.6 / 100} \times 100 = 92.0810\%$ $PPD\ 2011 = \frac{\$1}{119.4 / 100} \times 100 = 83.7521\%$ Step 3: $\Delta\% = \frac{0.837521 - 0.920810}{0.920810} \times 100 = -9.0452\%$	Calculator Instructions <table border="1"> <thead> <tr> <th>OLD</th><th>NEW</th><th>%CH</th><th>#PD</th></tr> </thead> <tbody> <tr> <td>0.920810</td><td>0.837521</td><td>Answer: -9.045188</td><td>1</td></tr> </tbody> </table>		OLD	NEW	%CH	#PD	0.920810	0.837521	Answer: -9.045188	1
OLD	NEW	%CH	#PD								
0.920810	0.837521	Answer: -9.045188	1								
Present	From March 2006 to March 2011, your purchasing power declined by 9.0452%. This means that if in March 2006 you were able to buy approximately 92 items with a certain sum of money, in March 2011 you were only able to buy about 83 items with the same amount of money.										

Real Income

Another application of the consumer price index allows you to assess your real earnings. With the cost of living (inflation) always changing, it is difficult to assess how much more or less money you are truly making. **Real income** allows you to remove the effects of inflation from your income, allowing for a fair comparison of earnings at different points in time. For example, if your gross earnings last year were \$50,000 and you received a \$1,500 raise this year, how much more are you truly earning if the CPI changed from 114.6 to 116.6 over the same time frame?



The Formula

RI is Real Income: This is the true amount of your income once inflation has been removed. The income represents the true amount of your earnings corresponding to the CPI base year. Thus, if the CPI is based in 2002, then these are your real earnings for 2002.

Nominal Income: The nominal income reflects your current gross earnings inclusive of any inflation since the base period of the CPI.

Formula 4.5 – Real Income:
$$RI = \frac{\text{Nominal Income}}{\text{CPI} / 100}$$

CPI is Consumer Price Index: The index value used is the CPI that corresponds to the same point in time at which you earned the nominal

/ 100 is Decimal Conversion: Convert the CPI to decimal format before completing the division.



How It Works

Follow these steps to calculate your real income:

Step 1: Identify the nominal income. Determine the CPI value that corresponds to the same time frame as the nominal income and note the base year for interpretation purposes. In the event that the real income is known and you are solving for a different unknown, you must identify the known variables in this step.

Step 2: Calculate the real income by applying Formula 4.5.

Assume your gross earnings were \$40,000 when the CPI was 111.7. Currently your income is \$42,000 and the CPI is 113.3. Calculate your real income to assess how your income has changed.

Step 1: In this case, you have two nominal incomes to convert for comparison purposes. In the past, nominal income = \$40,000 and CPI = 111.7. Currently, nominal income = \$42,000 and CPI = 113.3.

Step 2: In the past, $RI = \frac{\$40,000}{111.7 / 100} = \$35,810.21$. Currently, $RI = \frac{\$42,000}{113.3 / 100} = \$37,069.73$. Comparing the amounts, you see that your real income expressed in 2002 dollars has risen by $\$37,069.73 - \$35,810.21 = \$1,259.52$. Notice that your nominal income rose by \$2,000. This means that of your \$2,000 increase in income, you really are only making \$1,259.82 more. The other \$740.18 represents your increased cost of living.

Example 4.3D: Did You Get a Raise?

Now that your boss has conducted your performance review, the time has come to discuss your raise. Currently, you earn \$81,250 annually. Your boss has offered you a new salary of \$83,000 annually. You know last year that the CPI was 104.1 and this year it sits at 106.6. Assess the salary offer.

Plan	To assess the offer, remove inflation from your nominal incomes, allowing for a fair comparison between the two salary amounts. Calculate the real income (RI) for each salary.	
Understand	What You Already Know Step 1: You know the following information about your old and new salaries: Old Nominal Income = \$81,250 Old CPI = 104.1 New Nominal Income = \$83,000 New CPI = 106.6 Both CPI amounts are based on the year 2002.	How You Will Get There Step 2: Apply Formula 4.5 to each salary.

Perform	$\text{Step 2: RI Last Year} = \frac{\$81,250}{104.1 / 100} = \$78,049.95$ $\text{RI This Year} = \frac{\$83,000}{106.6 / 100} = \$77,861.16$ $\$77,861.16 - \$78,049.95 = -\$188.79$
Present	Although it appears as if you are receiving a \$1,750 wage increase, when you factor in the increased cost of living you are actually earning \$188.79 less than you did last year. Perhaps you should discuss this matter further with your boss. Since the CPI had a percent change of 2.4015%, your income must rise to at least \$83,201.25 for you to break even.



Section 4.3 Exercises

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided. Round index numbers to one decimal and base values to the nearest integer.

	Chosen Quantity (\$)	Base Quantity (\$)	Base Value	Index Number
1.	\$559.99	\$539.99	100	?
2.	\$315,000.00	?	1,000	12,415.9
3.	\$114.30	\$112.50	?	101.6
4.	?	\$248.75	1,000	1,548.9

For questions 5–9, solve for the unknown variables (identified with a ?) based on the information provided.

	Consumer Price Index (CPI)	Purchasing Power of a Dollar (PPD)
5.	107.9	?
6.	?	80%

	Nominal Income (\$)	Consumer Price Index (CPI)	Real Income (RI)
7.	\$45,000.00	120.0	?
8.	?	105.0	\$29,523.81
9.	\$86,000.00	?	\$80,298.71

Applications

- In Regina, a 4 L bag of 2% milk has an average price of \$3.71. Prices in Toronto and Montreal are \$4.55 and \$5.40, respectively. Calculate an index of these prices using Regina as the base.
- If Sabrina is currently earning \$53,000 and the CPI changes from 105.9 to 107.6, how much money does she need to earn next year just to keep up with inflation?
- If the CPI rises from 103.4 to 108.8, how much money at the start is required to have the same purchasing power as \$100 at the end?
- George currently earns \$28,000 per year. As per his union salary grid, next year he will earn \$32,500 per year. If the CPI increases from 104.0 to 106.1 over the same time frame, how much of his raise is real income?
- Last year the purchasing power of a dollar was 84.3%. This year the purchasing power of a dollar is 81.4%. What is the percent change in the CPI between the two years?

Challenge, Critical Thinking, & Other Applications

15. An investor had \$200,000 invested in an S&P/TSX Composite Index portfolio in 2009. By 2011, the index rose to 14,136.50. If the index in 2009 was 8,694.90, how much money would the investor have in her portfolio in 2011, if she was able to match the market performance?
16. Over the years, Hannah's income has changed from \$36,000 to \$40,000 to \$45,000. If the CPI changed from 102.9 to 105.1 to 108.6 over the same time period, determine Hannah's percent change in real income from year to year.
17. The CPI rose from 102.6 in 2003 to 116.5 in 2010. The S&P/TSX Composite Index rose from 6,569.49 to 11,294.42 over the same time period. If an investor had \$125,000 invested in 2003, how much of the growth (in dollars) for the portfolio over the seven years represents real growth?
18. An enterprise has 136 employees earning an average income of \$42,250 per year. If the CPI rises from 103.7 to 107.2, by what amount do total wages rise if all employee wages are adjusted to match the CPI?
19. Using January 2003 as your base year with a base value = 100, compute a series of indexes for the S&P/TSX Composite Index from select years in 2003 to 2019. Interpret your results.

	2003	2007	2011	2015	2019
January index	6,569.49	13,034.12	13,551.99	14,673.50	15,540.60

20. Below is select information on the CPI from 2003 to 2018:

	2003	2008	2013	2018
May CPI	102.5	114.6	123.0	133.4

Using this information, determine the following:

- a. The purchasing power of a dollar for each year. Interpret your results.
- b. If you were earning \$34,500 in 2002, calculate the nominal income required each year to keep up with the changes in the CPI.

Chapter 4 Summary

Key Concepts Summary

Section 4.1: Gross Earnings (Off to Work You Go)

- Calculating salary and hourly gross earnings
- Discussion of employment contract characteristics for salary and hourly earners
- Calculating commission gross earnings
- Calculating piecework gross earnings

Section 4.2: Personal Income Tax (The Taxman Taketh)

- Provision of 2013 federal and provincial/territorial tax brackets and corresponding tax rates
- Calculation of income taxes

Section 4.3: Indexes (The Times Are Changing)

- What is an index and how is it calculated?
- Specialty indexes, including the consumer price index and the S&P/TSX Composite Index
- Using the CPI to calculate the purchasing power of a dollar
- Using the CPI to calculate real income

The Language of Business Mathematics

basic personal amount The amount of income for which the wage earner is granted a tax exemption.

commission An amount or a fee paid to an employee for performing or completing some form of transaction.

consumer price index A measure of the average price of a typical Canadian market basket, which is used to estimate inflation in Canada.

graduated commission A form of compensation where an employee is offered increasing rates of commission for higher levels of performance.

gross earnings The amount of money earned before any deductions are removed from a paycheque.

holiday earnings Earnings paid to an employee on a statutory holiday for which no work is performed.

hourly wage A variable compensation based on the time an employee has worked.

index A number used to compare two quantities sharing the same characteristic as measured under different circumstances.

index number The expression of a relationship between two quantities; it is a result of an index calculation.

market basket The average price of the goods and services that a typical Canadian household commonly purchases; used in calculating the consumer price index.

overtime Work time in excess of an employee's regular workday or regular workweek.

overtime or premium earnings Earnings determined by an employee's overtime rate of pay and that occur when regular hours are exceeded.

personal income tax A tax on earned income that is levied by both the federal and provincial/territorial governments.

piecework A form of compensation where an employee is paid on a per-unit basis.

progressive tax system A personal income tax system where the tax rate increases as the amount of income increases; however, the increased tax rates apply only to income amounts above a minimum threshold.

public holiday A provincially recognized day for which employees may or may not get a day of rest and may or may not receive pay depending on provincial employment standards.

purchasing power of a dollar The amount of goods and services that can be purchased with a dollar.

real income Income that has the effects of inflation removed from its amount.

regular earnings Earnings determined by an employee's regular rate of pay.

S&P/TSX Composite Index This index captures the equity prices of approximately 200 of Canada's largest companies on the Toronto Stock Exchange as measured by their market capitalization, representing approximately 70% of the Canadian market capitalization listed on the TSX.

salary A fixed compensation paid to a person on a regular basis for services rendered.

salary plus commission A form of compensation in which gross earnings combine a basic salary together with commissions on transactions.

statutory holiday A legislated day of rest with pay.

statutory holiday worked earnings Earnings paid to an employee on a statutory holiday at a premium rate for working on the statutory holiday.

straight commission A form of compensation where the employee's entire earnings are based on dollar transactions and calculated strictly as a percentage of the total transactions.

The Formulas You Need to Know

Symbols Used

CPI = consumer price index

GE = gross earnings

PPD = purchasing power of a dollar

RI = real income

Formulas Introduced

Formula 4.1: Salary and Hourly Gross Earnings:

$$GE = \text{Regular Earnings} + \text{Overtime Earnings} + \text{Holiday Earnings} + \text{Statutory Holiday Worked Earnings} \quad (\text{Section 4.1})$$

Formula 4.2 Annual Income Tax:

$$\text{Income Tax} = \sum (\text{Eligible Income in Tax Bracket} \times \text{Tax Bracket Rate}) \quad (\text{Section 4.2})$$

Formula 4.3 Index Numbers: $\text{Index Number} = \frac{\text{Chosen quantity}}{\text{Base quantity}} \times \text{Base value}$ (Section 4.3)

Formula 4.4: Purchasing Power of a Dollar: $\text{PPD} = \frac{\$1}{\text{CPI} / 100}$ (Section 4.3)

Formula 4.5: Real Income: $\text{RI} = \frac{\text{Nominal Income}}{\text{CPI} / 100}$ (Section 4.3)

Technology

Calculator

The percent change ($\Delta\%$) function is used in this chapter. See the end of Chapter 3 for a full discussion of this function.



Chapter 4 Review Exercises

For any questions involving income tax calculations, refer to the tables previously introduced in this chapter for federal income taxes and provincial/territorial income taxes.

Mechanics

1. Jim's annual salary is \$45,670 paid weekly for 36.25 hours of work each week. Determine his total gross earnings for one pay period in which he worked 8.75 hours of overtime.
2. Tracy is paid 4% commission on any sales above \$150,000 plus a base salary of \$500. What are her gross earnings for a pay period where her sales were \$225,000?
3. Sisko has annual gross taxable earnings of \$99,990. Determine his annual federal income tax.

4. Jean-Luc works two part-time jobs in Ontario. If his annual wages from each position are \$17,444 and \$16,430, determine his annual provincial income tax.
5. If the CPI rises from 102.6 to 110.8, determine the percent change in the purchasing power of a dollar.
6. Since Maybelline graduated from college in 2004, her salary has changed from \$23,100 in 2004 to \$27,800 in 2009. Over that time, the CPI has increased from 105.0 to 114.9. Express Maybelline's salary in real dollars and determine the percent change in her real earnings.

Applications

7. You have been given two job offers. The first employer is offering \$17.50 per hour. The second employer is offering a \$33,000 annual salary. Both employers expect that you might be able to earn an additional two hours of overtime per week. Treat any holiday earnings as regular earnings and suppose there are no statutory holiday worked earnings. If both jobs have 7.5 regular hour workdays five times per week for 52 weeks, which employer is offering the higher annual gross earnings? By what amount in dollars?
8. Juanita is an architect. She is paid \$41.25 per hour and normally works eight-hour days, Monday through Friday. Her employer has agreed to pay her overtime at double her regular rate of pay and any time worked on a statutory holiday at 2.5 times her regular pay. What are Juanita's gross earnings in a single week from Sunday to Saturday where she worked 4, 9, 8, 11.5, 8, 0, and 6 hours, respectively, where Monday was a statutory holiday and her employer has decided to pay her in lieu of another day off?
9. Lorelei works on the production line for eight hours each day. She earns \$9.50 per hour plus \$0.13 per kilogram of sardines that she packs in excess of 100 kg per day. What are her daily gross earnings in a single day where she manages to pack 200, 250 g cans of sardines per hour?
10. Determine whether a wage earner in British Columbia or Ontario pays the least provincial income taxes on gross taxable earnings of \$58,000. What percentage more does the wage earner with the higher income tax pay?
11. If Henri had \$200,000 invested in his S&P/TSX Composite Index portfolio in 1975, how much would the portfolio be worth in March of 2011 when the S&P/TSX Composite Index was 14,116.10?
12. In the province of Saskatchewan, you are paid an hourly wage of \$11.25 and you work 7.5-hour days for five days per week. Assuming a 52-week year, determine the amount of provincial income tax deducted from your biweekly paycheck.

Challenge, Critical Thinking, & Other Applications

13. Karen is paid \$12.00 per hour for her work as a cashier at an Alberta retail outlet. Her employer pays a standard overtime rate. Her regular workday is eight hours and 40 hours per week.
 - a. What are her gross earnings in a single week from Sunday to Saturday where Karen worked 12, 8, 12, 8, 0, 8, and 8 hours, respectively?
 - b. If this represents a typical week for Karen, what are the federal and provincial income taxes that should be deducted from her biweekly paycheques (assume 52 weeks in a year)?
14. Living in Nova Scotia, Rufus pays \$410.50 per biweekly pay period in total federal and provincial income tax. What are his gross taxable earnings?
15. Gilligan is paid a graduated commission of $1\frac{7}{8}\%$ on his first \$50,000 in sales, $3\frac{1}{2}\%$ on the next \$75,000 in sales, $4\frac{1}{8}\%$ on the next \$75,000 in sales, and $5\frac{1}{5}\%$ on all sales above that amount. If his total gross earnings were \$9,516.25, determine his sales during the pay period.
16. Last year your annual salary was \$58,150. This year your salary was increased in accordance with the CPI. Last year, the CPI was 108.9 and this year it is 111.1. Determine the amount of federal income tax that is deducted from your semi-monthly paycheck throughout the current year.

17. As a sales representative, you average \$350,000 per month in sales. Your commission is calculated as 1% of the first \$120,000, 2% on the next \$120,000, and 3% for anything higher. If you live in Ontario, determine both of the federal and provincial income taxes that are removed from your monthly commission cheques.
18. Rufaro is the payroll administrator for New Wave Industries. There are four employees on the payroll. Buck and Roger receive annual salaries of \$41,000 and \$85,555, respectively. Luke receives piecework wages of \$2.50 per item and produces 45 items per day five days per week. Leah is paid a base salary of \$200 per week and receives 6% commission on average weekly sales of \$10,000. Assuming a 52-week year, determine the annual total federal income taxes that New Wave Industries must remit to the federal government on behalf of its employees.
19. For every province in Western Canada (British Columbia, Alberta, Saskatchewan, and Manitoba), determine the total annual income tax deducted from an annual gross taxable income of \$66,660. Express the total income tax as a percentage of gross income.
20. For a weekly pay period, you know the following information about the payroll at Hopper Industries. Friday is a statutory holiday.
- Christie receives an annual salary of \$44,880 based on a 36-hour workweek comprised of four, nine-hour days. She has two hours of overtime at time-and-a-half and did not work the statutory holiday.
 - Marie is paid \$16.75 per hour based on eight-hour days and a 40-hour workweek. All premium wages are paid at regulatory minimums. From Sunday to Saturday, she worked 8, 0, 8, 8, 8, 3, and 8 hours, respectively. She will not receive another day off in lieu for the statutory holiday.
 - Norm receives a graduated commission of 2.25% on his first \$100,000 in sales and 4.15% on anything higher. His sales during one week were \$123,000. Ignore his holiday or statutory earnings in your calculations.
 - Garth is paid a straight piecework wage of \$1.12 per unit produced. He worked a total of 30 hours during the week, producing one unit every five minutes. Ignore his holiday or statutory earnings in your calculations.

For all employees combined, determine the total regular earnings, overtime earnings, holiday earnings, and holiday worked earnings that Hopper Industries must report on its weekly company payroll report.

Chapter 4 Case Study

Payroll Planning

The Situation

As Lightning Wholesale reviews the payroll for one of its divisions, it needs to produce a report on the overall payroll for last year. The tables below list each employee along with his or her compensation for last year. The report needs to include details about each employee along with overall company figures.

The Data

Last Year's Employee Compensation Information:

Name	Annual Salary	Hourly Wage	Average O/T Hours per Pay Period	Average Sales per Pay Period
Jack	\$39,000		2	
Gail				\$169,500
Jennifer		\$16.98	5	
Evelyn				\$134,000
Indiana	\$76,500		3	
Hakim		\$34.94	4	

Per Pay Period Graduated Commission Scale:

Commission Scale	Commission Rate
\$0.00–\$50,000.00	1.01%
\$50,000.01–\$100,000.00	2.98%
\$100,000.01 and up	4.44%

Important Information

- A normal workweek for any employee is 7.5 hours per day from Monday through Friday.
- Lightning Wholesale pays overtime at time-and-a-half and is never open on statutory holidays.
- Lightning Wholesale issues payroll every two weeks.
- Lightning Wholesale is situated in Ontario.
- Federal and provincial income tax information can be found in Section 4.2 tables from this chapter.
- A 52-week year is assumed.

Your Tasks

1. Create a report on each employee and summarize the information in a table.
 - a. Calculate the annual regular earnings.
 - b. Calculate the annual overtime earnings.
 - c. Calculate the annual gross earnings.
 - d. Calculate the annual federal income taxes.
 - e. Calculate the annual provincial income taxes.
 - f. Calculate the employee's take-home pay after deducting taxes.
 - g. Calculate the totals of all of the above.

Chapter 5: Marketing and Accounting Fundamentals

(Keeping Your Nose above Water)

When a new retail store opens in your neighbourhood strip mall, you wonder whether it will still be running six months or a year from now. Do its owners know how much merchandise they have to sell to cover their costs? Will any money be left over to help grow the business?

While the numbers vary, approximately 15% of new companies in Canada go out of business in their first year of operations. This number rises to 38% by the third year and to a staggering 49% over the first five years.¹ Why do so many new businesses fail? Economic conditions, the fierceness of competition, changing consumer tastes, or even changes in taxes all contribute to the problem, but one of the most common reasons businesses fail is poor tracking of their basic financial numbers.

So if you want your own business to succeed, you need to understand its financial side. You need to be able to answer questions like the following:

- How does the money leave the business through costs and expenses?
- How does the money come into the business through revenue, which is determined by both the price tags on the merchandise and the volume at which that merchandise sells?
- What is the minimum amount of revenue that is needed to cover the costs?
- What is the resulting profitability of the business?

The next three chapters introduce the concepts behind the key numbers in any business.

- This chapter introduces the global model of the relationship between costs and revenues. It shows the overall cost structure of a business and introduces models for developing total costs and total revenues. It explains how you calculate net income and contribution margins. Armed with all of this knowledge, you can then answer the break-even question: How much does the business need to sell to cover its costs?
- Chapter 6 explores merchandising, including methods of calculating the cost of an individual product along with setting its associated retail price, and the impact of markup and markdown decisions.
- Chapter 7 delves into cost elements that the accounting department typically handles, including sales and property taxes, the mathematics of invoicing, and how to handle currency conversions properly.

Outline of Chapter Topics

5.1: Cost-Revenue-Net Income Analysis (Need to Be in the Know)

5.2: Break-Even Analysis (Sink or Swim)

¹ Eileen Fisher and Rebecca Reuber, *The State of Entrepreneurship in Canada: February, 2010*, (Ottawa, ON: Small Business and Tourism Branch, Industry Canada), 9, [www.ic.gc.ca/eic/site/061.nsf/vwapj/SEC-EEC_eng.pdf/\\$file/SEC-EEC_eng.pdf](http://www.ic.gc.ca/eic/site/061.nsf/vwapj/SEC-EEC_eng.pdf/$file/SEC-EEC_eng.pdf).

5.1: Cost-Revenue-Net Income Analysis

(Need to Be in the Know)

The end of the month is approaching, and bills are coming due. As you sit at your kitchen table trying to figure out your budget for next month, you wonder whether you will be able to afford that concert you had been planning on attending. Some of your costs remain unchanged from month to month, such as your rent, Internet service, gym membership, and insurance. Other costs tend to fluctuate with your usage, such as your utilities, cellphone bill, vehicle fuel, and the amount of money spent on recreational activities. Together, these regular and irregular costs total to next month's costs.

Examining a few recent paycheque stubs, you calculate the average monthly net income you bring home from your hourly cashier position at Sobeys. The exact amount of each paycheque, of course, depends on how many hours you work. Besides your short-term costs, you need to start saving for next year's tuition. Therefore, your budget needs to include regular deposits into your savings account to meet that goal. Once you have put your bills, paycheques, and goals together, you hope that your budget will balance. If there is a shortfall, you will have to miss out on those concert tickets.

Budgeting at work is no different in principle from your home budget. Businesses also need to recognize the different types of costs they incur each month, some of which remain the same and some of which fluctuate. Businesses need to pay for these costs by generating revenues, which correspond to your paycheque. As with your education goals, businesses also require profits to grow. A business needs to understand all of these numbers so it can plan its activities realistically.

This section explores the various types of costs and establishes a model relating total costs to total revenues to determine total profitability levels. You will then apply this model to see how the sale of an individual product contributes to covering costs and how each product individually contributes to overall profitability.

Types of Costs

A **cost** is an outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services. Costs can come in three forms:

1. A **fixed cost** is a cost that does not change with the level of production or sales (call this "output" for short). In other words, whether the business outputs nothing or outputs 10,000 units, these costs remain the same. Some examples include rent, insurance, property taxes, salaries unrelated to production (such as management), production equipment, office furniture, and much more. **Total fixed costs** are the sum of all fixed costs that a business incurs.
2. A **variable cost** is a cost that changes with the level of output. In other words, if the business outputs nothing there is no variable cost. However, if the business outputs just one unit (or more) then a cost appears. Some examples include material costs of products, production labour (hourly or piecework wages), sales commissions, repairs, maintenance, and more. **Total variable costs** are the sum of all variable costs that a business incurs at a particular level of output.
3. A **blended cost** is a cost that comprises both fixed cost and variable cost components. In other words, a portion of the total cost remains unchanged while another portion depends on the output. For calculation purposes, you must separate a blended cost into its fixed and variable cost components. A few examples will illustrate the concept of blended costs:
 - Residential natural gas bills from a gas company include a fixed charge per month of \$14 plus charges for cubic metres of actual consumption based on transportation, distribution, and primary and supplemental gas rates. In this situation, the \$14 is a fixed cost while the actual consumption of natural gas is a variable cost.
 - A cellphone bill includes a fixed charge for the phone service plus any additional charges for usage, such as long distance, texting, or data.
 - If employees are paid a salary plus commission, then their salaries represent fixed costs while their commissions are a variable cost.



The Formula

In calculating business costs, fixed costs are commonly calculated on a total basis only since the business incurs these costs regardless of any production. However, variable costs are commonly calculated both on a total and per-unit basis to reveal the overall cost along with the cost associated with any particular unit of output. When these variable costs are assigned

on an individual basis it is called a **unit variable cost**. The calculation of unit variable cost has a further benefit because it allows managers to explore how the total business costs vary at different levels of output.

VC is Unit Variable Cost: The unit variable cost is an adapted simple average formula (Formula 3.3) with specific definitions for the data and the quantity. The end result of the calculation is the typical or average variable cost associated with an individual unit of output. Being a dollar cost, the unit variable cost is rounded to two decimals.

TVC is Total Variable Cost: The total variable costs in dollars that were incurred at a particular level of output. In the simple average formula, this is represented by the symbol Σx , which stands for the total of all pieces of data.

Formula 5.1 – Unit Variable Cost:
$$VC = \frac{TVC}{n}$$

n is Level of Output: In the simple average formula, n represented the number of pieces of data. For this chapter, the definition is further specified to represent the total number of units produced or sold or the total output that incurred the total variable costs.



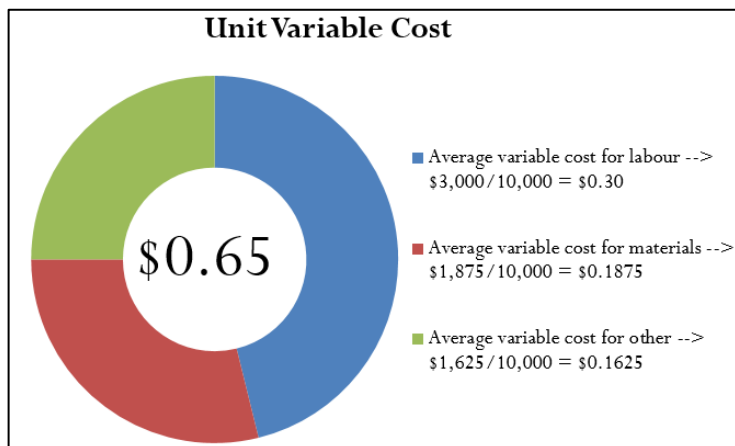
How It Works

Follow these steps to calculate the unit variable cost:

Step 1: Identify all fixed, variable, and blended costs, along with the level of output. For variable costs, understand any important elements of how the cost is structured. For blended costs, separate the costs into variable and fixed components.

Step 2: Calculate the total variable cost (TVC) by totalling all variable costs at the indicated level of output. This involves taking any known unit variable costs and multiplying each by the level of output.

Step 3: Divide the total variable cost by the total level of output by applying Formula 5.1.



Assume a company produces 10,000 units and wants to know its unit variable cost. It incurs production labour costs of \$3,000, material costs of \$1,875, and other variable costs totalling \$1,625.

Step 1: In this case, all costs are variable costs (production labour and material costs are always variable). The level of output is $n = 10,000$ units.

Step 2: Total all variable costs together to get $TVC = \$3,000 + \$1,875 + \$1,625 = \$6,500$.

Step 3: Apply Formula 5.1 to arrive at $VC = \$6,500 / 10,000 = \$0.65$. This means that, on average, the variable cost associated with one unit of production is \$0.65.



Important Notes

The definitions of variable, fixed, and blended costs along with their typical associated examples represent a simplified view of how the real world operates. The complexities involved in real-world business costs complicate the fundamentals of managing

a business. Therefore, here is how this textbook addresses the complexities of atypical operations, changing fixed costs, and decreasing unit variable costs:

1. **Atypical Operations.** Although there are "normal" ways that businesses operate, there are also businesses that have atypical operations. What is a fixed cost to one business may be a variable cost to another. For example, rent is usually a fixed cost. However, some rental agreements include a fixed base cost plus a commission on the operational output. These rental agreements form blended costs. This textbook does not venture into any of these atypical costs and instead focuses on common cost categorizations.
2. **Changing Fixed Costs.** In real-world applications, fixed costs do not remain flat at all levels of output. As output increases, fixed costs tend to move upwards in steps. For example, at a low level of output only one manager (on salary) may be needed. As output increases, eventually another manager needs to be hired, perhaps one for every 20,000 units produced. In other words, up to 20,000 units the fixed costs would be constant, but at 20,001 units the fixed costs take a step upwards as another manager is added. The model presented in this textbook does not address these upward steps and treats fixed costs as a constant at all levels of output.
3. **Decreasing Unit Variable Costs.** Production tends to realize efficiencies as the level of output rises, resulting in the unit variable cost dropping. This is commonly known as achieving **economies of scale**. As a consumer, you often see a similar concept in your retail shopping. If you purchase one can of soup, it may cost \$1. However, if you purchase a bulk tray of 12 cans of soup it may cost only \$9, which works out to 75¢ per can. This price is lower partly because the retailer incurs lower costs, such as fewer cashiers to sell 12 cans to one person than to sell one can each to 12 different people. Now apply this analogy to production. Producing one can of soup costs 75¢. However, a larger production run of 12 soup cans may incur a cost of only \$6 instead of \$9 because workers and machines can multitask. This means the unit variable cost would decrease by 25¢ per can. However, the model in this textbook assumes that unit variable costs always remain constant at any given level of output.

Example 5.1A: Knowing Your Costs

You are considering starting your own home-based Internet business. After a lot of research, you have gathered the following financial information:

Dell computer	\$214.48 monthly lease payments
Office furniture (desk and chair)	\$186.67 monthly rental
Shaw high-speed Internet connection	\$166.88 per month
Your wages	\$30 per hour
Utilities	\$13 per month plus \$0.20 per hour usage
Software (and ongoing upgrades)	\$20.00 per month
Business licences and permits	\$27.00 per month
Google click-through rate	\$10.00 per month + \$0.01 per click payable as total clicks per sale

Generating and fulfilling sales of 430 units involves 80 hours of work per month. Based on industry response rates, your research also shows that to achieve your sales you require a traffic volume of 34,890 Google clicks. On a monthly basis, calculate the total fixed cost, total variable cost, and unit variable cost.

Plan	Calculate the monthly total fixed costs (TFC), total variable costs (TVC), and the unit variable cost (VC).	
Understand	What You Already Know There are three different types of costs. Other known information includes: $n = 430$ units Hours = 80 Google clicks = 34,890	How You Will Get There Step 1: Sort the costs into fixed and variable. Separate the components for blended costs. You can total the fixed costs to arrive at the total fixed costs, or TFC. Step 2: Calculate variable costs, then total them to arrive at the total variable cost, or TVC. Step 3: Apply Formula 5.1 to calculate the unit variable cost.

Perform	Step 1:			
	Fixed Costs		Variable Costs	
	Dell computer	\$214.48	Wages	\$30.00 per hour
	Office furniture	\$186.67	Utilities (blended cost)	\$0.20 per hour
	Shaw high-speed Internet	\$166.88	Google clicks (blended cost)	\$0.01 per click
	Utilities (blended cost)	\$13.00 only		
	Software	\$20.00		
	Business licences/permits	\$27.00		
	Google clicks (blended cost)	\$10.00 only		
	TOTAL FIXED COSTS	TFC = \$638.03		
Step 2: $TVC = (\$30.00 \times \text{hours}) + (\$0.20 \times \text{hours}) + (\$0.01 \times \text{Google clicks})$ $= (\$30.00 \times 80) + (\$0.20 \times 80) + (\$0.01 \times 34,890) = \$2,764.90$				
Step 3: $VC = \frac{\$2,764.90}{430} = \$6.43$				
Present	Seven components make up the fixed costs: the computer, furniture, Internet service, fixed utilities, software, licences/permits, and the fixed Google cost, all totalling \$638.03. Three components make up the variable costs: wages, hourly utilities, and Google clicks, totalling \$2,764.90. The average unit variable cost is \$6.43 per unit sold.			

Net Income Using a Total Revenue and Total Cost Approach

Most businesses are "for-profit" businesses, meaning that they operate to make money. In Example 5.1A, you figured out the total fixed costs, total variable costs, and the unit variable cost for your Internet business. However, you left unanswered one of the most important questions in business: If you sell the planned 430 units, are you profitable? Is there any money left after you pay for all of those fixed and variable costs? You must also remember that 430 units is just an estimate. What happens if you sell only 350 units? What happens if you sell 525 units? What happens if you decide to pay yourself a higher wage?

There are no guarantees in business, and the future is always uncertain. Successful business managers plan for the future and perform many "what-if" scenarios to answer questions such as those above. This section develops a model for calculating total net income based on total revenues and total costs. The model allows managers to analyze various scenarios and determine the impact on profitability.



The Formula

NI is Net Income: Net income is the amount of money left over after all costs are deducted from all revenues. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss since the costs are exceeding the revenues. Note that many companies use the terms *net earnings* or *net profit* instead of the term *net income*. Net income is based on a certain level of output. This model assumes that the number of units that are produced or purchased (for resale) by the company exactly matches the number of units that are output or sold by the company. Therefore, the model does not consider inventory and its associated costs.

$n(S)$ is Total Revenue: This term in the formula calculates how much money or gross income the sale of the product at a certain output level brings into the organization. **Total revenue** is the entire amount of money received by a company for selling its product, calculated by multiplying the quantity sold by the selling price.

$TFC + n(VC)$ is Total Costs: This term in the formula calculates how much money is spent to generate the revenue. **Total cost** is the sum of all costs for the company, including both the total fixed costs and total variable costs. Two terms make up the costs: Total fixed costs (TFC) are a constant since these costs do not change with the level of output; total variable costs, represented mathematically by $n(VC)$, are the level of output multiplied by the unit variable cost.

Formula 5.2 – Net Income Using a Total Revenue and Cost Approach:

$$NI = n(S) - (TFC + n(VC))$$

n is Level of Output: The number of units produced or sold or the output that incurred all of the variable costs.

S is Unit Selling Price: The unit selling price of the product.

TFC is Total Fixed Costs: The total of all costs that are not affected by the level of output.

VC is Unit Variable Cost: From Formula 5.1, this is the average variable cost associated with an individual unit of output.



How It Works

Follow these steps to calculate net income using a total revenue and total income approach:

Step 1: Calculate the total revenue. This requires identifying the unit selling price of the product and multiplying it by the level of sales.

Step 2: Calculate your total costs. This requires identifying and separating costs into fixed and variable components. Fixed costs are totalled to arrive at the total fixed cost. Total variable costs are either known or can be calculated through multiplying the unit variable cost by the level of output.

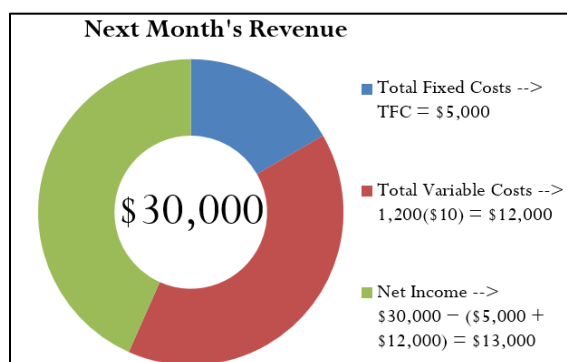
Step 3: Calculate the net income by applying Formula 5.2.

For example, assume that last month a company incurred total variable costs of \$10,000 in the course of producing 1,000 units. For next month it forecasts total fixed costs of \$5,000 and all variable costs remaining unchanged. Projected production for next month is 1,200 units selling for \$25 each. You want to estimate next month's net income.

Step 1: Using Formula 5.2, you calculate total revenue from $n(S)$, or the total level of output multiplied by the price of the product. If you project sales of 1,200 units (n) at \$25 each (S), then the total forecasted revenue is $1,200(\$25) = \$30,000$.

Step 2: Total fixed costs, or TFC, are \$5,000. To get the total variable costs, you must resolve $n(VC)$. You calculate the unit variable cost, or VC, with Formula 5.1. Using the current month figures, you see $VC = \$10,000 \div 1,000 = \$10$. If the projected level of output is 1,200 units, then the total variable costs are $1,200(\$10) = \$12,000$.

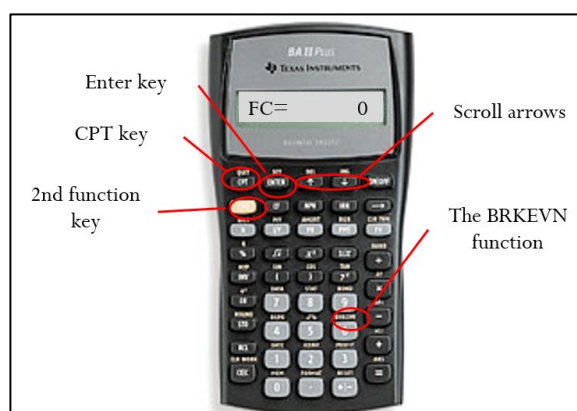
Step 3: Applying Formula 5.2 you have $NI = \text{Total Revenue} - \text{Total Costs} = \$30,000 - (\$5,000 + \$12,000) = \$13,000$. Based on the numbers, you forecast net income of \$13,000 for next month.



Important Notes

The Texas Instruments BAII Plus calculator is programmed with a version of Formula 5.2. The function is called Brkevn, and you access it by pressing 2nd and then the number six key. The relationship between the formula symbols and the calculator symbols is displayed in the table below.

Variable	Formula 5.2 Notation	Calculator Notation
Total fixed cost	TFC	FC
Unit variable cost	VC	VC
Price per unit	S	P (for price)
Net income	NI	PFT (for profit)
Level of output	n	Q (for quantity)



To solve Formula 5.2 for the PFT or any other variable, enter data into all of the above variables except one. Keying in a variable requires inputting the value and pressing Enter. Use $\uparrow$ and $\downarrow$ to scroll through the display. When you are ready, scroll to the unknown variable and press CPT.



Paths To Success

An easy way to remember Formula 5.2 is to understand what the formula represents. As explained, the calculation of $n(S)$ multiplies quantity by price to produce the total revenue. The $(TFC + n(VC))$ takes the total fixed costs and adds the total variable costs (which is a function of quantity multiplied by unit variable cost) to arrive at the total cost. Therefore, Formula 5.2 expressed more simply is:

$$\text{Net Income} = \text{Total Revenue} - \text{Total Cost}$$



Give It Some Thought

In the following situations, explain what would happen to net income and why.

1. The selling price is raised.
2. The hourly wages of production workers are increased to match the increase in the consumer price index.
3. The level of output decreases.
4. Your insurance company lowers your insurance premiums because your company has had no claims in the past year.

Solutions:

1. Net income increases because the total revenues increase with no similar increase in costs.
2. Net income decreases because the hourly wages of production workers are variable costs. If the costs go up, then less money is left over.
3. Net income decreases because both total revenue and total variable costs drop; however, the fixed costs remain the same, so there is less revenue to cover proportionally higher costs.
4. Net income increases because the insurance premium is a fixed cost that has become smaller. With lower costs, more money is left over.

Example 5.1B: Running Your Internet Business

Recall from Example 5.1A that you are considering starting your own home-based Internet business. The following information is known:

$$FC = \$638.03 \quad VC = \$6.43 \quad \text{forecasted } n = 430$$

Based on these numbers, calculate:

- a. The forecasted net income if your price per unit is \$10.
- b. The dollar change in net income if you decide to pay yourself a higher wage of \$35.38 per hour instead of \$30.00 per hour while still working 80 hours. Note the total variable costs excluding wages were \$364.90.
- c. The dollar change in net income if sales are 30% lower than your initial forecast, ignoring part (b) calculations.

Plan	a. You need to determine net income (NI) using the revenue and cost numbers. b. Recalculate the total variable cost (TVC) to determine the impact of the higher wage. Then calculate the change in net income (NI). c. Modify the level of output (n) and calculate the change in net income (NI).	
Understand	What You Already Know The costs, level of output, and price are known: $FC = \\$638.03$ $VC = \\$6.43$ forecasted $n = 430$ $S = \\$10$ For part (b), the total variable costs excluding wages are known: TVC (excluding wages) = \$364.90 New wage = \$35.38 Hours = 80 For part (c), the change in the level of output is known: $\Delta\% = -30\%$	How You Will Get There a. Step 1: Calculate total revenue at the forecasted output. Step 2: Calculate total costs at the forecasted output. Step 3: Apply Formula 5.2. b. Step 1: Total revenue is unchanged since the output has not changed. Step 2: Fixed costs are unchanged. Recalculate the total variable costs by adding the new wages to the total variable costs excluding wages. By rearranging Formula 5.1, you can see $TVC = n(VC)$. Therefore, substitute TVC in place of $n(VC)$ in Formula 5.2. Step 3: Apply Formula 5.2 to recalculate the new net income. Compare to net income in part (a) and determine the change. c. Step 1: The revised level of output is a percent change calculation where $\Delta\% = -30\%$ and $OLD = 430$. Solve Formula 3.1 for New. Once New is known, recalculate total revenue. Step 2: Fixed costs are unchanged. Recalculate total variables costs using the revised level of output. Step 3: Apply Formula 5.2 to recalculate the new net income. Compare to net income in part (a) and determine the change.

Perform	a.	Step 1: Total revenue = $n(S) = 430(\$10) = \$4,300$ Step 2: $TFC = \$638.03$; $TVC = n(VC) = 430(\$6.43) = \$2,764.90$ Step 3: $NI = \$4,300 - (\$638.03 + \$2,764.90) = \$897.07$	Calculator Instructions <table><tr><td></td><td>FC</td><td>VC</td><td>P</td><td>PFT</td><td>Q</td></tr><tr><td>a.</td><td>638.03</td><td>6.43</td><td>10</td><td>Answer: 897.07</td><td>430</td></tr><tr><td>b.</td><td>✓</td><td>$3195.3 \div 430 =$</td><td>✓</td><td>Answer: 466.67</td><td>✓</td></tr><tr><td>c.</td><td>✓</td><td>6.43</td><td>✓</td><td>Answer: 436.54</td><td>301*</td></tr></table> *First use Δ% function to calculate this number: <table><tr><td></td><td>OLD</td><td>NEW</td><td>%CH</td><td>#PD</td></tr><tr><td>c.</td><td>430</td><td>Answer: 301</td><td>-30</td><td>1</td></tr></table>		FC	VC	P	PFT	Q	a.	638.03	6.43	10	Answer: 897.07	430	b.	✓	$3195.3 \div 430 =$	✓	Answer: 466.67	✓	c.	✓	6.43	✓	Answer: 436.54	301*		OLD	NEW	%CH	#PD	c.	430	Answer: 301	-30	1
		FC		VC	P	PFT	Q																														
	a.	638.03		6.43	10	Answer: 897.07	430																														
	b.	✓		$3195.3 \div 430 =$	✓	Answer: 466.67	✓																														
c.	✓	6.43	✓	Answer: 436.54	301*																																
	OLD	NEW	%CH	#PD																																	
c.	430	Answer: 301	-30	1																																	
b.	Step 1: Total revenue = $\$4,300$ Step 2: $TFC = \$638.03$; $TVC = \$364.90 + (\$35.38 \times 80 \text{ hours}) = \$3,195.30$ Step 3: $NI = \$4,300 - (\$638.03 + \$3,195.30) = \$466.67$ Change in $NI = \$466.67 - \$897.07 = -\$430.40$																																				
c.	Step 1: $-30\% = \frac{\text{New}-430}{430} \times 100$ $-129 = \text{New} - 430$ $301 = \text{New}$ Total revenue = $301(\$10) = \$3,010$ Step 2: $TFC = \$638.03$; $TVC = 301(\$6.43) = \$1,935.43$ Step 3: $NI = \$3,010 - (\$638.03 + \$1,935.43) = \$436.54$ Change in $NI = \$436.54 - \$897.07 = -\$460.53$																																				
Present	Under the initial scenario, a net income of \$897.07 on sales of 430 units is forecast. If you pay yourself a higher wage of \$35.38 per hour, your net income decreases by \$430.40 to \$466.67. If your forecast is inaccurate and is lower by 30%, then your net income is reduced by \$460.53, resulting in a lower net income of \$436.54.																																				

Net Income Using a Total Contribution Margin Approach

In Example 5.1B, you learned that if you sell the projected 430 units of product for your Internet business, the total net income is \$897.07. What if you sold 431 units of the product? How would your net income change? Clearly it should rise, but by how much? One approach to answering this question is to rerun the numbers through Formula 5.2, revising the total revenues and total variable costs to calculate a new net income. This new net income can then be compared against the existing net income to determine how it changed. This is a multistep approach and involves a lot of work. An alternative approach explored in this section involves using a unit contribution margin to calculate the net income. The benefit of this approach is that with minimal calculations you can easily assess the impact of any change in the level of output.

In accounting and marketing, the **contribution margin** is the amount that each unit sold adds to the net income of the business. This approach allows you to understand the impact on net income of each unit sold, and Section 5.2 will explain its further benefit of allowing for a straightforward calculation of a break-even point. The contribution margin determines on a per-unit basis how much money is left over after unit variable costs are removed from the price of the product. This leftover money is then available to pay for the fixed costs. Ultimately, when all fixed costs have been paid for, the leftover money becomes the profits of the business. If not enough money is left over to pay for the fixed costs, then the business has a negative net income and loses money.



The Formula

The difference between the increase in the total revenue and the increase in the total variable costs is how much the sale of an individual product contributes toward the change in your net income. Formula 5.3 expresses this relationship.

CM is Unit Contribution Margin: This is the amount of money that remains available to pay for fixed costs once the unit variable cost is removed from the selling price of the product.

S is Unit Selling Price: The unit selling price of the product.

Formula 5.3 – Unit Contribution: $CM = S - VC$

VC is Unit Variable Cost: The typical or average variable cost associated with an individual unit of output, as determined from Formula 5.1.

If you have no units sold, your net income is negative and equal to the total fixed costs associated with your business, since there is no offsetting revenue to pay for those costs. With each unit sold, the contribution margin of each product is available to pay off the fixed costs. Formula 5.4 expresses this relationship when calculating net income.

NI is Net Income: The amount of money left over after all costs have been paid is the net income. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss.

n is Level of Output: The number of units produced or sold or the output that incurred all of the variable costs.

Formula 5.4 – Net Income Using Total Contribution Margin Approach: $NI = n(CM) - TFC$

n(CM) is Total Contribution Margin: This term in the formula calculates how much money is left over to pay the total fixed costs. Use the contribution margin calculated in Formula 5.3 and multiply it by the level of output to determine the total monies remaining after all variable costs are paid.

CM is Unit Contribution Margin: The amount of money per unit remaining after variable costs have been paid. It is available to cover fixed costs. Calculate the unit contribution margin by taking the unit selling price and subtracting the unit variable cost as per Formula 5.3.

TFC is Total Fixed Cost: The total of all costs that are not affected by the level of output.



How It Works

Follow these steps to calculate the net income using a contribution margin approach:

Step 1: If unit information is known, apply Formula 5.3 and calculate the unit contribution margin by subtracting the unit variable cost from the selling price. This may or may not require you to use Formula 5.1 to calculate the unit variable cost.

Step 2: Calculate the total contribution margin by multiplying the contribution margin with the associated level of output.

Step 3: Calculate the total fixed costs by adding all fixed costs.

Step 4: Based on the level of output, calculate the net income by applying Formula 5.4.

For example, using the contribution margin approach, calculate the net income for a product that sells for \$75, has unit variable costs of \$31, total fixed costs of \$23,000, and total sales of 800 units.

Step 1: The unit contribution margin is calculated from Formula 5.3. If the product sells for \$75 and has unit variable costs of \$31, then $CM = \$75 - \$31 = \$44$. This means that every unit sold has \$44 left over to contribute toward fixed costs.

Step 2: Now convert that into a total contribution margin. The first part of Formula 5.4 calculates total contribution margin through $n(\text{CM})$. With sales of 800 units, the total contribution margin is $800(\$44) = \$35,200$.

Step 3: Total fixed costs are known: $\text{TFC} = \$23,000$.

Step 4: Apply Formula 5.4, which translates to $\text{Net Income} = \text{Total contribution margin} - \text{Total fixed costs}$, or $\$35,200 - \$23,000 = \$12,200$.

**TOP
SECRET**

Paths To Success

When you work with Formula 5.4, sometimes unit information may not be known. Instead, you might just have a single aggregate number representing the total contribution margin for which somebody has already taken the total revenue and subtracted the total variable costs. In this case, skip step 1 and take the provided total contribution margin as the answer for step 2 with no calculations necessary.

Example 5.1C: The Contribution Margin for Your Internet Business														
Continuing with Examples 5.1A and 5.1B, calculate the unit contribution margin and net income using the contribution margin approach. From the previous examples, recall the unit variable cost is \$6.43, unit selling price is \$10, total fixed costs are \$638.03, and the projected sales are 430 units.														
Plan	Calculate the unit contribution margin (CM) followed by the net income (NI) using the contribution margin approach.													
Understand	What You Already Know The cost, price, and sales information are known: $\text{VC} = \$6.43$ $\text{S} = \$10$ $\text{TFC} = \$638.03$ $n = 430$	How You Will Get There Step 1: Calculate the unit contribution margin using Formula 5.3. Step 2: Calculate the total contribution margin by multiplying the unit contribution margin times the level of output. Step 3: Determine the total fixed costs. Step 4: To calculate net income, apply Formula 5.4.												
Perform	Step 1: $\text{CM} = \$10.00 - \$6.43 = \$3.57$ Step 2: $n(\text{CM}) = 430(\$3.57) = \$1,535.10$ Step 3: $\text{TFC} = \$638.03$ Step 4: $\text{NI} = \$1,535.10 - \$638.03 = \$897.07$	Calculator Instructions <table border="1"> <thead> <tr> <th>FC</th><th>VC</th><th>P</th><th>PFT</th><th>Q</th></tr> </thead> <tbody> <tr> <td>638.03</td><td>6.43</td><td>10</td><td>Answer: 897.07</td><td>430</td></tr> </tbody> </table>			FC	VC	P	PFT	Q	638.03	6.43	10	Answer: 897.07	430
FC	VC	P	PFT	Q										
638.03	6.43	10	Answer: 897.07	430										
Present	When a product sells for \$10, \$6.43 goes toward paying for the variable costs of your business, leaving \$3.57 as your unit contribution margin. This means that for every unit increase in sales, your net income rises by \$3.57. Thus, if you sell 430 units you have a total contribution margin of \$1,535.10, which results in a net income of \$897.07 after removing fixed costs of \$638.03.													

Contribution Rates

It is difficult to compare different products and their respective dollar amount contribution margins if their selling prices and costs vary widely. For example, how do you compare a unit contribution margin of \$1,390 (selling price of \$2,599.99) on a big screen television to a unit contribution margin of \$0.33 on a chocolate bar (selling price of \$0.67)? On a per-unit basis, which contributes relatively more to fixed costs? To facilitate these comparisons, the products must be placed on equal terms, requiring you to convert all dollar amount contribution margins into percentages. A **contribution rate** is a contribution margin expressed as a percentage of the selling price.



The Formula

Your choice between two formulas for calculating a contribution rate depends on whether you have unit information or only aggregate information. Both formulas are adaptations of Formula 2.2 (Rate, Portion, Base).

If unit information is known, including the unit variable cost and unit selling price, then calculate the contribution rate using unit information as expressed in Formula 5.5.

CR is Contribution Rate: This represents the percentage of the unit selling price that is available to pay for all of the fixed costs of the business. When all fixed costs are paid for, this percentage is the portion of the selling price that will remain as profit. In Formula 2.2, this is the rate.

CM is Unit Contribution: Calculated through Formula 5.3, this is the unit dollar amount that is left over after you subtract the unit variable costs from the unit selling price of the product. In Formula 2.2, this is the portion.

Formula 5.5: Contribution Rate if Unit Information Is Known):
$$\text{CR} = \frac{\text{CM}}{\text{S}} \times 100$$

S is Unit Selling Price: The unit selling price of the product. In Formula 2.2, this is the base.

× 100 is Percent Conversion: The contribution rate is always expressed as a percentage.

If any unit information, including the unit variable cost or unit selling price, is unknown or unavailable, then you cannot apply Formula 5.5. Sometimes only aggregate information is known. When total revenue and total variable costs for any product are known or can at least be calculated, then you must calculate the contribution rate from the aggregate information as expressed in Formula 5.6.

CR is Contribution Rate: This represents the percentage of the unit selling price that is available to pay for all of the fixed costs of the business.

TVC is Total Variable Costs: This is the total cost associated with the level of output. When total variable costs are subtracted from the total revenue, the remainder represents the portion of money left over to pay the fixed costs.

Formula 5.6: Contribution Rate if Aggregate Information Is Known:
$$\text{CR} = \frac{\text{TR} - \text{TVC}}{\text{TR}} \times 100$$

TR is Total Revenue: This is the total amount of money that the company has received from the sale of the product.

× 100 is Percent Conversion: The contribution rate is always expressed as a percentage.



How It Works

Follow these steps to calculate a contribution rate:

Step 1: Identify the required variables and calculate the margin, if needed.

- If unit information is known, this requires you to calculate the unit contribution margin. Otherwise, calculate the unit contribution margin by applying Formula 5.3. You must identify the unit selling price.
- If aggregate information is known, you need to identify total revenue and total variable costs.

Step 2: Calculate the contribution rate.

- If unit information is known, apply Formula 5.5.
- If only aggregate information is known, apply Formula 5.6.

As an example of these steps, recall earlier that you wanted to compare the relative contributions of the big screen television and the chocolate bar. The television sells for \$2,599.99 and has a unit contribution margin of \$1,390. The chocolate bar sells for \$0.67 and has a unit contribution margin of \$0.33. Notice that the information being provided is on a per-unit basis. Calculate the contribution rate of each product.

Step 1: The contribution margins are known. For the television, $CM = \$1,390$, and for the chocolate bar, $CM = \$0.33$.

Step 2: Applying Formula 5.5, the television $CR = \$1,390.00 \div \$2,599.99 \times 100 = 53.4617\%$, while the chocolate bar $CR = \$0.33 \div \$0.67 \times 100 = 49.2537\%$. It is now evident from the contribution

rate that 4.208% more of the television's selling price is available to pay for fixed costs as compared to the chocolate bar's price.

Now change the facts. This time, assume there is no unit information. Instead, the television's total revenue is \$129,999.50 and associated total variable costs are \$60,499.50. The chocolate bar has total revenue of \$3,886.00 with total variable costs of \$1,972.00. Based on this information, you are to determine the product with the higher contribution rate.

Step 1: The aggregate numbers are known for both products. For the television, $TR = \$129,999.50$ and $TVC = \$60,499.50$. For the chocolate bar, $TR = \$3,886.00$ and $TVC = \$1,972.00$.

Step 2: Applying Formula 5.6, the television $CR = \frac{\$129,999.50 - \$60,499.50}{\$129,999.50} \times 100 = 53.4617\%$, while the chocolate bar $CR = \frac{\$3,886 - \$1,972}{\$3,886} \times 100 = 49.2537\%$. You have reached the same conclusion as above.



Give It Some Thought

In each of the following situations, what would happen to the contribution rate and why?

1. The selling price is raised.
2. The hourly wages of production workers are increased to match the increase in the consumer price index.
3. The level of output decreases.
4. Your insurance company lowers your insurance premiums because your company has had no claims in the past year.

Solutions:

1. The contribution rate increases, since raising the price increases the unit contribution margin.
2. The contribution rate decreases, since total variable costs rise, eroding some of the unit contribution margin.
3. There is no effect on the contribution rate. The level of output is not a factor in calculating the contribution rate.
4. There is no effect on the contribution rate since the insurance premiums are a fixed cost. Fixed costs are not a factor in calculating contribution rates.

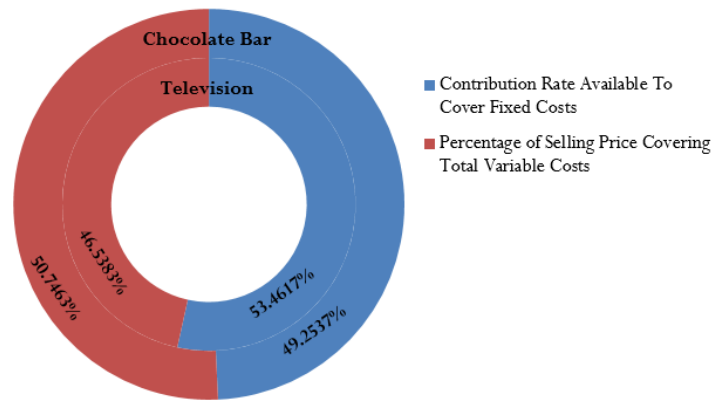
Example 5.1D: Understanding the Contribution Rate of Your Internet Business

Continuing with your ongoing Internet business from the three previous examples, calculate the contribution rate using both the unit and aggregate methods and show how they arrive at the same number. Remember that you are selling products for \$10 each, your unit contribution margin is \$3.57, total revenue is projected at \$4,300, and total variable costs are \$2,764.90.

Plan

Your goal is to calculate the contribution rate (CR) using both the unit and aggregate methods.

Comparing the Contribution Rates



Understand	What You Already Know Step 1: Prices, margins, revenue, and costs are known: $S = \$10$ $CM = \$3.57$ $TR = \$4,300$ $TVC = \$2,764.90$	How You Will Get There Step 2a: Using the unit method, apply Formula 5.5. Step 2b: Using the aggregate method, apply Formula 5.6.
Perform	Step 2a: $CR = \frac{\$3.57}{\$10.00} \times 100 = 35.7\%$ Step 2b: $CR = \frac{\$4,300.00 - \$2,764.90}{\$4,300.00} \times 100 = 35.7\%$	
Present	Under both the unit and aggregate method, your contribution rate equals 35.7%. This means that every time you sell a \$10 product, 35.7% of the revenue remains after recovering the cost of the product.	

Putting It All Together

You have studied costs, volume, and net income in this section. So far, you have considered each of these concepts separately while you worked through the various applications. It is time to put the types of costs, unit variable cost, net income, sales, contribution margin, and contribution rate together so that you can see all of these concepts in a single scenario. Look at the following example.

Example 5.1E: Running a Pizza Delivery Business			
In the commercial section of the newspaper you come across an ad for a pizza delivery business for sale. Upon inquiry, you discover that the owner, who wants to sell the business and then retire, has four salaried employees and owns two delivery vehicles. He invites you to look through his books, where you acquire the following information:			
Employees:		Business:	
Owner's salary	\$5,000 per month	Phone	\$85 per month
Employee salaries and premiums	\$2,000 per month each	Business insurance	\$3,600 per year
		Building lease	\$2,000 per month
Delivery Vehicles:		Operations:	
Car insurance	\$1,200 per year per vehicle	Pizza ingredients, materials, and packaging	\$19,125 per month
Fuel	\$1,125 per month per vehicle	Selling price of pizza	\$9 per pizza
Oil changes	\$225 per month per vehicle	Average number of pizzas sold	4,500 per month
Vehicle maintenance and repairs	\$562.50 per month per vehicle		
Assume that this is all of the key information. You need to understand this business and therefore want to determine: - A unit variable cost - A typical monthly net income for the business - The contribution margin per pizza - The contribution rate			
Plan	All calculations require the sorting of costs into fixed and variable components, which will be your first step. You then calculate a unit variable cost (VC) using the historical operations information. Once you obtain this, you can calculate the net income (NI), contribution margin (CM), and contribution rate (CR).		

Understand	What You Already Know You have unit information on costs, volume, and revenue, as listed above.	How You Will Get There a. Unit variable cost: Step 1: Sort the costs into fixed and variable categories. Total the monthly fixed costs. Step 2: Total the monthly variable costs. Step 3: Calculate the unit variable cost by applying Formula 5.1. b. Net income: Step 1: Calculate total revenue at the level of output. Step 2: Calculate total costs at the level of output. Step 3: Calculate monthly net income by applying Formula 5.2. c. and d. Contribution margin and rate: Step 1: Calculate the contribution margin by applying Formula 5.3. Step 2: Calculate the contribution rate by applying Formula 5.5.																															
	a. Unit variable cost: Steps 1 and 2: <table><thead><tr><th colspan="2">Fixed Costs per Month</th><th colspan="2">Variable Costs per Month</th></tr></thead><tbody><tr><td>Owner's salary</td><td>\$5,000</td><td>Fuel</td><td>\$1,125 × 2 vehicles = \$2,250</td></tr><tr><td>Employee salaries and premiums</td><td>\$2,000 × 4 = \$8,000</td><td>Oil changes</td><td>\$225 × 2 vehicles = \$450</td></tr><tr><td>Car insurance</td><td>\$1,200 × 2 cars ÷ 12 months = \$200</td><td>Vehicle maintenance and repairs</td><td>\$562.50 × 2 vehicles = \$1,125</td></tr><tr><td>Phone</td><td>\$85</td><td>Pizza ingredients, materials, and packaging</td><td>\$19,125</td></tr><tr><td>Business insurance</td><td>\$3,600 ÷ 12 months = \$300</td><td>TOTAL VARIABLE COSTS</td><td>TVC = \$22,950 per month</td></tr><tr><td>Building lease</td><td>\$2,000</td><td></td><td></td></tr><tr><td>TOTAL FIXED COSTS</td><td>TFC = \$15,585 per month</td><td></td><td></td></tr></tbody></table> Step 3: $VC = \frac{\\$2,250 + \\$450 + \\$1,125 + \\$19,125}{4,500 \text{ pizzas}} = \\$5.10 \text{ per pizza}$ b. Net income: Step 1: Total Revenue = 4,500(\$9) = \$40,500 Step 2: TFC = \$15,585; TVC = 4,500(\$5.10) = \$22,950 Step 3: NI = \$40,500 – (\$15,585 + \$22,950) = \$40,500 – \$38,535 = \$1,965 c. and d. Contribution margin and rate: Step 1: CM = \$9.00 – \$5.10 = \$3.90 Step 2: CR = $\frac{\\$3.90}{\\$9.00} \times 100 = 43.\bar{3}\%$		Fixed Costs per Month		Variable Costs per Month		Owner's salary	\$5,000	Fuel	\$1,125 × 2 vehicles = \$2,250	Employee salaries and premiums	\$2,000 × 4 = \$8,000	Oil changes	\$225 × 2 vehicles = \$450	Car insurance	\$1,200 × 2 cars ÷ 12 months = \$200	Vehicle maintenance and repairs	\$562.50 × 2 vehicles = \$1,125	Phone	\$85	Pizza ingredients, materials, and packaging	\$19,125	Business insurance	\$3,600 ÷ 12 months = \$300	TOTAL VARIABLE COSTS	TVC = \$22,950 per month	Building lease	\$2,000			TOTAL FIXED COSTS	TFC = \$15,585 per month	
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Perform	Calculator Instructions <table><tr><td>FC</td><td>VC</td><td>P</td><td>PFT</td><td>Q</td></tr><tr><td>15585</td><td>5.1</td><td>9</td><td>Answer: 1,965</td><td>4500</td></tr></table>		FC	VC	P	PFT	Q	15585	5.1	9	Answer: 1,965	4500																					
	FC	VC	P	PFT	Q																												
15585	5.1	9	Answer: 1,965	4500																													
Present	This is a profitable business. For an average month, the business incurs \$15,585 in fixed costs along with \$22,950 in variable costs, which works out to a unit variable cost of \$5.10 for each of the 4,500 pizzas sold. With total revenue of \$40,500, after removing both fixed and variable costs, net income is \$1,965 per month. Every pizza, after paying for the unit variable costs, has \$3.90, or 43.3%, left over.																																
	Pizza by the Numbers - Total Fixed Costs -- >\$15,585 per month - Total Variable Costs --> \$22,950 per month - Net Income --> \$1,965 per month																																



Section 5.1 Exercises

In each of the following questions, round all of the money and percentages to two decimals.

Mechanics

- Classify each of the following costs as fixed costs, variable costs, or blended costs. If a cost is blended, separate it into its fixed and variable components.
 - Natural gas bill for \$15 per month plus $\$0.33/\text{m}^2$.
 - A chief executive officer salary of \$240,000 per year.
 - An author earning a royalty of 5% of sales.
 - Placing a commercial on television for \$300,000.
 - A cellphone bill for \$40 per month plus $\$0.25/\text{minute}$ for long distance.
 - Hourly production worker wages of \$18/hr.
 - Sales staff who are compensated at a salary of \$1,000 per month plus 15% of sales.

For questions 2–7, solve for the unknown variables (identified with a ?) based on the information provided.

	Total Fixed Costs	Total Variable Costs	Unit Variable Cost	Selling Price	Total Revenue	Level of Output	Net Income	Contribution Rate	Unit Contribution Margin
2.	\$5,000	\$6,600	?	\$13	?	?	\$4,000	?	\$7.50
3.	\$2,000	?	\$5	\$10	\$10,000	?	?	?	?
4.	?	?	?	\$75	\$60,000	?	\$14,500	35%	?
5.	\$18,000	\$45,000	?	?	\$84,600	1,800	?	?	?
6.	?	?	?	?	\$78,000	3,000	\$18,000	?	\$13
7.	?	\$94,050	\$75.24	?	?	?	−\$19,500	38%	?

Applications

- In the current period, a packing company in Salmon Arm, British Columbia, had fixed costs of \$228,000 and a total cost of \$900,000 while maintaining a level of output of 6,720 units. Next period sales are projected to rise by 20%. What total cost should the packing company project?
- Fred runs a designer candle-making business out of his basement. He sells the candles for \$15 each, and every candle costs him \$6 to manufacture. If his fixed costs are \$2,300 per month, what is his projected net income or loss next month, for which he forecasts sales of 225 units?
- A college print shop leases an industrial photo copier for \$1,500 per month plus 1.5¢ for every page. Additional printing costs are estimated at 2¢ per page, which covers toner, paper, labour, and all other incurred costs. If copies are made for students at 10¢ each, determine the following:
 - How does net income change with every 100 copies sold?
 - What is the monthly net income if, on average, the shop makes 25,000 copies for students each month?
- Gayle is thinking of starting her own business. Total fixed costs are \$19,000 per month and unit variable costs are estimated at \$37.50. From some preliminary studies that she completed, she forecasts sales of 1,400 units at \$50 each, 1,850 units at \$48 each, 2,500 units at \$46 each, and 2,750 units at \$44 each. What price would you recommend Gayle set for her products?
- Last year, a children's store clothing franchise had total sales of \$743,000. If its total fixed costs were \$322,000 and net income was \$81,000, what was its contribution rate?
- What level of output would generate a net income of \$15,000 if a product sells for \$24.99, has unit variable costs of \$9.99, and total fixed costs of \$55,005?

14. In the current year, a small hotel franchise had sales of \$1,800,000, fixed costs of \$550,000, and total variable costs of \$750,000. Next year, sales are forecast to increase by 25% but costs will remain the same. How much will net income change (in dollars)?

Challenge, Critical Thinking, & Other Applications

15. A chemical company reported the following on its income statement for one of its divisions:

Sales	\$6,000,000
Total Fixed Costs	\$2,000,000
Total Variable Costs	\$3,200,000
Total Costs	<u>\$5,200,000</u>
Net Income	\$800,000

Calculate the total contribution margin in dollars and the contribution rate for this division.

16. A consumer goods company is budgeting for next year. For one of its brands, it projects it will operate at 80% production capacity next year and forecasts the following:

Sales	\$80,000,000
Total Fixed Costs	\$20,000,000
Total Variable Costs	\$50,000,000
Total Costs	<u>\$70,000,000</u>
Net Income	\$10,000,000

Determine the net income if sales are higher than expected and it realizes 90% production capacity.

17. Through market research, a marketing manager determines that consumers are willing to pay 5% more for the company's product. However, some of their customers would not like this price increase, so the level of output would drop by 5%. Should the marketing manager leave things as they are or increase the price by 5%? Justify your solution.
18. Francesca is a departmental manager in the women's wear department for a department store. She believes that if she places her line of \$100 dresses on sale at 20% off, she would see her sales rise by 75%. The contribution rate on her dresses is 50%. On a strictly financial basis, should she place the dresses on sale?

Use the following information for questions 19 and 20:

TFC = \$3,200,000.00	S = \$99.97
TVC = \$5,009,440.00	n = 131,000

19. Calculate the following information: unit variable cost, total revenue, net income, unit contribution margin, total contribution margin, and contribution rate.
20. Determine a new value for net income if the following situations occur:
- Fixed costs rise by 10%.
 - The selling price is lowered by 25% during a sale, resulting in 50% more volume.
 - Fixed costs are lowered by 5%, unit variable costs rise by 3%, the price is lowered by 5%, and the level of output rises 10%.

5.2: Break-Even Analysis

(Sink or Swim)

Should you start up the Internet business described in the last section? Right now, all you have are some projected costs and a forecasted level of sales. You imagine you are going to sell 400 units. Is that possible? Is it reasonable to forecast this many sales?

Now you may say to yourself, “400 units a month . . . that's about 13 per day. What's the big deal?” But let's gather some more information. What if you looked up your industry in Statistics Canada data and learned that the product in question sells just 1,000 units per month in total? Statistics Canada also indicates that there are eight existing companies selling these products. How does that volume of 400 units per month sound now? Unless you are revolutionizing your industry, it is unlikely you will receive a 40% market share in your first month of operations. With so few unit sales in the industry and too many competitors, you might be lucky to sell 100 units. If this is the case, are you still profitable?

Simply looking at the fixed costs, variable costs, potential revenues, contribution margins, and typical net income is not enough. Ultimately, all costs in a business need to be recovered through sales. Do you know how many units have to be sold to pay your bills? The answer to this question helps assess the feasibility of your business idea.

What Is Break-Even Analysis?

If you are starting your own business and head to the bank to initiate a start-up loan, one of the first questions the banker will ask you is your break-even point. You calculate this number through **break-even analysis**, which is the analysis of the relationship between costs, revenues, and net income with the sole purpose of determining the point at which total revenue equals total cost. This **break-even point** is the level of output (in units or dollars) at which all costs are paid but no profits are earned, resulting in a net income equal to zero. To determine the break-even point, you can calculate a break-even analysis in two different ways, involving either the number of units sold or the total revenue in dollars. Each of these two methods is discussed in this section.

Method 1: Break-Even Analysis in Units

In this method, your goal is to determine the level of output that produces a net income equal to zero. This method requires unit information, including the unit selling price and unit variable cost.

It is helpful to see the relationship of total cost and total revenue on a graph. Assume that a company has the following information:

$$\text{TFC} = \$400 \quad S = \$100 \quad \text{VC} = \$60$$

The graph shows dollar information on the y -axis and the level of output on the x -axis. Here is how you construct such a graph:

1. Plot the total costs:

- a. At zero output you incur the total fixed costs of \$400. Denote this as Point 1 (0, \$400).
- b. As you add one level of output, the total cost rises in the amount of the unit variable cost. Therefore, total cost is $\text{TFC} + n(\text{VC}) = \$400 + 1(\$60) = \$460$. Denote this as Point 2 (1, \$460).
- c. As you add another level of output (2 units total), the total cost rises once again in the amount of the unit variable cost, producing $\$400 + 2(\$60) = \$520$. Denote this as Point 3 (2, \$520).
- d. Repeat this process for each subsequent level of output and plot it onto the figure. The red line plots these total costs at all levels of output.

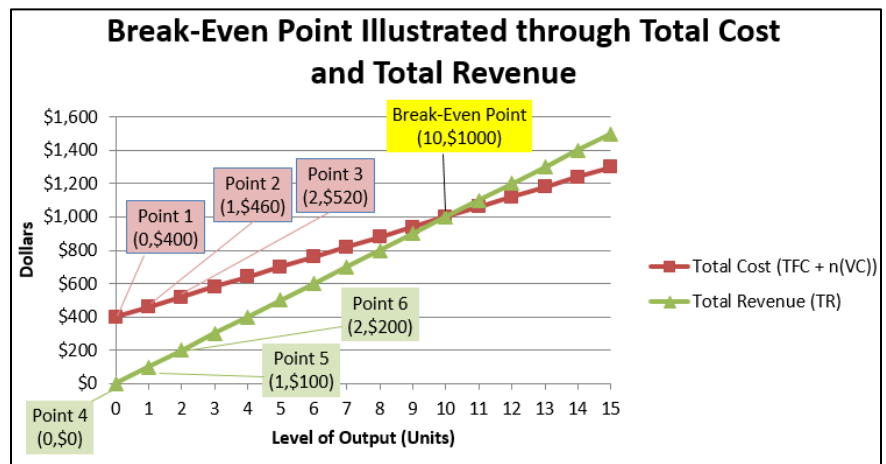
2. Plot the total revenue:

- a. At zero output, there is no revenue. Denote this as Point 4 (0, \$0).

b. As you add one level of output, total revenue rises by the selling price of the product. Therefore, total revenue is $n(S) = 1(\$100) = \$100$. Denote this as Point 5 (1, \$100).

c. As you add another level of output (2 units total), the total revenue rises once again in the amount of the selling price, producing $2(\$100) = \$200$. Denote this as Point 6 (2, \$200).

d. Repeat this process for each subsequent level of output and plot it onto the figure. The green line plots the total revenue at all levels of output.



The purpose of break-even analysis is to determine the point at which total cost equals total revenue. The graph illustrates that the break-even point occurs at an output of 10 units. At this point, the total cost is $\$400 + 10(\$60) = \$1,000$, and the total revenue is $10(\$100) = \$1,000$. Therefore, the net income is $\$1,000 - \$1,000 = \$0$; no money is lost or gained at this point.



The Formula

Recall that Formula 5.2 states that the net income equals total revenue minus total costs. In break-even analysis, net income is set to zero, resulting in

$$0 = n(S) - (TFC + n(VC))$$

Rearranging and solving this formula for n gives the following:

$$0 = n(S) - TFC - n(VC)$$

$$TFC = n(S) - n(VC)$$

$$TFC = n(S - VC)$$

$$\frac{TFC}{(S - VC)} = n$$

Formula 5.3 states that $CM = S - VC$; therefore, the denominator becomes just CM. The calculation of the break-even point using this method is thus summarized in Formula 5.7.

n is Breakeven Level of Output (Units): This is the level of output in units that produces a net income equal to zero.

TFC is Total Fixed Costs: The total of all costs that are not affected by the level of output.

Formula 5.7: Unit Break-Even: $n = \frac{TFC}{CM}$

CM is Unit Contribution Margin: The amount of money left over per unit after you have recovered your variable costs. Calculate this by taking the unit selling price and subtracting the unit variable cost (Formula 5.3). You use this money to pay off your fixed costs.

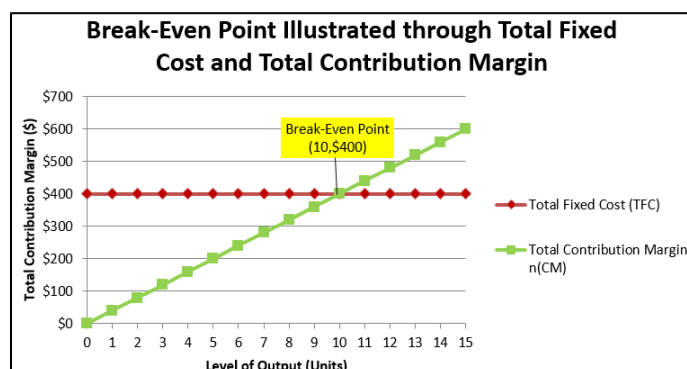


How It Works

Follow these steps to calculate the break-even point in units:

Step 1: Calculate or identify the total fixed costs (TFC).

Step 2: Calculate the unit contribution margin (CM) by applying any needed techniques or formulas.



Step 3: Apply Formula 5.7.

Continuing with the example that created the graph on the previous page:

Step 1: Total fixed costs are known, $TFC = \$400$.

Step 2: The unit contribution margin is $\$100 - \$60 = \$40$. For each unit sold this is the amount left over that can be applied against total fixed costs.

Step 3: Applying Formula 5.7 results in $n = \$400 \div \$40 = 10$ units.



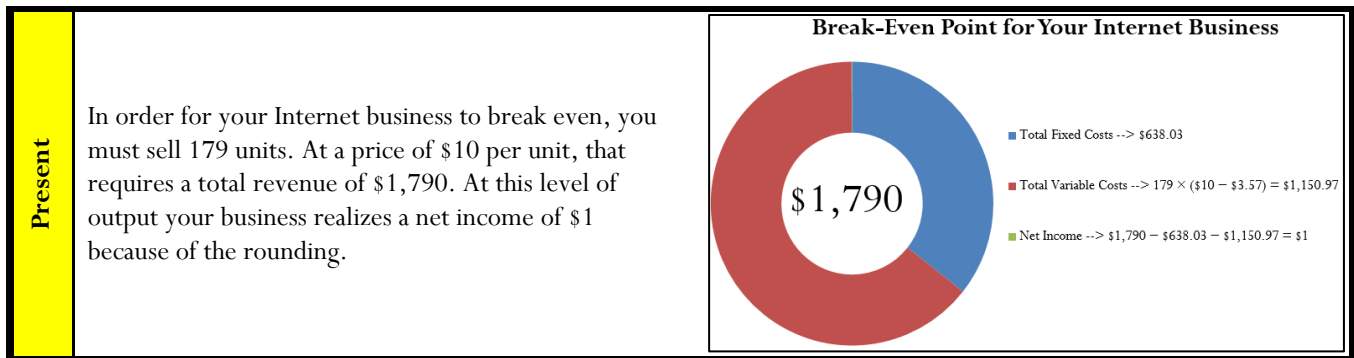
Important Notes

When you calculate the break-even units, the formula may produce a number with decimals. For example, a break-even point might be 324.39 units. How should you handle the decimal? A partial unit cannot be sold, so the rule is always to round the level of output up to the next integer, regardless of the decimal. Why? The main point of a break-even analysis is to show the point at which you have recovered all of your costs. If you round the level of output down, you are 0.39 units short of recovering all of your costs. In the long-run, you always operate at a loss, which ultimately puts you out of business. If you round the level of output up to 325, all costs are covered and a tiny dollar amount, as close to zero as possible, is left over as profit. At least at this level of output you can stay in business.

Example 5.2A: The Break-Even Units for Your Planned Internet Business

Recall from Section 5.1 the Internet business explored throughout Examples 5.1A to 5.1D. Now let's determine the break-even point in units. As previously calculated, the total fixed costs are \$638.03 and the unit contribution margin is \$3.57.

Plan	Calculate the break-even point in units sold, or n at break-even.													
Understand	What You Already Know Step 1: The total fixed costs are known: $TFC = \$638.03$. Step 2: The contribution rate is known: $CM = \$3.57$.		How You Will Get There Step 3: Apply Formula 5.7.											
	Step 3: $= \frac{\$638.03}{\$3.57} = 178.719888$. Round this up to 179.		Calculator Instructions <table><tr><td>FC</td><td>VC</td><td>P</td><td>PFT</td><td>Q</td></tr><tr><td>638.03</td><td>6.43</td><td>10</td><td>0</td><td>Answer: 178.719888</td></tr></table>			FC	VC	P	PFT	Q	638.03	6.43	10	0
FC	VC	P	PFT	Q										
638.03	6.43	10	0	Answer: 178.719888										



Method 2: Break-Even Analysis in Dollars

The income statement of a company does not display unit information. All information is aggregate, including total revenue, total fixed costs, and total variable costs. Typically, no information is listed about unit selling price, unit variable costs, or the level of output. Without this unit information, it is impossible to apply Formula 5.7.

The second method for calculating the break-even point relies strictly on aggregate information. As a result, you cannot calculate the break-even point in units. Instead, you calculate the break-even point in terms of aggregate dollars expressed as total revenue.



The Formula

To derive the break-even point in dollars, once again start with Formula 5.2, where total revenue at break-even less total fixed costs and total variable costs must equal a net income of zero:

$$NI = TR - (TFC + TVC)$$

$$0 = TR - (TFC + TVC)$$

Rearranging this formula for total revenue gives:

$$0 = TR - TFC - TVC$$

$$TR = TFC + TVC$$

Thus, at the break-even point the total revenue must equal the total cost. Substituting this value into the numerator of Formula 5.6 gives you:

$$CR = (TR - TVC) / TR \times 100$$

$$CR = ((TFC + TVC) - TVC) / TR \times 100$$

$$CR = TFC / TR \times 100$$

A final rearrangement results in Formula 5.8, which expresses the break-even point in terms of total revenue dollars.

TR is Total Revenue at Break-even: The total amount of dollars that the company must earn as revenue to pay for all of the fixed and variable costs. At this level of revenue, the net income equals zero.

TFC is Total Fixed Costs: The total of all costs that are not affected by the level of output.

Formula 5.8: Dollar Break-Even:

$$TR = \frac{TFC}{CR}$$

CR is Contribution Rate: The percentage of the selling price, expressed in decimal format, available to pay for fixed costs.



How It Works

Follow these steps to calculate the break-even point in total revenue dollars:

Step 1: Calculate or identify the total fixed costs (TFC).

Step 2: Calculate the contribution rate (CR), by applying any needed techniques or formulas. If not provided, typically the CR is calculated using Formula 5.6, which requires aggregate information only.

Step 3: Apply Formula 5.8 to calculate the break-even point in dollars.

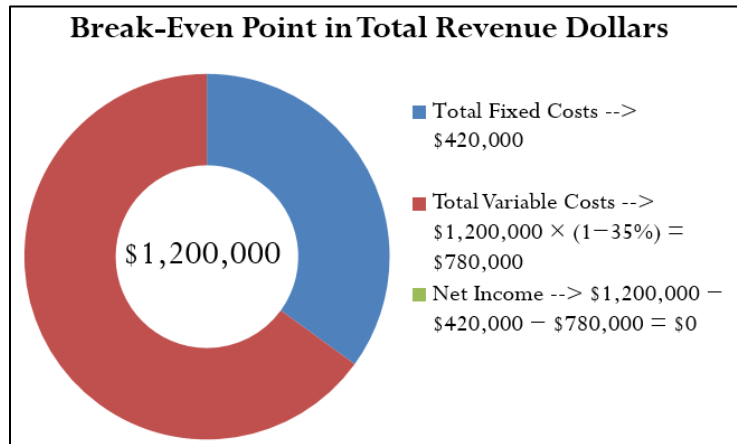
Assume that you are looking at starting your own business. The fixed costs are generally easier to calculate than the variable costs. After running through the numbers, you determine that your total fixed costs are \$420,000, or $TFC = \$420,000$. You are not sure of your variable costs but

need to gauge your break-even point. Many of your competitors are publicly traded companies, so you go online and pull up their annual financial reports. After analyzing their financial statements, you see that your competitors have a contribution rate of 35%, or $CR = 0.35$, on average. What is your estimate of your break-even point in dollars?

Step 1: Total fixed costs are $TFC = \$420,000$.

Step 2: The estimated contribution rate is $CR = 0.35$.

Step 3: Applying Formula 5.8 results in $TR = \$420,000 \div 0.35 = \$1,200,000$. If you average a similar contribution rate, you require total revenue of \$1,200,000 to cover all costs, which is your break-even point in dollars.



Important Notes

You need to be very careful with the interpretation and application of a break-even number. In particular, the break-even must have a point of comparison, and it does not provide information about the viability of the business.

Break-Even Points Need to Be Compared. The break-even number by itself, whether in units or dollars, is meaningless. You need to compare it against some other quantity (or quantities) to determine the feasibility of the number you have produced. The other number needs to be some baseline that allows you to grasp the scope of what you are planning. This baseline could include but is not limited to the following:

- Industry sales (in units or dollars)
- Number of competitors fighting for market share in your industry
- Production capacity of your business

For example, in your Internet business the break-even point is 179 units per month. Is that good? In the section opener, you explored a possibility where your industry had total monthly sales of 1,000 units and you faced eight competitors. A basic analysis shows that if you enter the industry and if everyone split the market evenly, you would have sales of 1,000 divided by nine companies, equal to 111 units each. To just pay your bills, you would have to sell almost 61% higher than the even split and achieve a 17.9% market share. This doesn't seem very likely, as these other companies are already established and probably have satisfied customers of their own that would not switch to your business.

Break-Even Points Are Not Green Lights. A break-even point alone cannot tell you to do something, but it can tell you *not* to do something. In other words, break-even points can put up red lights, but at no point does it give you the green light. In the above scenario, your break-even of 179 units put up a whole lot of red lights since it does not seem feasible to obtain. However, what if your industry sold 10,000 units instead of 1,000 units? Your break-even would now be a 1.79% market

share (179 units out of 10,000 units), which certainly seems realistic and attainable. This does not mean “Go for it,” however. It just means that from a strictly financial point of view breaking even seems possible. Whether you can actually sell that many units depends on a whole range of factors beyond just a break-even number. For instance, if your Google ad is written poorly you might not be able to generate that many sales. The break-even analysis cannot factor in this non-quantitative variable, and for that reason it cannot offer a “go ahead.”



Give It Some Thought

What would happen to the break-even point in each of the following situations? Would it increase, decrease, or remain the same?

1. The unit contribution margin increases.
2. The total fixed costs increase.
3. The contribution rate decreases.

Solutions:
 1. Decrease. In Formula 5.7, the denominator is larger, producing a lower break-even.
 2. Increase. In both Formulas 5.7 and 5.8, the numerator is larger, producing a higher break-even.
 3. Increase. In Formula 5.8, the denominator is smaller, producing a higher break-even.

Example 5.2B: Determining the Break-Even Dollars

In the annual report to shareholders, a manufacturing company reported total gross sales of \$7,200,000, total variable costs of \$4,320,000, and total fixed costs of \$2,500,000. Determine its break-even point in dollars.

Plan	Calculate the dollar break-even point, which is the total revenue (TR) at break-even.	
Understand	What You Already Know Step 1: The total fixed costs are known: $TFC = \$2,500,000$. Other known information includes the following: $TR = \$7,200,000$ $TVC = \$4,320,000$	How You Will Get There Step 2: Calculate the contribution rate by applying Formula 5.6. Step 3: Apply Formula 5.8.
Perform	Step 2: $CR = \frac{\$7,200,000 - \$4,320,000}{\$7,200,000} \times 100 = \frac{\$2,880,000}{\$7,200,000} \times 100 = 40\%$ Step 3: $TR = \frac{\$2,500,000}{40\%} = \frac{\$2,500,000}{0.4} = \$6,250,000$	
Present	The manufacturing company achieves its break-even point at \$6,250,000 in total revenue. At this point, total fixed costs are \$2,500,000 and total variable costs are \$3,750,000, producing a net income of zero.	



Section 5.2 Exercises

In each of the following questions, round all of the money and percentages to two decimals unless otherwise specified.

Mechanics

1. Franklin has started an ink-jet print cartridge refill business. He has invested \$2,500 in equipment and machinery. The cost of refilling a cartridge including labour, ink, and all other materials is \$4. He charges \$14.95 for his services. How many cartridges does he need to refill to break even?

2. Hasbro manufactures a line of children's pet toys. If it sells the toy to distributors for \$2.30 each while variable costs are 75¢ per toy, how many toys does it need to sell to recover the fixed cost investment in these toys of \$510,000? What total revenue would this represent?
3. You are thinking of starting your own business and want to get some measure of feasibility. You have determined that your total fixed costs would be \$79,300. From annual business reports and competitive studies, you estimate your contribution rate to be 65%. What is your break-even in dollars?
4. If a business has total revenue of \$100,000, total variable costs of \$60,000, and total fixed costs of \$20,000, determine its break-even point in dollars.
5. If the break-even point is 15,000 units, the selling price is \$95, and the unit variable cost is \$75, what are the company's total fixed costs?
6. Louisa runs a secretarial business part time in the evenings. She takes dictation or handwritten minutes and converts them into printed word-processed documents. She charges \$5 per page for her services. Including labour, paper, toner, and all other supplies, her unit variable cost is \$2.50 per page. She invested \$3,000 worth of software and equipment to start her business. How many pages will she need to output to break even?
7. If your organization has a contribution rate of 45% and knows the break-even point is \$202,500, what are your organization's total fixed costs?
8. What is the unit contribution margin on a product line that has fixed costs of \$1,800,000 with a break-even point of 360,000 units?

Applications

9. Ashley rebuilds old laptops as a home hobby business. Her variable costs are \$125 per laptop and she sells them for \$200. She has determined that her break-even point is 50 units per month. Determine her net income for a month in which she sells 60 units.

10. A snowboard company reported the following figures last week for its new snowboard:

Sales	\$70,000
Total Fixed Costs	\$19,000
Total Variable Costs	\$35,000
Total Costs	<u>\$54,000</u>
Net Income	\$16,000

If the above numbers represent 70% operational capacity, express the weekly break-even point in dollars as a percentage of maximum capacity.

11. Shardae is starting a deluxe candy apple business. The cost of producing one candy apple is \$4.50. She has total fixed costs of \$5,000. She is thinking of selling her deluxe apples for \$9.95 each.
 - a. Determine her unit break-even point at her selling price of \$9.95.
 - b. Shardae thinks her price might be set too high and lowers her price to \$8.95. Determine her new break-even point.
 - c. An advertising agency approaches Shardae and says people would be willing to pay the \$9.95 if she ran some "upscale" local ads. They would charge her \$1,000. Determine her break-even point.
 - d. If she wanted to maintain the same break-even units as determined in (a), what would the price have to be to pay for the advertising?
12. Robert is planning a wedding social for one of his close friends. Costs involve \$865 for the hall rental, \$135 for a liquor licence, \$500 for the band, and refreshments and food from the caterer cost \$10 per person. If he needs to raise \$3,000 to help his friend with the costs of his wedding, what price should he charge per ticket if he thinks he can fill the social hall to its capacity of 300 people?
13. Boston Beer Company, the brewer of Samuel Adams, reported the following financial information to its shareholders:

Total Revenue	\$388,600,000
Total Variable Costs	\$203,080,000
Total Fixed Costs	\$182,372,000
Total Costs	<u>\$385,452,000</u>
Net Income	\$3,148,000

If this represented sales of 2,341,000 barrels of beer, determine its break-even point in units and dollars.

14. In the beverage industry, PepsiCo and The Coca-Cola Company are the two big players. The following financial information, in millions of dollars, was reported to its shareholders:

	PepsiCo	The Coca-Cola Company
Total Revenue	\$14.296	\$21.807
Total Variable Costs	\$7.683	\$12.663
Total Fixed Costs	\$6.218	\$8.838
Total Costs	<u>\$13.901</u>	<u>\$21.501</u>
Net Income	\$0.395	\$0.306

Compare the break-even points in total dollars between the two companies based on these reports.

Challenge, Critical Thinking, & Other Applications

15. Calculate the following:
- By what percentage does the unrounded unit break-even point change if the unit contribution margin increases by 1% while all other numbers remain the same?
 - By what percentage does the unrounded unit break-even point change if total fixed costs are reduced by 1% while all other numbers remain the same?
 - What do the solutions to the above questions illustrate?
16. A school board is trying to raise funds to replace its old playground equipment with a modern, child-safe structure. The ABC playground equipment company has quoted the school a cost of \$49,833 for its 20 m × 15 m megastructure. To raise the funds, the school wants to sell coupon books. These books retail for \$15.00 each and cost \$8.50 to purchase. How many books must the school sell to raise funds for the new playground?
17. The Puzzle Company had total revenue of \$4,750,000, total fixed costs of \$1,500,000, and total variable costs of \$2,750,000. If the company desires to earn a net income of \$1,000,000, what total sales volume is needed to achieve the goal?
18. Whirlpool Corporation had annual sales of \$18.907 billion with a net income of \$0.549 billion. Total variable costs amounted to \$16.383 billion.
- Determine Whirlpool Corporation's break-even in dollars.
 - If Whirlpool Corporation managed to increase revenues by 5% the following year while implementing cost-cutting measures that trimmed variable costs by 2%, determine the percent change in the break-even dollars.

Use the following information for questions 19 and 20:

$$S = \$100 \quad VC = \$60 \quad TFC = \$250,000$$

19. Calculate the current break-even point in both units and dollars.
20. A production manager is trying to control costs but is faced with the following tradeoffs under four different situations:
- Total fixed costs are reduced by 15%, but unit variable costs will rise by 5%.
 - Unit variable costs are reduced by 10%, but fixed costs will rise by 5%.
 - Total fixed costs are reduced by 20%, but unit variable costs will rise by 10%.
 - Unit variable costs are reduced by 15%, but fixed costs will rise by 15%.
- Based strictly on break-even calculations, which course of action would you recommend she pursue?

Chapter 5 Summary

Key Concepts Summary

Section 5.1: Cost-Revenue-Net Income Analysis (Need to Be in the Know)

- The three different types of costs
- How to determine the unit variable cost
- Calculating net income when you know total revenue and total cost
- Calculating net income when you know the contribution margin
- Comparing varying contribution margins by calculating a contribution rate
- Integrating all of the concepts together

Section 5.2: Break-Even Analysis (Sink or Swim)

- Explanation of break-even analysis
- How to calculate the break-even point expressed in the level of output
- How to calculate the break-even point expressed in total revenue dollars

The Language of Business Mathematics

blended cost A cost that comprises both fixed cost and variable cost components.

break-even analysis An analysis of the relationship between costs, revenues, and net income with the sole purpose of determining the point at which total revenue equals total cost.

break-even point A quantity that represents the level of output (in units or dollars) at which all costs are paid but no profits are earned, resulting in a net income equal to zero.

contribution margin The amount each unit sold adds to the net income of the business.

contribution rate A contribution margin expressed as a percentage of the selling price.

cost An outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services.

economies of scale The principle that, as production levels rise, per-unit variable costs tend to become lower as efficiencies are achieved.

fixed cost A cost that does not change with the level of production or sales.

net income The amount of money left over after all costs are deducted from all revenues.

total fixed cost The sum of all fixed costs that a business incurs.

total cost The sum of all costs for the company, including both the total fixed costs and total variable costs.

total revenue The entire amount of money received by a company for selling its product, calculated by multiplying the quantity sold by the selling price.

total variable cost The sum of all variable costs that a business incurs at a particular level of output.

unit variable cost The assignment of total variable costs on a per-unit basis.

variable cost A cost that changes with the level of production or sales.

The Formulas You Need to Know

Symbols Used

CM = unit contribution margin

CR = contribution rate

n = number of pieces of data, which in this chapter is the level of output

NI = net income

S = unit selling price of the product

TFC = total fixed cost

TR = total revenue

TVC = total variable cost

VC = unit variable cost

Formulas Introduced

Formula 5.1 Unit Variable Cost: $VC = \frac{TVC}{n}$ (Section 5.1)

Formula 5.2 Net Income Using a Total Revenue and Cost Approach: $NI = n(S) - (TFC + n(VC))$ (Section 5.1)

Formula 5.3 Unit Contribution Margin: $CM = S - VC$ (Section 5.1)

Formula 5.4 Net Income Using Total Contribution Margin Approach: $NI = n(CM) - TFC$ (Section 5.1)

Formula 5.5 Contribution Rate if Unit Information Is Known: $CR = \frac{CM}{S} \times 100$ (Section 5.1)

Formula 5.6 Contribution Rate if Aggregate Information Is Known: $CR = \frac{TR - TVC}{TR} \times 100$ (Section 5.1)

Formula 5.7 Unit Break-Even: $n = \frac{TFC}{CM}$ (Section 5.2)

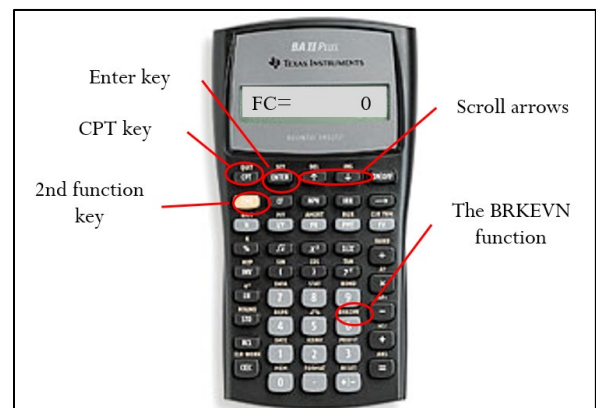
Formula 5.8 Dollar Break-Even: $TR = \frac{TFC}{CR}$ (Section 5.2)

Technology

Calculator

Net Income or Break-Even

- 2nd Brkevn to open the function
- To clear the memory before entering any data, press 2nd CLR Work after opening the function. Use $\uparrow$ and $\downarrow$ to scroll through the function.
- Enter four of the following variables and press Enter after each to save:
 - FC = Total fixed costs
 - VC = Unit variable cost
 - P = Selling price per unit
 - PFT = Net income
 - Q = Level of output
- Press CPT on the unknown variable (when it is on the screen display) to compute.



Chapter 5 Review Exercises

In each of the following questions, round all of the money and percentages to two decimals unless otherwise specified.

Mechanics

1. Logitech manufactures a surround sound multimedia speaker system for personal computers. The average selling price is \$120 per unit, with unit variable costs totalling \$64. Fixed costs associated with this product total \$980,000. Calculate the net income or loss if 25,000 units are sold.
2. Microsoft sells its wireless laser desktop mouse and keyboard for \$70. Unit variable costs are \$45.60 and fixed costs associated with this product total \$277,200.

- a. What is the break-even point in units?
 - b. What is the net income or loss if 40,000 units are sold?
3. Frances has a home-based cosmetics business. She purchases makeup from her supplier for an average price of \$12 and sells it to her customers on average for \$20. She assigns \$100 of her monthly computer leasing costs to this product line and spends \$260 per month advertising this product line. In hosting parties, she estimates her labour cost at \$2 per unit sold.
- a. What is her net income or loss if 100 units are sold?
 - b. What is her net income or loss if 35 units are sold?
 - c. What is the break-even point in both units and dollars?
 - d. What is the new unit break-even point if Frances increases her advertising costs by 50% per month?

Applications

4. Brynne is developing a business plan. From market research she has learned that her customers are willing to pay \$49.95 for her product. Unit variable costs are estimated at \$23.75, and her monthly total fixed costs are \$12,000. To earn a net income of at least \$5,000, what minimum level of monthly sales (in units) does she need?
5. Advanced Microcomputer Systems had annual sales of \$31,979,000 with total variable costs of \$20,934,000 and total fixed costs of \$6,211,000.
- a. What is the company's contribution rate?
 - b. What annual revenue is required to break even?
 - c. In the following year, a competitor dealt a severe blow to sales and total revenues dropped to \$15,000,000. Calculate the net income.
6. In the North American automotive markets, Toyota and Honda are two of the big players. The following financial information (all numbers in millions of dollars) for each was reported to its shareholders:

	<u>Toyota</u>	<u>Honda</u>
Total Revenue	\$247,734	\$101,916
Total Variable Costs	\$204,135	\$75,532
Total Fixed Costs	\$20,938	\$24,453
Total Costs	<u>\$225,073</u>	<u>\$99,985</u>
Net Income	\$22,661	\$1,931

Compare the break-even points in total dollars between the two companies based on these reports.

7. Lagimodiere Industry's unit variable costs are 45% of the unit selling price. Annual fixed costs are \$1.5 million.
- a. What total revenue results in an income of \$250,000?
 - b. Calculate the break-even dollars for Lagimodiere Industry.
 - c. If revenues exceed the break-even dollars by 20%, determine the net income.

Challenge, Critical Thinking, & Other Applications

8. Svetlana and Yoric are each thinking of starting the same type of business. Svetlana is young and into technology and mechanization. She has a large fixed cost of \$15,000, but her advanced production machinery minimizes her labour, resulting in a low unit variable cost of \$25. Yoric is a little older and believes the best quality items are handmade. He has a low fixed cost of \$8,000 but incurs more labour costs, resulting in a high unit variable cost of \$60. Regardless of who makes the product, market research shows that consumers would pay \$100 for the product.
- a. Calculate the unit break-even for both Svetlana and Yoric.
 - b. Calculate the net income or loss for both if unit sales are 50% higher than the break-even point.
 - c. Calculate the net income or loss for both if unit sales are 50% lower than the break-even point.
 - d. Comment on your findings from the above solutions.

9. A surfboard manufacturer lost \$500,000 last year during a recession. Total revenue was \$5,000,000 and total variable costs were 40% of sales. The production facility ran at 50% capacity. The production manager wants to know the following:
- What is the percent capacity required to break even?
 - When the economy recovers this year, if the plant runs at 100% capacity what net income could the company realize?
 - There is a possibility that sales could be so strong this year that the plant may be required to run at 120% capacity by offering a lot of overtime to its production workers. This would result in total variable costs rising by 35%. On a strictly financial basis, should the production manager plan to exceed capacity or should he advise top management to freeze production at 100% capacity? Justify your answer.
10. The following annual information is known about a company:

Material inputs to manufacturing	\$17,400,000	Marketing and advertising	\$3,500,000
Management salaries	\$2,750,000	Production worker wages	\$9,990,000
Supplies needed for manufacturing	\$8,345,000	Utilities used in production activities	\$6,235,000
Utilities used in nonproduction activities	\$2,222,000	Property taxes	\$2,100,000
# of units sold	75,132	Unit price	\$749

- Calculate the unit variable cost, total fixed costs, unit contribution margin, contribution rate, net income, and break-even in both units and dollars.
- The marketing research manager shows that if the company lowers its price by 5%, unit sales will rise by 5%. Should the company lower its price? Justify your decision.
- Assume that the price remained the same. Due to economic pressures, fixed costs are forecasted to rise by 6% and variable costs by 4% next year. At the same level of output, by what percentage must the selling price rise to maintain the same level of net income achieved in (a)?

Chapter 5 Case Study

Forecasting the Impact of Revenue and Cost Changes

The Situation

Lightning Wholesale has just wrapped up the 2023 fiscal year of operations. As with all businesses, revenues, costs, and expenses are forecasted to change in 2024. Top management needs to know the impact of these changes on its 2024 operations. This is best completed by performing a year-over-year analysis of 2023 to 2024.

The Data

This table shows the financial figures for three of the company's operating divisions.

Three Operating Divisions*			
	Sporting Goods	Outdoor Clothing	Indoor Clothing
Sales Revenue	\$11,555	\$10,111	\$5,200
Cost of Goods Sold	\$9,585	\$6,154	\$3,531
Operating Expenses			
Sales & Marketing	\$582	\$485	\$356
General & Administrative	\$49	\$35	\$42
Payroll	\$269	\$326	\$184

*All figures are in thousands of dollars.

Important Information

- All variable costs are accounted for in the cost of goods sold.
- All operating expenses are treated as fixed costs.
- The table below shows the percent changes forecast for 2024.

Three Operating Divisions			
	Sporting Goods	Outdoor Clothing	Indoor Clothing
Sales Revenue	+8%	+4%	−2.3%
Cost of Goods Sold	+4%	+6%	+2.1%
Operating Expenses			
Sales & Marketing	+6%	0%	+3%
General & Administrative	−2%	+1.5%	+2%
Payroll	+2.1%	+3.5%	+2.5%

Your Tasks

1. Top management needs to understand the current 2023 year of operations. Perform the following activities:
 - a. For each division, calculate the net income, total contribution margin, and contribution rate.
 - b. For each division, calculate the break-even point in dollars.
 - c. For all three divisions combined, calculate the total revenue dollars required to break even.
2. Forecast the changes for the 2024 operating year to help management understand how the business is changing.
 - a. For each division, calculate the forecasted net income, total contribution margin, and contribution rate. Express these figures as a percentage of the 2023 numbers.
 - b. For each division, forecast the new break-even point in dollars. Express the forecasted break-even points as a percentage of the 2023 break-even points.
 - c. For all three divisions combined, calculate the total forecasted revenue dollars required to break even and express the total break-even point as a percentage of the 2023 total break-even point.
3. Write a report to management about your findings for each department and the company as a whole for the 2024 year.

Chapter 6: Marketing Applications

(What Is It Going to Cost Me?)

When you buy a new tech device, it is very important that the right price is set. The price should

- be seen as fair by you the buyer,
- pay for the costs (plastics, battery, buttons, circuit boards, etc.) and expenses (employees, factory, electricity, distribution) of making the tech device, and
- allow the seller's business to make some extra money as profit so that it can grow its business further.

If your business is selling a product, you will need to pay close attention to price adjustments because they affect profitability. Various discounts, like putting items on sale, may increase sales while lowering the amount of profit per transaction. How do you know where to set the balance to maximize profit overall?

As a student in a business program, consider this chapter essential to the success of any business. Whether your pricing strategy is high or low, the company must ensure that it can still pay its bills as a minimum requirement. And that requires careful juggling of many factors. If it fails to manage its pricing properly, the company will go bankrupt!

This chapter will make you a smarter business professional and a wiser consumer. You shop retail almost every day and regularly purchase goods and services. If you understand how product pricing works, you can make sense of “deals.” You can easily explain why the same product sells for two different prices at two different stores.

In this chapter, you must learn the language of marketers to perform merchandising mathematics involving product costs, expenses, prices, markups, markdowns, and ultimately profitability. Once the study of the various pricing components is complete, we will see how the various pieces of the pricing puzzle fit together into a cohesive merchandising environment.

Outline of Chapter Topics

- 6.1: Figuring Out the Cost: Discounts (How Much?)
- 6.2: Markup: Setting the Regular Price (Need to Stay in Business)
- 6.3: Markdown: Setting the Sale Price (Everybody Loves a Sale)
- 6.4: Merchandising (How Does It All Come Together?)

6.1: Figuring Out the Cost: Discounts

(How Much?)

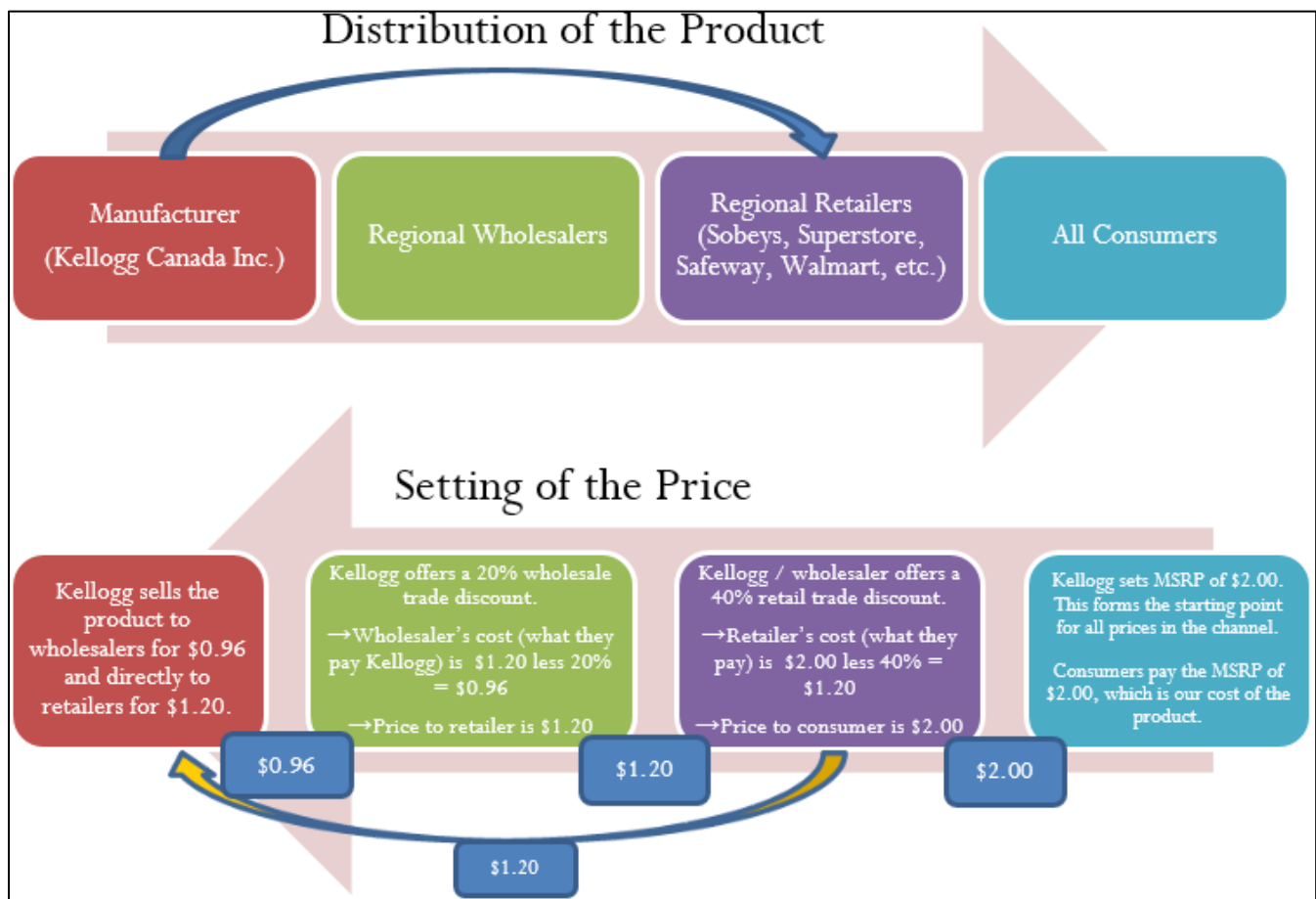
You mutter in exasperation, “Why can’t they just set one price and stick with it?” Your mind boggles at all the competing discounts you encounter at the mall in your search for that perfect Batman toy for your nephew. Walmart is running their Rollback promotion and is offering a Batmobile for 25% off, regularly priced at \$49.99. Toys R’ Us has an outlet in the parking lot where the regular price for the same toy is \$59.99, but all Batman products are being cleared out at 40% off. You head over to The Bay for a warehouse clearance event that has the same toy priced at \$64.99 but at 35% off. It is also Bay Days, which means you can scratch and win a further 10% to 20% off the sale price. You go to Dairy Queen for a Blizzard to soothe your headache while you figure things out.

The **cost** of a product is the amount of money required to obtain the merchandise. If you are a consumer, the ticketed price tag on the product is your cost. If you are a reseller (also known as a *middleman* or *intermediary*), what you pay to your supplier for the product is your cost. If you are a manufacturer, then your cost equals all of the labour, materials, and production expenditures that went into creating the product.

A **discount** is a reduction in the price of a product. As a consumer, you are bombarded with discounts all the time. Retailers use various terms for discounts, including *sales* or *clearance*. If your business purchases a product from a supplier, any discount it receives lowers how much the business pays to acquire the product. When a business buys products, the price paid is the cost to the business. Therefore, a lower price means a lower cost.

If your business is the one selling the product, any discount offered lowers the selling price and reduces revenue per sale. Since the revenue must cover all costs and expenses associated with the product, the lower price means that the business reduces profits per sale. In business, it is common practice to express a discount as a percentage off the regular price.

How Distribution and Pricing Work



Start with distribution in the top half of the figure and work left to right. As an example, let's look at a manufacturer such as Kellogg Canada Inc. (which makes such products as Pop-Tarts, Eggo Waffles, and Rice Krispies). Kellogg's Canadian production plant is located in London, Ontario. To distribute its products to the rest of Canada, Kellogg Canada uses various regional wholesalers. Each wholesaler then resells the product to retailers in its local trade area; however, some retailers (such as the Real Canadian Superstore) are very large, and Kellogg Canada distributes directly to these organizations, bypassing the wholesaler as represented by the blue arrow. Finally, consumers shop at these retailers and acquire Kellogg products.

The relationship of distribution to pricing is illustrated in the bottom half of the figure, working right to left. For now, focus on understanding how pricing works; the mathematics used in the figure will be explained later in this chapter. Kellogg Canada sets a **manufacturer's suggested retail price**, known as the MSRP. This is a recommended retail price based on consumer market research. Since grocery retailers commonly carry thousands or tens of thousands of products, the MSRP helps the retailer to determine the retail price at which the product should be listed. In this case, assume a \$2.00 MSRP, which is the price consumers will pay for the product.

The retailer must pay something less than \$2.00 to make money when selling the product. Kellogg Canada understands its distributors and calculates that to be profitable most retailers must pay approximately 40% less than the MSRP. Therefore, it offers a 40% discount. If the retailer purchases directly from Kellogg, as illustrated by the yellow arrow, the price paid by the retailer to acquire the product is \$2.00 less 40%, or \$1.20. Smaller retailers acquire the product from a wholesaler for the same price. Thus, the retailer's cost equals the wholesaler's price (or Kellogg Canada's price if the retailer purchases it directly from Kellogg).

The wholesaler's price is \$1.20. Again, Kellogg Canada, knowing that the wholesaler must pay something less than \$1.20 to be profitable, offers an additional 20% discount exclusively to the wholesaler. So the price paid by the wholesaler to acquire the product from Kellogg Canada is \$1.20 less 20%, or \$0.96. This \$0.96 forms Kellogg Canada's price to the wholesaler, which equals the wholesaler's cost.

In summary, this discussion illustrates two key pricing concepts:

1. Companies higher up in the distribution channel pay lower prices than those farther down the channel. Companies receive discounts off the MSRP based on their level in the distribution system. This may result in multiple discounts, such as a wholesaler receiving both the retailer's discount and an additional discount for being a wholesaler.
2. One organization's price becomes the next organization's cost (assuming the typical distribution channel structure):
 - Manufacturer's Price = Wholesaler's Cost
 - Wholesaler's Price = Retailer's Cost
 - Retailer's Price = Consumer's Cost

Types of Discounts

You will perform discount calculations more effectively if you understand how and why single pricing discounts and multiple pricing discounts occur. Businesses or consumers are offered numerous types of discounts, of which five of the most common are trade, quantity, loyalty, sale, and seasonal.

1. **Trade Discounts.** A **trade discount** is a discount offered to businesses only based on the type of business and its position in the distribution system (e.g., as a retailer, wholesaler, or any other member of the distribution system that resells the product). Consumers are ineligible for trade discounts. In the discussion of the figure, two trade discounts are offered. The first is a 40% retail trade discount, and the second is a 20% wholesale trade discount. Typically, a business that is higher up in the distribution system receives a combination of these trade discounts. For example, the wholesaler receives both the 40% retail trade discount and the 20% wholesale trade discount from the MSRP. The wholesaler's cost is calculated as an MSRP of \$2.00 less 40% less 20% = \$0.96.
2. **Quantity Discounts.** A **quantity discount** (also called a *volume discount*) is a discount for purchasing larger quantities of a certain product. If you have ever walked down an aisle in a Real Canadian Superstore, you probably noticed many shelf tags that indicate quantity discounts, such as "Buy one product for \$2" or "Take two products for \$3." Many Shell gas stations offer a Thirst Buster program in which customers who purchase four Thirst Busters within a three-month period get the fifth one free. If the Thirst Busters are \$2.00 each, this is equivalent to buying five drinks for \$10.00 less a \$2.00 quantity discount.
3. **Loyalty Discounts.** A **loyalty discount** is a discount that a seller gives to a purchaser for repeat business. Usually no time frame is specified; that is, the offer is continually available. As a consumer, you see this regularly in marketing programs such as Air Miles or with credit cards that offer cash back programs. For example, Co-op gas stations in

Manitoba track consumer gasoline purchases through a loyalty program and mails an annual loyalty discount cheque to its customers, recently amounting to 12.5¢ per litre purchased. In business-to-business circles, sellers typically reward loyal customers by deducting a loyalty discount percentage, commonly ranging from 1% to 5%, from the selling price.

4. **Sale Discounts.** A **sale discount** is a temporary lowering of the price from a product's regular selling price. Businesses put items on sale for a variety of reasons, such as selling excess stock or attracting shoppers. You see such promotional events all the time: LED monitors are on sale at Best Buy; Blu-Ray discs are half off at Walmart; The Brick is having a door crasher event Saturday morning.
5. **Seasonal Discounts.** A **seasonal discount** is a discount offered to consumers and businesses for purchasing products out of season. At the business level, manufacturers tend to offer seasonal discounts encouraging retailers, wholesalers, or distributors to purchase products before they are in season. Bombardier Inc. manufactures Ski-Doos, which are sold in Canada from approximately November through March—a time of year when most of the country has snow and consumers would want to buy one. To keep production running smoothly from April through October, Bombardier could offer seasonal discounts to its wholesalers and retailers for the coming winter season. At a retail level, the examples are plentiful. On November 1 most retailers place their Halloween merchandise on seasonal discount to clear out excess inventory, and many retailers use Boxing Day (or Boxing Week) to clear their out-of-season merchandise.

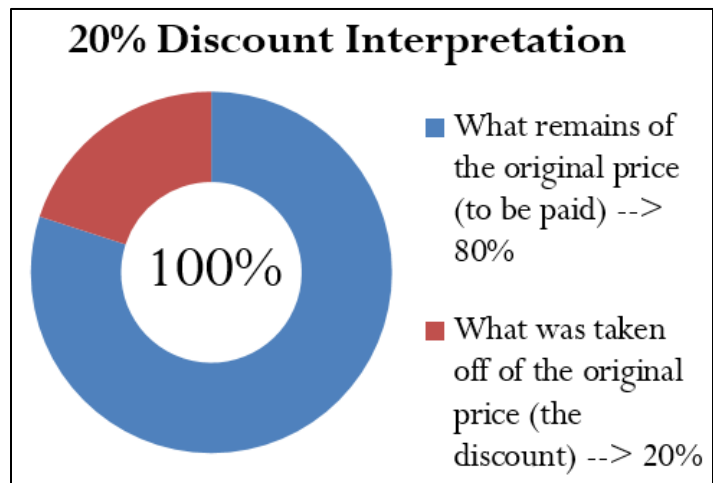
Single Discounts

Let's start by calculating the cost when only one discount is offered. Later in this section you will learn how to calculate a cost involving multiple discounts.



The Formula

Figuring out the price after applying a single discount is called a net price calculation. When a business calculates the net price of a product, it is interested in what you still have to pay, not in what has been removed. Note in Formula 6.1 below that you take 1 and subtract the discount rate to determine the rate owing. If you are eligible for a 20% discount, then you must pay 80% of the list price, as illustrated in the figure to the right.



N is Net Price: The **net price** is the price of the product after the discount is removed from the list price. It is a dollar amount representing what price remains after you have applied the discount.

Formula 6.1 – Single Discount:
$$N = L \times (1 - d)$$

L is List Price: The **list price** is the normal or regular dollar price of the product before any discounts. It is the **Manufacturer's Suggested Retail Price (MSRP)** - a price for a product that has been published or advertised in some way), or any dollar amount before you remove the discount.

d is Discount Rate: The discount rate represents the percentage (in decimal format) of the list price that is deducted.

Formula 6.1 once again applies Formula 2.2 on rate, portion, and base, where the list price is the base, the $(1 - d)$ is the rate, and the net price represents the portion of the price to be paid.

Notice that Formula 6.1 requires the discount to be in a percentage (decimal) format; sometimes a discount is expressed as a dollar amount, though, such as "Save \$5 today." Formulas 6.2a and 6.2b relate the discount dollar amount to the list price, discount percent, and net price. Choose one formula or the other depending on which variables are known.

D\$ is Discount Amount: Determine the discount amount in one of two ways, depending on what information is known. If the list price and discount rate are known, apply Formula 6.2a. If the list price and net price are known, apply Formula 6.2b. Either of these formulas can be algebraically manipulated to solve for any other unknown variable as well.

Formula 6.2a – Discount Amount: $D\$ = L \times d$

Formula 6.2b – Discount Amount: $D\$ = L - N$

L is List Price: The dollar amount of the price before any discounts.

N is Net Price: The dollar amount of the price after you have deducted all discounts.

d is Discount Rate: The percentage (in decimal format) of the list price to be deducted. This time you are interested in figuring out the amount of the discount, therefore you do not take it away from 1.



How It Works

Follow these steps to calculate the net price involving a single discount. These steps are adaptable if the net price is a known variable and one of the other variables is unknown.

Step 1: Identify any known variables, including list price, discount rate, or discount amount.

Step 2: If the list price is known, skip this step. Otherwise, solve for list price using an appropriate formula.

Step 3: Calculate the net price.

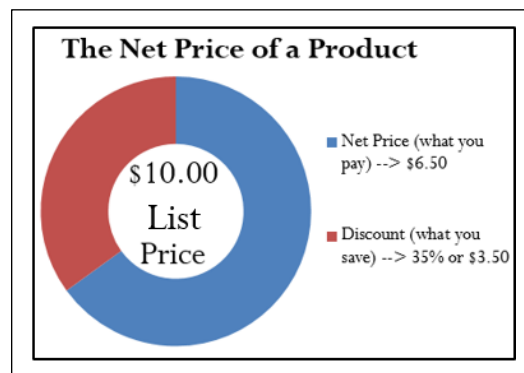
- If the list price and discount are known, apply Formula 6.1.
- If the list price and discount amount are known, apply Formula 6.2b and rearrange for N.

Assume a product sells for \$10 and is on sale at 35% off the regular price. Calculate the net price for the product.

Step 1: The list price of the product is $L = \$10$. It is on sale with a discount rate of $d = 0.35$.

Step 2: List price is known, so this step is not needed.

Step 3: Applying Formula 6.1 results in a new price of $N = \$10 \times (1 - 0.35) = \$6.50$. Note that if you are interested in learning the discount amount, you apply Formula 6.2b to calculate $D\$ = \$10 - \$6.50 = \$3.50$.



Important Notes

You can combine Formula 6.1 and either version of Formula 6.2 in a variety of ways to solve any single discount situation for any of the three variables. As you deal with increasingly complicated pricing formulas, your algebraic skills in solving linear equations and substitution become very important.

Many of the pricing problems take multiple steps that combine various formulas, so you need to apply the PUPP model systematically. In any pricing problem, you must understand which variables are provided and match them up to the known formulas. To get to your end goal, you must look for formulas in which you know all but one variable. In these cases, solving for variables will move you forward toward solving the overall pricing problem.

If you find you cannot produce a formula with only one unknown variable, can you find two formulas with the same two unknowns? If so, recall from Section 2.5 that you can use your algebraic skills to find the roots of the two equations simultaneously. Alternatively, you can solve one formula for a variable then substitute it into the other formula, allowing you to isolate the remaining variable. Throughout the examples in this chapter you will see many applications of these algebraic skills.



Things To Watch Out For

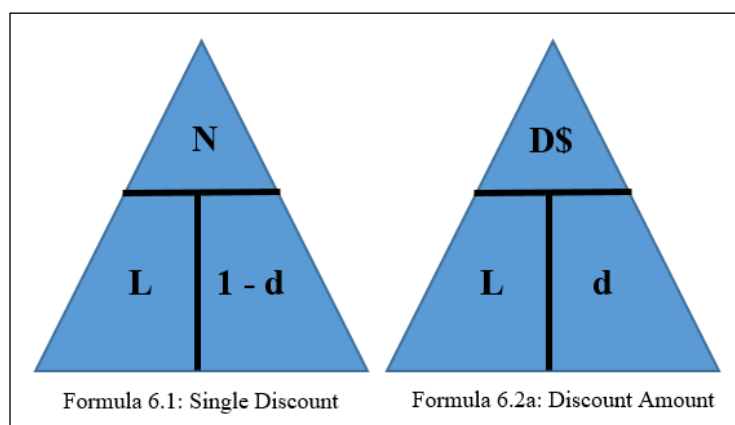
Remember to apply the rounding rules discussed in Chapter 2:

1. Until you arrive at the final solution, avoid rounding any interim numbers unless you have some special reason to do so.
2. Round all dollar amounts to the nearest cent. If the dollar amount has no cents, you may write it either without the cents or with the “.00” at the end.
3. Round all percentages to four decimals when in percent format.

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When working with single discounts, you are not always solving for the net price. Sometimes you must calculate the discount percent or the list price. At other times you know information about the discount amount but need to solve for list price, net price, or the discount rate. The triangle technique discussed in Section 2.3 can remind you how to rearrange the formulas for each variable, as illustrated in the figure to the right.



Give It Some Thought

1. Will you pay more than, less than, or exactly \$10.00 for a product if you are told that you are paying:
 - a. a net price of \$10.00 when there is a discount of 25%?
 - b. a list price of \$10.00 when there is a discount of 25%?
2. If an item is subject to a 40% discount, will the net price be more than or less than half of the list price of the product?

Solutions:
 1a. Exactly \$10. The net price is the price after the discount.
 1b. Less than \$10. The discount needs to be removed from the list price.
 2. More than half. A 40% discount means that you will pay 60% of the list price.

Example 6.1A: Determining the Retailer's Net Price for a Pair of Jeans

A manufacturer that sells jeans directly to its retailers uses market research to find out it needs to offer a 25% trade discount. In doing so, the retailers will then be able to price the product at the MSRP of \$59.99. What price should retailers pay for the jeans?

Plan	Calculate how much a retailer should pay for the jeans after the regular price has been discounted to accommodate the trade discount. This is called the net price for the product, or N.	
Understand	What You Already Know Step 1: The list price and the discount rate are known: $L = \$59.99$ $d = 0.25$	How You Will Get There Step 2: List price is known, so skip this step. Step 3: Apply Formula 6.1.

Perform	Step 3: $N = \$59.99 \times (1 - 0.25) = \$59.99 \times 0.75 = \$44.99$
Present	The manufacturer should sell the jeans to the retailers for \$44.99.

Example 6.1B: Determining the List Price of a Jacket

Winners pays a net price of \$27.50 for a winter jacket after receiving a retail trade discount of 45%. What was the MSRP of the jacket?

Plan	Calculate the MSRP for the jacket before Winners received the retail trade discount to arrive at the net price. This is called the list price for the product, or L.	
Understand	What You Already Know Step 1: The net price and the discount rate are known: $N = \$27.50$ $d = 0.45$	How You Will Get There Step 2: List price is the unknown variable; skip this step. Step 3: Apply Formula 6.1, rearranging for L.
Perform	Step 3: $\$27.50 = L(1 - 0.45)$ $L = \$27.50 \div (1 - 0.45)$ $L = \$27.50 \div 0.55 = \$50.00$	
Present	The MSRP, or list price, of the winter jacket is \$50.00.	

Example 6.1C: Determining the Discount Percent and Discount Amount

You are shopping for a new environmentally friendly water bottle. The price tag reads \$14.75, which is \$10.24 off the regular price. Determine the discount rate applied.

Plan	You need to find out how the sale price translates into the discount rate, or d.	
Understand	What You Already Know Step 1: The discount amount and net price are known: $D\$ = \$10.24$ $N = \$14.75$	How You Will Get There Step 2: Use Formula 6.2b to calculate the list price, rearranging for L. Step 3: Convert the discount amount into a percentage by applying Formula 6.2a, rearranging for d.
Perform	Step 2: List price: $\$10.24 = L - \$14.75$, which results in $L = \$24.99$ Step 3: Discount rate: $d = D\$ \div L$ $d = \frac{\$10.24}{\$24.99} = 0.409764 \text{ or } 40.9764\%$	
Present	The water bottle today has been reduced in price by the amount of \$10.24. This represents a sale discount of 40.9674%.	

Multiple Discounts

You are driving down the street when you see a large sign at Old Navy that says, “Big sale, take an additional 25% off already reduced prices!” In other words, products on sale (the first discount) are being reduced by an additional 25% (the second discount). Because Formula 6.1 handles only a single discount, you must use an extended formula in this case.



The Formula

Businesses commonly receive more than one discount when they make a purchase. Consider a transaction in which a business receives a 30% trade discount as well as a 10% volume discount. First, you have to understand that this is not a $30\% + 10\% = 40\%$ discount. The second discount is always applied to the net price after the first discount is applied. Therefore, the second discount has a smaller base upon which it is calculated. If there are more than two discounts, you deduct each subsequent discount from continually smaller bases. Formula 6.3 expresses how to calculate the net price when multiple discounts apply.

N is Net Price: The dollar amount of the price after all discounts have been deducted.

L is List Price: The dollar amount of the price before any discounts.

Formula 6.3 – Multiple Discounts:

$$N = L \times (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$$

d_1 is First Discount, d_2 is Second Discount, d_n is nth Discount: When there is more than one discount, you must extend beyond Formula 6.1 by multiplying another discount expression. These discounts are represented by the same d symbol; however, each discount receives a subscript to make its symbol unique. Therefore, the first discount receives the symbol of d_1 , the second discount receives the symbol d_2 , and so on. Recall that the symbol n represents the number of pieces of data (a count), so you can expand or contract this formula to the exact number of discounts being offered.

It is often difficult to understand exactly how much of a discount is being received when multiple discounts are involved. Often it is convenient to summarize the multiple discount percentages into a single percentage. This makes it easier to calculate the net price and aids in understanding the discount benefit. Simplifying multiple percent discounts into a single percent discount is called finding the **single equivalent discount**. Whether you apply the multiple discounts or just the single equivalent discount, you arrive at the same net price. The conversion of multiple discount percentages into a single equivalent discount percent is illustrated in Formula 6.4.

d_{equiv} (or just d) is the **single equivalent discount rate** that is equal to the series of multiple discounts. Recall that taking $(1 - d)$ calculates what you pay. Therefore, if you take 1, which represents the entire amount, and reduce it by what you pay, the rate left over must be what you did not pay. In other words, it is the discount rate.

Formula 6.4 – Single Equivalent Discount:

$$d_{\text{equiv}} = 1 - (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$$

d_1 is First Discount, d_2 is Second Discount, d_n is nth Discount: This is the same notation as in Formula 6.3. Since there are multiple discounts, each discount receives a numerical subscript to give it a unique identifier. You can expand or contract the formula to the exact number of discounts being offered.



How It Works

Refer back to the steps in calculating net price. The procedure for calculating a net price involving a single discount extends to a more generic procedure involving multiple discounts. As with the single discount procedures, you can adapt the model if the net price is known and one of the other variables is unknown. Follow these steps to calculate the net price involving any number of discounts:

Step 1: Identify any known variables, including list price, discount rate(s), or discount amount.

Step 2: If the list price is known, skip this step. Otherwise, solve for list price.

- If only one discount is involved, apply Formula 6.2a.
- If more than one discount is involved, the discount amount represents the total discount amount received from all of the discounts combined. This requires you first to convert the multiple discount rates into an equivalent single discount rate using Formula 6.4 and then to apply Formula 6.2a.

Step 3: Calculate the net price.

- If the list price and only a single known discount rate are involved, apply Formula 6.1.
- If the list price and multiple discount rates are known and involved, apply Formula 6.3.
- If the list price and the total discount amount are known, apply Formula 6.2b and rearrange for N .

Assume a product with an MSRP of \$100 receives a trade discount of 30% and a volume discount of 10%.

Calculate the net price.

Step 1: The list price and discounts are $L = \$100$, $d_1 = 0.30$ and $d_2 = 0.10$.

Step 2: List price is known, so skip this step.

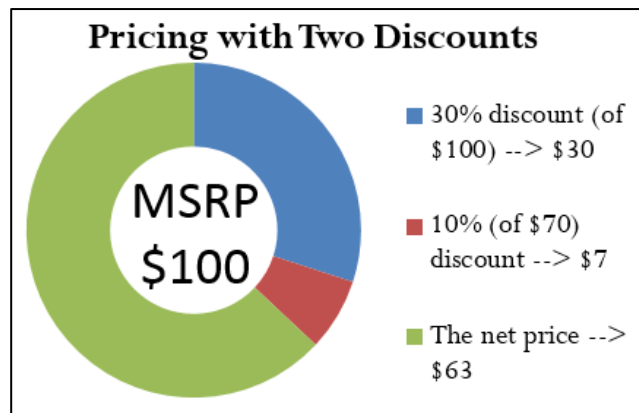
Step 3: Apply Formula 6.3 to calculate the net price:

$$N = \$100 \times (1 - 0.30) \times (1 - 0.10) = \$63$$

The net price is \$63, which is illustrated to the right.

If you are solely interested in converting multiple discounts into a single equivalent discount, you need only substitute into Formula 6.4. In the above example, the product received a trade discount of 30% and a volume discount of 10%. To calculate the single equivalent discount, apply Formula 6.4:

$$\begin{aligned} d_{\text{equiv}} &= 1 - (1 - 0.30) \times (1 - 0.10) \\ d_{\text{equiv}} &= 1 - (0.70)(0.90) \\ d_{\text{equiv}} &= 1 - 0.63 = 0.37 \end{aligned}$$



Therefore, whenever discounts of 30% and 10% are offered together, the single equivalent discount is 37%. Whether it is the multiple discounts or just the single equivalent discount that you apply to the list price, the net price calculated is always the same.



Important Notes

Order of Discounts: The order of the discounts *does not* matter in determining the net price. Remember from the rules of BEDMAS that you can complete multiplication in any order. Therefore, in the above example you could have arrived at the \$63 net price through the following calculation:

$$\$100 \times (1 - 0.10) \times (1 - 0.30) = \$63$$

The order of the discounts *does* matter if trying to interpret the value of any single discount. If the trade discount is applied before the quantity discount and you are wanting to know the quantity discount amount, then the quantity discount needs to be second. Thus,

$$\$100 \times (1 - 0.30) = \$70$$

$$\$70 \times 0.10 = \$7$$

which is the amount of the quantity discount.

Price Does Not Affect Single Equivalent Discount: Notice in Formula 6.4 that the list price and the net price are not involved in the calculation of the single equivalent discount. When working with percentages, whether you have a net price of \$6.30 and a list price of \$10, or a net price of \$63 and a list price of \$100, the equivalent percentage always remains constant at 37%.



Things To Watch Out For

A common mistake when working with multiple discounts is to add the discounts together to calculate the single equivalent discount. This mistaken single discount is then substituted into Formula 6.1 to arrive at the wrong net price. Remember that if two discounts of 30% and 10% apply, you cannot sum these discounts. The second discount of 10% is applied on a smaller price tag, not the original price tag. To calculate the net price you must apply Formula 6.3.

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If you happen to know any two of the net price (N), list price (L), or the total discount amount (D\$), then you could also use Formula 6.2 to solve for the single equivalent discount, d_{equiv} . For example, if you know the net price is \$63 and the total discount amount for all discounts is \$37, you could use Formula 6.2b to figure out that the list price is \$100, then convert the discount amount into a percentage using Formula 6.2a. This method will also produce a single equivalent discount of 37%.

Another method of calculating the single equivalent discount is to recognize Formula 6.2a as an application of Formula 3.1 involving percent change. The variable d is a discount rate, which you interpret as a negative percent change. The discount amount, D\$, is the difference between the list price (representing the Old price) and the net price (representing the New price after the discount). Therefore, Formula 6.2a can be rewritten as follows:

$$D\$ = L \times d$$

becomes

$$\text{New} - \text{Old} = \text{Old} \times \Delta\% \text{ or}$$

$$\frac{\text{New} - \text{Old}}{\text{Old}} = \Delta\%$$

Therefore, any question about a single equivalent discount where net price and list price are known can be solved as a percent change. Using our ongoing net price example, you have:

$$\frac{\$63 - \$100}{\$100} = -0.37 \text{ or } -37\%; \text{ this is a discount of } 37\%.$$



Give It Some Thought

If you are offered discounts in the amount of 25%, 15%, 10%, and 5%, will your total discount percent be 55%, less than 55%, or more than 55%?

Solution:
Less than 55%. Each percent discount is calculated from a smaller base.

Example 6.1D: Retailer Purchasing Ski-Doods with Multiple Discounts

A retail dealership purchases some Ski-Doods to stock in its stores. Examining the merchandising terms of the manufacturer, Bombardier, the dealership notices that it would be eligible to receive a 35% trade discount, 15% volume discount, and 3% loyalty discount. Because it is June and Ski-Doods are out of season, Bombardier offers a seasonal discount of 12% for purchases made before June 30. If the MSRP for the Ski-Doo is \$12,399.00 and the dealership purchases this item on June 15, what price would it pay?

Plan	You are looking for the net price that the retail dealership will pay for the Ski-Doo, or N.												
Understand	What You Already Know Step 1: The retail dealership is eligible for all four discounts (it qualifies for the seasonal discount since it is purchasing before June 30). Therefore, $L = \\$12,399.00$, $d_1 = 0.35$, $d_2 = 0.15$, $d_3 = 0.03$, and $d_4 = 0.12$	How You Will Get There Step 2: You know the list price, so skip this step. Step 3: Apply Formula 6.3.											
Perform	Step 3: $N = \\$12,399 \times (1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12)$ $N = \\$12,399 \times 0.65 \times 0.85 \times 0.97 \times 0.88 = \\$5,847.54$												
Present	After all four discounts, the retail dealership could purchase the Ski-Doo for \$5,847.54. Ski-Doo Pricing with Four Discounts <table><tbody><tr><td>■ 35% discount (of \$12,399) --></td><td>\$4,339.65</td></tr><tr><td>■ 15% discount (of \$8,059.45*) --></td><td>\$1,208.90</td></tr><tr><td>■ 3% discount (of \$6,850.45*) --></td><td>\$205.51</td></tr><tr><td>■ 12% discount (of \$6,644.93*) --></td><td>\$797.39</td></tr><tr><td>■ What you pay (net price) --></td><td>\$5,847.54</td></tr></tbody></table> *Note: For illustrative purposes these numbers have been rounded to two decimals. In practice, we would need to retain all decimals throughout our calculations.			■ 35% discount (of \$12,399) -->	\$4,339.65	■ 15% discount (of \$8,059.45*) -->	\$1,208.90	■ 3% discount (of \$6,850.45*) -->	\$205.51	■ 12% discount (of \$6,644.93*) -->	\$797.39	■ What you pay (net price) -->	\$5,847.54
■ 35% discount (of \$12,399) -->	\$4,339.65												
■ 15% discount (of \$8,059.45*) -->	\$1,208.90												
■ 3% discount (of \$6,850.45*) -->	\$205.51												
■ 12% discount (of \$6,644.93*) -->	\$797.39												
■ What you pay (net price) -->	\$5,847.54												

Example 6.1E: Reducing Multiple Discounts to a Single Equivalent Discount

The retail dealership in Example 6.1D purchases more products subject to the same discounts. It needs to simplify its calculations. Using the information from Example 6.1D, what single equivalent discount is equal to the four specified discounts?

Plan	You are looking for a single equivalent discount that is equal to the four discount percentages, or d_{equiv} (or just d).

Understand	What You Already Know You know the discount rates: $d_1 = 0.35$, $d_2 = 0.15$, $d_3 = 0.03$, and $d_4 = 0.12$	How You Will Get There Apply Formula 6.4.
Perform	$d_{\text{equiv}} = 1 - (1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12)$ $d_{\text{equiv}} = 1 - (0.65)(0.85)(0.97)(0.88) = 1 - 0.471614 = 0.528386$ or 52.8386%	
Present	The retail dealership can apply a 52.8386% discount to all the products it purchases.	

Example 6.1F: Making a Smart Consumer Purchase

You are shopping on Boxing Day for an 80" television. You have just one credit card in your wallet, a cashback Visa card, which allows for a 1% cash rebate on all purchases. While scanning flyers for the best deal, you notice that Visions is selling the television for \$5,599.99 including taxes, while Best Buy is selling it for \$5,571.99 including taxes. However, because of a computer glitch Best Buy is unable to accept Visa today. Where should you buy your television?

Plan	You want to know which store you should buy the television at. You must calculate the net price (N) for each of the stores.	
Understand	What You Already Know Step 1: You know the list price for each of the stores. You also know the discount available from Visa. Thus, $L_{\text{Best Buy}} = \$5,571.99$ with no discounts since Visa cannot be used there $L_{\text{Visions}} = \$5,599.99$, $d = 0.01$, since you can use your Visa card there	How You Will Get There Step 2: List price is known, so skip this step. Step 3: Apply Formula 6.1.
Perform	Step 3: Best Buy: No discounts apply, so the list price equals the net price and $N = \$5,571.99$. Visions: $N = \$5,599.99 \times (1 - 0.01) = \$5,599.99 \times 0.99 = \$5,543.99$	
Present	The net price for Visions is \$5,543.99. You save $\$5,571.99 - \$5,543.99 = \$28.00$ by purchasing your TV at Visions.	

Example 6.1G: Understanding the Price

An advertisement claims that at 60% off, you are saving \$18. However, today there is an additional 20% off. What price should you pay for this item? What percent savings does this represent?

Plan	You are looking for how much you should pay after the discounts (N), and the single equivalent percentage that represents the two discounts (d_{equiv} or just d).	
Understand	What You Already Know Step 1: You know the discount amount for the first discount only, as well as the two discount rates: $D\$_1 = \$18$, $d_1 = 0.60$, and $d_2 = 0.20$.	How You Will Get There Step 2: Calculate the list price by applying Formula 6.2a and rearranging for L. Step 3: To calculate the net price, apply Formula 6.3. Step 4: To calculate the single equivalent discount, apply Formula 6.4.

Perform	Step 2: $\\$18.00 = L \times 0.60$ $L = \\$18.00 \div 0.6 = \\$30.00$ Step 3: $N = \\$30 \times (1 - 0.60) \times (1 - 0.20) = \\$30 \times 0.40 \times 0.80 = \\$9.60$ Step 4: $d_{\text{equiv}} = 1 - (1 - 0.60) \times (1 - 0.20) = 1 - (0.40)(0.80) = 1 - 0.32 = 0.68$ or 68%
Present	You should pay \$9.60 for the item, which represents a 68% savings.



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6.2: Markup: Setting the Regular Price

(Need to Stay in Business)

As you wait in line to purchase your Iced Caramel Macchiato at Starbucks, you look at the pricing menu and think that \$4.99 seems like an awful lot of money for a frozen coffee beverage. Clearly, the coffee itself doesn't cost anywhere near that much. But then gazing around the café, you notice the carefully applied colour scheme, the comfortable seating, the high-end machinery behind the counter, and a seemingly well-trained barista who answers customer questions knowledgeably. Where did the money to pay for all of this come from? You smile as you realize your \$4.99 pays not just for the macchiato, but for everything else that comes with it.

The process of taking a product's cost and increasing it by some amount to arrive at a selling price is called **markup**. This process is critical to business success because every business must ensure that it does not lose money when it makes a sale. From the consumer perspective, the concept of markup helps you make sense of the prices that businesses charge for their products or services. This in turn helps you to judge how reasonable some prices are (and hopefully to find better deals).

The Components in a Selling Price

Before you learn to calculate markup, you first have to understand the various components of a selling price. Then, in the next section, markup and its various methods of calculation will become much clearer.

When your business acquires merchandise for resale, this is a monetary outlay representing a cost. When you then resell the product, the price you charge must recover more than just the product cost. You must also recover all the selling and operating expenses associated with the product, which is often referred to as overhead. Ultimately, you also need to make some money, or profit, as a result of the whole process.



The Formula

Most people think that marking up a product must be a fairly complex process. It is not. Formula 6.5 illustrates the relationship between the three components of cost, expenses (also known as overhead), and profits in calculating the selling price.

S is Selling Price: Once you calculate what the business paid for the product (cost), the bills it needs to cover (expenses), and how much money it needs to earn (profit), you arrive at a selling price by summing the three components.

C is Cost: The cost is the amount of money that the business must pay to purchase or manufacture the product. If manufactured, the cost represents all costs incurred to make the product. If purchased, this number results from applying an appropriate discount formula from Section 6.1. There is a list price from which the business will deduct discounts to arrive at the net price. The net price paid for the product equals the cost of the product. If a business purchases or manufactures a product for \$10 then it must sell the product for at least \$10. Otherwise, it fails to recover what was paid to acquire or make the product in the first place—a path to sheer disaster!

Formula 6.5: The Selling Price of a Product: $S = C + E + P$

E is Expenses: Expenses (also called overhead) are the financial outlays involved in selling the product. Beyond just purchasing the product, the business has many more bills to pay, including wages, taxes, leases, equipment, electronics, insurance, utilities, fixtures, décor, and many more. These expenses must be recovered and may be calculated as

- A fixed dollar amount per unit
- A percentage of the product cost. For example, if a business forecasts total merchandise costs of \$100,000 for the coming year and total business expenses of \$50,000, then it may set a general guideline of adding 50% ($\$50,000 \div \$100,000$) to the cost of a product to cover expenses.
- A percentage of the product selling price based on a forecast of future sales. For example, if a business forecasts total sales of \$250,000 and total business expenses of \$50,000, then it may set a general guideline of adding 20% ($\$50,000 \div \$250,000$) of the selling price to the cost of a product to cover expenses.

P is Profit: Profit is the amount of money that remains after a business pays all of its costs and expenses. A business needs to add an amount above its costs and expenses to allow it to grow. If it adds too much profit, though, the product's price will be too high, in which case the customer may refuse to purchase it. If it adds too little profit, the product's price may be too low, in which case the customer may perceive the product as shoddy and once again refuse to purchase it. Many businesses set general guidelines on how much profit to add to various products. As with expenses, this profit may be expressed as

- A fixed dollar amount per unit
- A percentage of the product cost
- A percentage of the selling price



How It Works

Follow these steps to solve pricing scenarios involving the three components:

Step 1: Four variables are involved in Formula 6.5. Identify the known variables. Note that you may have to calculate the product's cost by applying the single or multiple discount formulas (Formulas 6.1 and 6.3, respectively). Pay careful attention to expenses and profits to capture how you calculate these amounts.

Step 2: Apply Formula 6.5 and solve for the unknown variable.

Assume a business pays a net price of \$75 to acquire a product. Through analyzing its finances, the business estimates expenses at \$25 per unit, and it figures it can add \$50 in profit. Calculate the selling price.

Step 1: The net price paid for the product is the product cost. The known variables are $C = \$75$, $E = \$25$, and $P = \$50$.

Step 2: According to Formula 6.5, the unit selling price is $S = C + E + P = \$75 + \$25 + \$50 = \$150$.



Important Notes

In applying Formula 6.5 you must adhere to the basic rule of linear equations requiring all terms to be in the same unit. That is, you could use Formula 6.5 to solve for the selling price of an individual product, where the three components are the unit cost, unit expenses (unit overhead), and unit profit. When you add these, you calculate the unit selling price. Alternatively, you could use Formula 6.5 in an aggregate form where the three components are total cost, total expenses (total overhead), and total profit. In this case, the selling price is a total selling price, which is more commonly known as total revenue. But you cannot mix individual components with aggregate components.



Things To Watch Out For

The most common mistake in working with pricing components occurs in the "Understand" portion of the PUPP model. It is critical to identify and label information correctly. You have to pay attention to details such as whether you are expressing the expenses (overhead) and/or profit in dollar format or as a percentage of either cost or selling price. Systematically work your way through the information provided piece by piece to ensure that you do not miss an important detail.



Give It Some Thought

1. What three components make up a selling price? In what units are these components commonly expressed?
2. In what three ways are expenses and profits expressed?
3. What is the relationship between net price and cost?

Solutions:

1. Cost, expenses (also called overhead), and profit. They are expressed either per unit or as a total.
2. A specific dollar amount, a percentage of cost, or a percentage of the selling price.
3. The net price paid for a product is the same as the cost of the product.

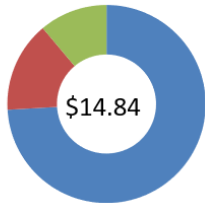
Example 6.2A: Setting a Price on Fashion in Dollars

Mary's Boutique purchases a dress for resale at a cost of \$23.67. The owner determines that each dress must contribute \$5.42 to the expenses of the store. The owner also wants this dress to earn \$6.90 toward profit. What is the regular selling price for the dress?

Plan	You are looking for the regular selling price for the dress, or S.	
Understand	What You Already Know Step 1: The unit cost of the dress and the unit expense and the unit profit are all known: $C = \$23.67$, $E = \$5.42$, $P = \$6.90$	How You Will Get There Step 2: Apply Formula 6.5.
Perform	Step 2: $S = \$23.67 + \$5.42 + \$6.90 = \$35.99$	
Present	Mary's Boutique will set the regular price of the dress at \$35.99.	

Example 6.2B: Setting the Price Using Percentage of Cost

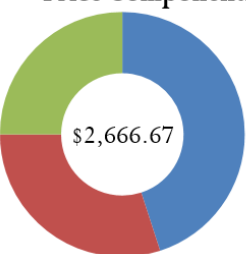
John's Discount Store just completed a financial analysis. The company determined that expenses average 20% of the product cost and profit averages 15% of the product cost. John's Discount Store purchases fake household plants from its supplier for an MSRP of \$19.99 less a trade discount of 45%. What will be the regular selling price?

Plan	You are looking for the regular selling price for the fake household plants, or S.							
Understand	What You Already Know Step 1: The list price, discount rate, expenses, and profit are known: L = \$19.99 d = 0.45 E = 20% of cost, or 0.20C P = 15% of cost, or 0.15C	How You Will Get There Step 1 (continued): Although the cost of it is not directly known, you do know the MSRP (list price) and the trade discount. The cost is equal to the net price. Apply Formula 6.1. Step 2: To calculate the selling price, apply Formula 6.5.						
Perform	Step 1: $N = \$19.99 \times (1 - 0.45) = \$19.99 \times 0.55 = \$10.99 = C$ Step 2: $S = \$10.99 + 0.20C + 0.15C$ $S = \$10.99 + 0.20(\$10.99) + 0.15(\$10.99)$ $S = \$10.99 + \$2.20 + \$1.65 = \$14.84$							
Present	John's Discount Store will sell the fake household plants for \$14.84.	The Price Components <table><tr><td>■ Cost --></td><td>\$10.99</td></tr><tr><td>■ Expenses --></td><td>\$2.20</td></tr><tr><td>■ Profit --></td><td>\$1.65</td></tr></table>	■ Cost -->	\$10.99	■ Expenses -->	\$2.20	■ Profit -->	\$1.65
■ Cost -->	\$10.99							
■ Expenses -->	\$2.20							
■ Profit -->	\$1.65							

Example 6.2C: Setting the Price Using Percentage of Selling Price

Based on last year's results, Benthall Appliance learned that its expenses average 30% of the regular selling price. It wants a 25% profit based on the selling price. If Benthall Appliance purchases a fridge for \$1,200, what is the regular unit selling price?

Plan	You are looking for the regular unit selling price for the fridge, or S.	
Understand	What You Already Know Step 1: The cost, expenses, and profit for the fridge are known: $E = 30\%$ of S, or $0.3S$ $P = 25\%$ of S, or $0.25S$ $C = \$1,200.00$	How You Will Get There Step 2: Apply Formula 6.5.
Perform	Step 2: $S = \$1,200.00 + 0.3S + 0.25S$ $S = \$1,200.00 + 0.55S$ $S - 0.55S = \$1,200.00$ $0.45S = \$1,200.00$ $S = \$2,666.67$	

Present	Benthal Appliance should set the regular selling price of the fridge at \$2,666.67.	Price Components for a Fridge  ■ Cost --> \$1,200.00 ■ Expenses --> $0.3(\\$2,666.67) = \\$800.00$ ■ Profit --> $0.25(\\$2,666.67) = \\$666.67$

Example 6.2D: Using Selling Price to Figure Out the Cost

If a company knows that its profits are 15% of the selling price and expenses are 30% of cost, what is the cost of tech device that has a regular selling price of \$39.99?

Plan	You are looking for the cost of the tech device, or C.	
Understand	What You Already Know Step 1: The expenses, profits, and the regular unit selling price are as follows: $S = \\$39.99$ $P = 15\% \text{ of } S, \text{ or } 0.15S$ $E = 30\% \text{ of cost, or } 0.3C$	How You Will Get There Step 2: Apply Formula 6.5, rearranging for C.
	Step 2: $\\$39.99 = C + 0.3C + 0.15(\\$39.99)$ $\\$39.99 = 1.3C + \\$6.00$ $\\$33.99 = 1.3C$ $\\$26.15 = C$	
Present	The cost of the tech device is \$26.15.	Price Components Of A Tech Device  ■ Cost --> \$26.15 ■ Expenses --> $0.3(\\$26.15) = \\$7.84$ ■ Profit --> $0.15(\\$39.99) = \\$6.00$

Example 6.2E: Determining Profitability

An agricultural wholesaler considers setting the regular unit selling price of its strawberries at \$3.99 per kilogram. If it purchases these strawberries from the farmer for \$2.99 per kilogram and expenses average 40% of product cost, does the agricultural wholesaler make any money?

Plan	In asking whether the agricultural wholesaler makes money, you are looking for the profit, or P.	
Understand	What You Already Know Step 1: The cost, expenses, and proposed regular unit selling price for the strawberries are as follows: $S = \\$3.99$ $C = \\$2.99$ $E = 40\% \text{ of cost, or } 0.4C$	How You Will Get There Step 2: Apply Formula 6.5, rearranging for P.

Perform	Step 2: $\$3.99 = \$2.99 + 0.4(\$2.99) + P$ $\$3.99 = \$4.19 + P$ $-\$0.20 = P$
Present	The negative sign on the profit means that the agricultural wholesaler would take a loss of \$0.20 per kilogram if it sells the strawberries at \$3.99. Unless it has a marketing reason or sound business strategy for doing this, the company should reconsider its pricing.

Calculating the Markup Dollars

Most companies sell more than one product, each of which has different price components with varying costs, expenses, and profits. Can you imagine trying to compare 50 different products, each with three different components? You would have to juggle 150 numbers! To make merchandising decisions more manageable and comparable, many companies combine expenses and profit together into a single quantity, either as a dollar amount or a percentage. This section focuses on the markup as a dollar amount.



The Formula

One of the most basic ways a business simplifies its merchandising is by combining the dollar amounts of its expenses and profits together as expressed in Formula 6.6.

M\$ is Markup Amount: Markup is taking the cost of a product and converting it into a selling price. The **markup amount** represents the dollar amount difference between the cost and the selling price.

Formula 6.6 – Markup Amount: $M\$ = E + P$

E is Expenses: The expenses associated with the product.

P is Profit: The profit earned when the product sells.

Note that since the markup amount (M\$) represents the expenses (E) and profit (P) combined, you can substitute the variable for markup amount into Formula 6.5 to create Formula 6.7, which calculates the regular selling price.

S is Selling Price: The regular selling price of the product.

Formula 6.7 – Selling Price Using Markup: $S = C + M\$$

C is Cost: The amount of money needed to acquire or manufacture the product. If the product is being acquired, the cost is the same amount as the net price paid.

M\$ is Markup Amount: From Formula 6.6, this is the single number that represents the total of the expenses and profits.



How It Works

Follow these steps when you work with calculations involving the markup amount:

Step 1: You require three variables in either Formula 6.6 or Formula 6.7. At least two of the variables must be known. If the amounts are not directly provided, you may need to calculate these amounts by applying other discount or markup formulas.

Step 2: Solve either Formula 6.6 or Formula 6.7 for the unknown variable.

Recall from Example 6.2D that the MP3 player's expenses are \$7.84, the profit is \$6.00, and the cost is \$26.15.

Calculate the markup amount and the selling price.

Step 1: The known variables are $E = \$7.84$, $P = \$6.00$, and $C = \$26.15$.

Step 2: According to Formula 6.6, the markup amount is the sum of the expenses and profit, or $M\$ = \$7.84 + \$6.00 = \$13.84$.

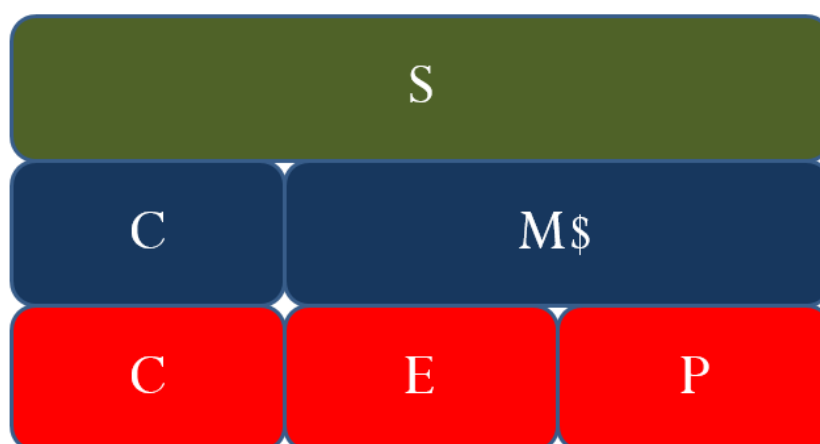
Step 2 (continued): Applying Formula 6.7, add the markup amount to the cost to arrive at the regular selling price, resulting in $S = \$26.15 + \$13.84 = \$39.99$

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Paths To Success

You might have already noticed that many of the formulas in this chapter are interrelated. The same variables appear numerous times but in different ways. To help visualize the relationship between the various formulas and variables, many students have found it helpful to create a markup chart, as shown to the right.

This chart demonstrates the relationships between Formulas 6.5, 6.6, and 6.7. It is evident that the selling price (the green line) consists of cost, expenses, and profit (the red line representing Formula 6.5); or it can consist of cost and the markup amount (the blue line representing Formula 6.7). The markup amount on the blue line consists of the expenses and profit on the red line (Formula 6.6).



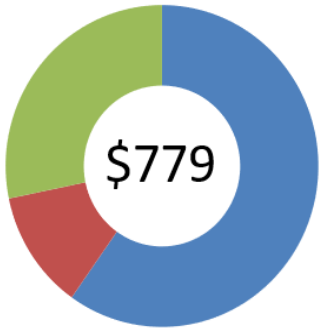
Example 6.2F: Markup as a Dollar Amount

A cellular retail store purchases a smartphone with an MSRP of \$779 less a trade discount of 35% and volume discount of 8%. The store sells the phone at the MSRP.

a. What is the markup amount?

b. If the store knows that its expenses are 20% of the cost, what is the store's profit?

Plan	First of all, you need to calculate the markup amount ($M\$$). You also want to find the store's profit (P).	
Understand	What You Already Know Step 1: The smartphone MSRP and the two discounts are known, along with the expenses and selling price: $L = \$270 \quad d_1 = 0.35 \quad d_2 = 0.08$ $E = 20\% \text{ of cost, or } 0.2C \quad S = \$779$	How You Will Get There Step 1 (continued): Calculate the cost of the smartphone by applying Formula 6.3. Step 2: Calculate the markup amount using Formula 6.7. Step 2 (continued): Calculate the profit by applying Formula 6.6, rearranging for P.

Perform	Step 1 (continued): $N = \\$779.00 \times (1 - 0.35) \times (1 - 0.08)$ $N = \\$779.00 \times 0.65 \times 0.92 = \\$465.84 = C$ Step 2: $\\$779.00 = \\$465.84 + M\\$ $\\$313.16 = M\\$ Step 2 (continued): $\\$313.16 = 0.2(\\$465.84) + P$ $\\$313.16 = \\$93.17 + P$ $\\$219.99 = P$
Present	The markup amount for the smartphone is \$313.16. When the store sells the phone for \$779.00, its profit is \$219.99. The Price Components For A Smartphone  ■ Cost --> \$465.84 ■ Expenses --> \$93.17 ■ Profit --> \$219.99

Calculating the Markup Percent

It is important to understand markup in terms of the actual dollar amount; however, it is more common in business practice to calculate the markup as a percentage. There are three benefits to converting the markup dollar amount into a percentage:

1. Easy comparison of different products having vastly different price levels and costs, to help you see how each product contributes toward the financial success of the company. For example, if a chocolate bar has a 50¢ markup included in a selling price of \$1, while a car has a \$1,000 markup included in a selling price of \$20,000, it is difficult to compare the profitability of these items. If these numbers were expressed as a percentage of the selling price such that the chocolate bar has a 50% markup and the car has a 5% markup, it is clear that more of every dollar sold for chocolate bars goes toward list profitability.
2. Simplified translation of costs into a regular selling price—a task that must be done for each product, making it helpful to have an easy formula, especially when a company carries hundreds, thousands, or even tens of thousands of products. For example, if all products are to be marked up by 50% of cost, an item with a \$100 cost can be quickly converted into a selling price of \$150.
3. An increased understanding of the relationship between costs, selling prices, and the list profitability for any given product. For example, if an item selling for \$25 includes a markup on selling price of 40% (which is \$10), then you can determine that the cost is 60% of the selling price (\$15) and that \$10 of every \$25 item sold goes toward list profits.

You can translate the markup dollars into a percentage using two methods, which express the amount either as a percentage of cost or as a percentage of selling price:

1. *Method 1: Markup as a Percentage of Cost.* This method expresses the markup rate using cost as the base. Many companies use this technique internally because most accounting is based on cost information. The result, known as the **markup on cost percentage**, allows a reseller to convert easily from a product's cost to its regular unit selling price.
2. *Method 2: Markup as a Percentage of Selling Price.* This method expresses the markup rate using the regular selling price as the base. Many other companies use this method, known as the **markup on selling price percentage**, since it allows for quick understanding of the portion of the selling price that remains after the cost of the product has been recovered. This percentage represents the list profits before the deduction of expenses and therefore is also referred to as the *list profit margin*.



The Formula

Both formulas are versions of Formula 2.2 on rate, portion, and base. The markup on cost percentage is expressed in Formula 6.8, while the markup on selling price percentage is expressed in Formula 6.9.

MoC% is Markup on Cost Percentage: This is the percentage by which the cost of the product needs to be increased to arrive at the selling price for the product.

M\$ is Markup Amount: The total dollars of the expenses and the profits; this total is the difference between the cost and the selling price.

Formula 6.8 – Markup On Cost Percentage:
$$\text{MoC}\% = \frac{\text{M\$}}{\text{C}} \times 100$$

C is Cost: The amount of money needed to acquire or manufacture the product. If the product is being acquired, the cost is the same amount as the net price paid.

× 100 is Percent Conversion: The markup on cost is always a percentage.

MoS% is Markup on Selling Price Percentage: This is the percentage of the selling price that remains available as list profits after the cost of the product is recovered.

M\$ is Markup Amount: The total dollars of the expenses and the profits; this total is the difference between the cost and the selling price.

Formula 6.9 – Markup On Selling Price Percentage:
$$\text{MoS}\% = \frac{\text{M\$}}{\text{S}} \times 100$$

S is Selling Price: The regular selling price of the product.

× 100 is Percent Conversion: The markup on cost is always a percentage.



How It Works

Refer back to the steps in calculating markup. The steps you must follow for calculations involving markup percent are almost identical to those for working with markup dollars:

Step 1: Three variables are required in either Formula 6.8 or Formula 6.9. For either formula, at least two of the variables must be known. If the amounts are not directly provided, you may need to calculate these amounts by applying other discount or markup formulas.

Step 2: Solve either Formula 6.8 or Formula 6.9 for the unknown variable.

Continuing to work with Example 6.2D, recall that the cost of the tech device is \$26.15, the markup amount is \$13.84, and the selling price is \$39.99. Calculate both markup percentages.

Step 1: The known variables are $C = \$26.15$, $M\$ = \$13.84$, and $S = \$39.99$.

Step 2: To calculate the markup on cost percentage, apply Formula 6.8:

$$\text{MoC}\% = \frac{\$13.84}{\$26.15} \times 100 = 52.9254\%$$

In other words, you must add 52.9254% of the cost on top of the unit cost to arrive at the regular unit selling price of \$39.99.

Step 2 (continued): To calculate the markup on selling price percentage, apply Formula 6.9:

$$\text{MoS}\% = \frac{\$13.84}{\$39.99} \times 100 = 34.6087\%$$

In other words, 34.6087% of the selling price represents list profits after the business recovers the \$26.15 cost of the tech device.



Important Notes

Businesses are very focused on profitability. Your Texas Instruments BAII Plus calculator is programmed with the markup on selling price percentage. The function is located on the second shelf above the number three. To use this function, open the window by pressing 2nd 3. You can scroll between lines using your ↑ and ↓ arrows. There are three variables:

- CST is the cost. Use the symbol C.
- SEL is the selling price. Use the symbol S.
- MAR is the markup on selling price percentage. Use the symbol MoS%.

As long as you know any two of the variables, you can solve for the third. Enter any two of the three variables (you need to press ENTER after each), making sure the window shows the output you are seeking, and press CPT.



Things To Watch Out For

Merchandising involves many variables. Nine formulas have been established so far, and a few more are yet to be introduced. Though you may feel bogged down by all of these formulas, just remember that you have encountered most of these merchandising concepts since you were very young and that you interact with retailers and pricing every day. This chapter merely formalizes calculations you already perform on a daily basis, whether at work or at home. The calculation of discounts is no different than going to Walmart and finding your favourite CD on sale. You know that when a business sells a product, it has to recoup the cost of the product, pay its bills, and make some money. And you have worked with percentages since elementary school.

Do not get stuck in the formulas. Think about the concept presented in the question. Change the scenario of the question and put it in the context of something more familiar. Ultimately, if you really have difficulties then look at the variables provided and cross-reference them to the merchandising formulas. Your goal is to find formulas in which only one variable is unknown. These formulas are solvable. Then ask yourself, “How does knowing that new variable help solve any other formula?”

You do not need to get frustrated. Just be systematic and relate the question to what you already know.

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The triangle method simplifies rearranging both Formulas 6.8 and 6.9 to solve for other unknown variables as illustrated in the figure to the right.

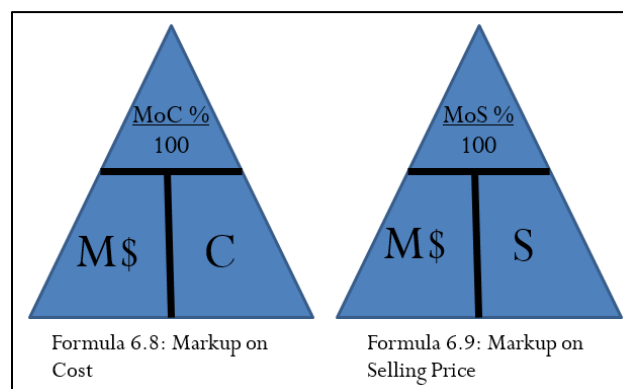
Sometimes you need to convert the markup on cost percentage to a markup on selling price percentage, or vice versa. Two shortcuts allow you to convert easily from one to the other:

$$\text{MoC}\% = \frac{\text{MoS}\%}{1 - \text{MoS}\%}$$

$$\text{MoS}\% = \frac{\text{MoC}\%}{1 + \text{MoC}\%}$$

Notice that these formulas are very similar. How do you remember whether to add or subtract in the denominator? In normal business situations, the MoC% is always larger than the MoS%. Therefore, if you are converting one to the other you need to identify whether you want the percentage to become larger or smaller.

- To calculate MoC%, you want a larger percentage. Therefore, make the denominator smaller by subtracting MoS% from 1.
- To calculate MoS%, you want a smaller percentage. Therefore, make the denominator larger by adding MoC% to 1.





Give It Some Thought

Answer the following true/false questions.

1. The markup on selling price percentage can be higher than 100%.
2. The markup dollar amount can be more than the selling price.
3. The markup on cost percentage can be higher than 100%.
4. The markup on cost percentage in most business situations is higher than the markup on selling price percentage.
5. If you know the markup on cost percentage and the cost, you can calculate a selling price.
6. If you know the markup on selling price percentage and the cost, you can calculate a selling price.

Solutions:

1. False. The markup amount is a portion of the selling price and therefore is less than 100%.
2. False. The markup amount plus the cost equals the selling price. It must be less than the selling price.
3. True. A cost can be doubled or tripled (or increased even more) to reach the price.
4. True. The base for markup on cost percentage is smaller, which produces a larger percentage.
5. True. You could combine Formulas 6.7 and 6.8 to arrive at the selling price.
6. True. You could convert the MoS% to a MoC% and solve as in the previous question.

Example 6.2G: Markup as a Percentage

A large national retailer wants to price a Texas Instruments BAII Plus calculator at the MSRP of \$39.99. The retailer can acquire the calculator for \$17.23.

- a. What is the markup on cost percentage?
- b. What is the markup on selling price percentage?

Plan	You have been asked to solve for the markup on cost percentage (MoC%) and the markup on selling price percentage (MoS%).							
Understand	What You Already Know Step 1: The regular unit selling price and the cost are provided: $S = \$39.99$, $C = \$17.23$	How You Will Get There Step 1 (continued): You need the markup dollars. Apply Formula 6.7, rearranging for M\$. Step 2: To calculate markup on cost percentage, apply Formula 6.8. Step 2 (continued): To calculate markup on selling price percentage, apply Formula 6.9.						
Perform	Step 1 (continued): $\$39.99 = \$17.23 + M\$$ $\$22.76 = M\$$ Step 2: $\text{MoC}\% = \frac{\$22.76}{\$17.23} \times 100 = 132.0952\%$ Step 2 (continued): $\text{MoS}\% = \frac{\$22.76}{\$39.99} \times 100 = 56.9142\%$	Calculator Instructions <table border="1"> <tr> <td>CST</td><td>SEL</td><td>MAR</td></tr> <tr> <td>17.23</td><td>39.99</td><td>Answer: 56.9142</td></tr> </table>	CST	SEL	MAR	17.23	39.99	Answer: 56.9142
CST	SEL	MAR						
17.23	39.99	Answer: 56.9142						
Present	The markup on cost percentage is 132.0952%. The markup on selling price percentage is 56.9142%.							

Break-Even Pricing

In running a business, you must never forget the “bottom line.” In other words, if you fully understand how your products are priced, you will know when you are making or losing money. Remember, if you keep losing money you will not stay in business for long! As revealed in Chapter 5, 15% of new businesses will not make it past their first year, and 49% fail in

their first five years. This number becomes even more staggering with an 80% failure rate within the first decade.¹ Do not be one of these statistics! With your understanding of markup, you now know what it takes to break even in your business. Recall from Chapter 5 that *break-even* means that you are earning no profit, but you are not losing money either. Your profit is zero.



The Formula

If the regular unit selling price must cover three elements—cost, expenses, and profit—then the regular unit selling price must exactly cover your costs and expenses when the profit is zero. In other words, if Formula 6.5 is modified to calculate the selling price at the break-even point (S_{BE}) with $P=0$, then

$$S_{BE} = C + E$$

This is not a new formula. It just summarizes that at break-even there is no profit or loss, so the profit (P) is eliminated from the formula.



How It Works

The steps you need to calculate the break-even point are no different from those you used to calculate the regular selling price. The only difference is that the profit is always set to zero.

Recall Example 6.2D that the cost of the tech device is \$26.15 and expenses are \$7.84. The break-even price (S_{BE}) is $\$26.15 + \$7.84 = \$33.99$. This means that if the tech device is sold for anything more than \$33.99, it is profitable; if it is sold for less, then the business does not cover its costs and expenses and takes a loss on the sale.

Example 6.2H: Knowing Your Break-Even Price

John is trying to run an eBay business. His strategy has been to shop at local garage sales and find items of interest at a great price. He then resells these items on eBay. On John's last garage sale shopping spree, he only found one item—a gaming console that was sold to him for \$100. John's vehicle expenses (for gas, oil, wear/tear, and time) amounted to \$40. eBay charges a \$2.00 insertion fee, a flat fee of \$2.19, and a commission of 3.5% based on the selling price less \$25. What is John's minimum list price for his gaming console to ensure that he at least covers his expenses?

Plan	You are trying to find John's break-even selling price (S_{BE}).	
Understand	What You Already Know Step 1: John's cost for the gaming console and all of his associated expenses are as follows: $C = \$100.00$ $E \text{ (vehicle)} = \$40.00$ $E \text{ (insertion)} = \$2.00$ $E \text{ (flat)} = \$2.19$ $E \text{ (commission)} = 3.5\%(S_{BE} - \$25.00)$	How You Will Get There Step 1 (continued): You have four expenses to add together that make up the E in the formula. Step 2: Formula 6.5 states $S = C + E + P$. Since you are looking for the break-even point, then P is set to zero and $S_{BE} = C + E$.
Perform	Step 1 (continued): $E = \$40.00 + \$2.00 + \$2.19 + 3.5\%(S_{BE} - \$25.00)$ $E = \$44.19 + 0.035(S_{BE} - \$25.00) = \$44.19 + 0.035S_{BE} - \$0.875$ $E = \$43.315 + 0.035S_{BE}$ Step 2: $S_{BE} = \$100.00 + \$43.315 + 0.035S_{BE}$ $S_{BE} = \$143.315 + 0.035S_{BE}$ $S_{BE} - 0.035S_{BE} = \$143.315$ $0.965S_{BE} = \$143.315$ $S_{BE} = \$148.51$	
Present	At a price of \$148.51 John would cover all of his costs and expenses but realize no profit or loss. Therefore, \$148.51 is his minimum price.	

¹ Statistics Canada, "Failure Rates for New Firms," *The Daily*, February 16, 2000, www.statcan.gc.ca/daily-quotidien/000216/dq000216b-eng.htm.



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6.3: Markdown: Setting the Sale Price

(Everybody Loves a Sale)

Flashy signs in a retail store announce, “40% off, today only!” Excitedly you purchase three tax-free products with regular price tags reading \$100, \$250, and \$150. The cashier processing the transaction informs you that your total is \$325. You are about to hand over your credit card when something about the total makes you pause. The regular total of all your items is \$500. If they are 40% off, you should receive a \$200 deduction and pay only \$300. The cashier apologizes for the mistake and corrects your total.

Although most retail stores use automated checkout systems, these systems are ultimately programmed by human beings. A computer system is only as accurate as the person keying in the data. A study by the Competition Bureau revealed that 6.3% of items at various retail stores scanned incorrectly. The average error spread is up to 13% around the actual product’s price!² Clearly, it is important for you as a consumer to be able to calculate markdowns.

Businesses must also thoroughly understand markdowns so that customers are charged accurately for their purchases. Businesses must always comply with the Competition Act of Canada, which specifically defines legal pricing practices. If your business violates this law, it faces severe penalties.

The Importance of Markdowns

A **markdown** is a reduction from the regular selling price of a product resulting in a lower price. This lower price is called the **sale price** to distinguish it from the selling price.

Many people perceive markdowns as a sign of bad business management decisions. However, in most situations this is not true. Companies must always attempt to forecast the future. In order to stock products, a reseller must estimate the number of units that might sell in the near future for every product that it carries. This is both an art and a science. While businesses use statistical techniques that predict future sales with a relative degree of accuracy, consumers are fickle and regularly change shopping habits. Markdowns most commonly occur under four circumstances:

1. *Clearing Out Excess or Unwanted Inventory.* In these situations, the business thought it could sell 100 units; however, consumers purchased only 20 units. In the case of seasonal inventory, such as Christmas items on Boxing Day, the retailer wishes to avoid packing up and storing the inventory until the next season.
2. *Clearing Out Damaged or Discontinued Items.* Selling a damaged product at a discount is better than not selling it at all. When products are discontinued, this leaves shelf space underused, so it is better to clear the item out altogether to make room for profitable items that can keep the shelves fully stocked.
3. *Increasing Sales Volumes.* Sales attract customers because almost everyone loves a deal. Though special marketing events such as a 48 hour sale reduce the profitability per unit, by increasing the volume sold these sales can lead to a greater profit overall.
4. *Promoting Add-On Purchases.* Having items on sale attracts customers to the store. Many times customers will not only purchase the item on sale but also, as long as they are on the premises, grab a few other items, which are regularly priced and very profitable. Like many others, you may have walked into Walmart to buy one item but left with five instead.



The Formula

Markdowns are no different from offering a discount. Recall from Section 6.1 that one of the types of discounts is known as a sale discount. The only difference here lies in choice of language. Markdowns are common, so you will find it handy to adapt the discount formulas to the application of markdowns, replacing the symbols with ones that are meaningful in merchandising. Formula 6.1, introduced in Section 6.1, calculates the net price for a product after it receives a single discount:

$$N = L \times (1 - d)$$

Formula 6.10 adapts this formula for use in markdown situations.

² Competition Bureau, Fair Business Practices Branch, *Price Scanning Report*, Table B, page 5, 1999, www.competitionbureau.gc.ca/epic/site/cb-bc.nsf/en/01288e.html.

S_{onsale} is Sale Price: The sale price is the price of the product after reduction by the markdown percent. Conceptually, the sale price is the same as the net price.

Formula 6.10 – The Sale Price Of A Product: $S_{\text{onsale}} = S \times (1 - d)$

S is Selling Price: The regular selling price of the product before any discounts. The higher price is the list price. In merchandising questions, this dollar amount may or may not be a known variable. If the selling price is unknown, you must calculate it using an appropriate formula or combination of formulas from either Section 6.1 or Section 6.2.

d is Markdown Rate: A markdown rate is the same as a sale discount rate. Therefore you use the same discount rate symbol from Section 6.1 to represent the percentage (in decimal format) by which you reduce the selling price. As in Formula 6.1, note that you are interested in calculating the sale price and not the amount saved. Thus, you take the markdown rate away from 1 to find out the rate owing.

In markdown situations, the selling price and the sale price are different variables. The sale price is always less than the selling price. In the event that a regular selling price has more than one markdown percent applied to it, you can extend Formula 6.10 in the same manner that Formula 6.3 calculated multiple discounts.

If you are interested in the markdown amount in dollars, recall that Formula 6.2 calculates the discount amount in dollars. Depending on what information is known, the formula has two variations:

$$\text{Formula 6.2a: } D\$ = L \times d$$

$$\text{Formula 6.2b: } D\$ = L - N$$

Formulas 6.11a and 6.11b adapt these formulas to markdown situations.

D\$ is Markdown Amount: You determine the markdown amount using either formula depending on what information is known. If you know the selling price and markdown percent, apply Formula 6.11a. If you know the selling price and sale price, apply Formula 6.11b.

Formula 6.11a – Markdown Amount: $D\$ = S \times d$

Formula 6.11b – Markdown Amount: $D\$ = S - S_{\text{onsale}}$

S is Selling Price: The regular selling price before you apply any markdown percentages.

S_{onsale} is Sale Price: The price after you have deducted all markdown percentages from the regular selling price.

d is Markdown Rate: the percentage of the selling price to be deducted (in decimal format). In this case, because you are interested in figuring out how much the percentage is worth, you do not take it away from 1 as in Formula 6.10.

The final markdown formula reflects the tendency of businesses to express markdowns as percentages, facilitating easy comprehension and comparison. Recall Formula 6.9 from Section 6.2, which calculated a markup on selling price percent:

$$\text{MoS}\% = \frac{M\$}{S} \times 100$$

Formula 6.12 adapts this formula to markdown situations.

d is Markdown Percentage: You always deduct a markdown amount from the regular selling price of the product. Therefore, you always express the markdown percent as a percentage of the selling price. Use the same symbol for a discount rate, since markdown rates are synonymous with sale discounts.

Formula 6.12 – Markdown Percentage:
$$d = \frac{D\$}{S} \times 100$$

S is Selling Price: The regular selling price of the product before any discounts.

×100 is Percent Conversion: The markdown is always expressed as a percentage.

D\$ is Markdown Amount: The total dollar amount deducted from the regular selling price.



How It Works

Follow these steps to calculate a markdown:

Step 1: Across all three markdown formulas, the four variables consist of the selling price (S), sale price (S_{onsale}), markdown dollars (D\$), and markdown rate (d). Identify which variables are known. Depending on the known information, you may have to calculate the selling price using a combination of discount and markup formulas.

Step 2: Apply one or more of Formulas 6.10, 6.11a, 6.11b, and 6.12 to calculate the unknown variable(s). In the event that multiple markdown rates apply, extend Formula 6.10 to accommodate as many markdown rates as required.

Recall from Section 6.2 the example of the MP3 player with a regular selling price of \$39.99. Assume the retailer has excess inventory and places the MP3 player on sale for 10% off. What is the sale price and markdown amount?

Step 1: The selling price and markdown percent are $S = \$39.99$ and $d = 0.10$, respectively.

Step 2: Apply Formula 6.10 to calculate the sale price, resulting in $S_{\text{onsale}} = \$39.99 \times (1 - 0.10) = \$35.99$.

Step 2 (continued): You could use either of Formulas 6.11a or 6.11b to calculate the markdown amount since the selling price, sale price, and markdown percent are all known. Arbitrarily choosing Formula 6.11a, you calculate a markdown amount of $D\$ = \$39.99 \times 0.10 = \$4.00$.

Therefore, if the retailer has a 10% off sale on the MP3 players, it marks down the product by \$4.00 and retails it at a sale price of \$35.99.



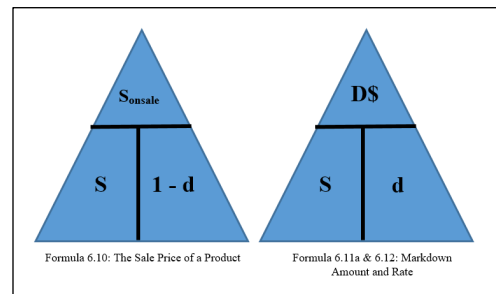
Things To Watch Out For

Just as in Section 6.2, avoid getting bogged down in formulas. Recall that the three formulas for markdowns are not new formulas, just adaptations of three previously introduced concepts. As a consumer, you are very experienced with endless examples of sales, bargains, discounts, blowouts, clearances, and the like. Every day you read ads in the newspaper and watch television commercials advertising percent savings. This section simply crystallizes your existing knowledge. If you are puzzled by questions involving markdowns, make use of your shopping experiences at the mall!

**TOP
SECRET**

Paths To Success

Three of the formulas introduced in this section can be solved for any variable through algebraic manipulation when any two variables are known. Recall that the triangle technique helps you remember how to rearrange these formulas, as illustrated here.



Example 6.3A: Determining the Sale Price and Markdown Amount

The MSRP for a video game is \$189.99. Most retail stores sell this product at a price in line with the MSRP. You have just learned that a local electronics retailer is selling the game for 45% off. What is the sale price for the video game and what dollar amount is saved?

Plan	There are two unknown variables. The first is the video game's sale price (S_{onsale}). The second is the markdown amount ($D\$$) that is realized at that sale price.	
Understand	What You Already Know Step 1: The regular selling price for the video game and the markdown rate are known: $S = \$189.99$ $d = 0.45$	How You Will Get There Step 2: Calculate the sale price by applying Formula 6.10. Step 2 (continued): Calculate the markdown amount by applying Formula 6.11b.
Perform	Step 2: $S_{\text{onsale}} = \$189.99 \times (1 - 0.45)$ $\quad = \$189.99 \times 0.55 = \$104.49$ Step 2 (continued): $D\$ = \$189.99 - \$104.49 = \$85.50$	
Present	The sale price for the video game is \$104.49. When purchased on sale, the video game is \$85.50 off of its regular price.	

Example 6.3B: Markdown Requiring Selling Price Calculation

A reseller acquires a tablet for \$650. Expenses are planned at 20% of the cost, and profits are set at 15% of the cost. During a special promotion, the tablet is advertised at \$100 off. What is the sale price and markdown percent?

Plan	The unknown variables for the tablet are the sale price (S_{onsale}) and the markdown rate (d).	
Understand	What You Already Know Step 1: The pricing elements of the tablet along with the markdown dollars are known: $C = \$650$ $E = 0.2C$ $P = 0.15C$ $D\$ = \$100$	How You Will Get There Step 1 (continued): Calculate the selling price of the product by applying Formula 6.5. Step 2: Calculate the markdown percent by applying Formula 6.12: Step 2 (continued): Calculate the sale price by applying Formula 6.11b, rearranging for S_{onsale} .
Perform	Step 1 (continued): $S = \$650 + 0.2(\$650) + 0.15(\$650) = \$877.50$ Step 2: $d = \frac{\$100}{\$877.50} \times 100 = 11.396\%$ Step 2 (continued): $\$100 = \$877.50 - S_{\text{onsale}}$ $\quad \quad \quad S_{\text{onsale}} = \$777.50$	
Present	When the tablet is advertised at \$100 off, it receives an 11.396% markdown and it will retail at a sale price of \$777.50.	

Never-Ending Sales

Have you noticed that some companies always seem to have the same item on sale all of the time? This is a common marketing practice. Recall the third and fourth circumstances for markdowns. Everybody loves a sale, so markdowns increase sales volumes for both the marked-down product and other regularly priced items.

For example, Michaels has a product line called the Lemax Village Collection, which has seasonal display villages for Christmas, Halloween, and other occasions. When these seasonal product lines come out, Michaels initially prices them at the regular unit selling price for a short period and then reduces their price. For Michaels, this markdown serves a strategic purpose. The company's weekly flyers advertising the Lemax Village Collection sale attract consumers who usually leave the store with other regularly priced items.



The Formula

If an item is on sale all the time, then businesses plan the pricing components with the sale price in mind. Companies using this technique determine the unit profitability of the product at the sale price and not the regular selling price. They adapt Formula 6.5 as follows:

$$S = C + E + P \quad \text{becomes} \quad S_{\text{onsale}} = C + E + P_{\text{onsale}}$$

where P_{onsale} represents the planned profit amount when the product is sold at the sale price. This is not a new formula, just a new application of Formula 6.5.



How It Works

Under normal circumstances, when businesses set their selling and sale prices they follow a three-step procedure:

1. Determine the product's cost, expenses, and profit amount.
2. Set the regular selling price of the product.
3. If a markdown is to be applied, determine an appropriate markdown rate or amount and set the sale price.

However, when a product is planned to always be on sale, businesses follow these steps instead to set the sale price and selling price:

Step 1: Set the planned markdown rate or markdown dollars. Determine the pricing components such as cost and expenses. Set the profit so that when the product is marked down, the profit amount is achieved. Alternatively, a planned markup on cost, markup on selling price, or even markup dollars may be set for the sale price.

Step 2: Calculate the sale price of the product. If cost, expenses, and profit are known, apply the adapted version of Formula 6.5. Alternatively, adapt and apply any of the other markup formulas (Formulas 6.6 through 6.9) with the understanding that the result is the sale price of the product and not the regular selling price.

Step 3: Using the known markdown rate or markdown amount, set the regular selling price by applying any appropriate markdown formula (Formulas 6.10 through 6.12).

Assume for the Michael's Lemax Village Collection that most of the time these products are on sale for 40% off. A particular village item costs \$29.99, expenses are \$10.00, and a planned profit of \$8.00 is achieved at the sale price. Calculate the sale price and the selling price.

Step 1: The known variables at the sale price are $C = \$29.99$, $E = \$10.00$, $P = \$8.00$, and $d = 0.40$.

Step 2: Adapting Formula 6.5, the sale price is $S_{\text{onsale}} = C + E + P_{\text{onsale}} = \$29.99 + \$10.00 + \$8.00 = \$47.99$. This is the price at which Michael's plans to sell the product.

Step 3: However, to be on sale there must be a regular selling price. Therefore, if the 40% off results in a price of \$47.99, apply Formula 6.10 and rearrange to get the selling price: $S = \$47.99 \div (1 - 0.40) = \$79.98$. Therefore, the product's selling price is \$79.98, which, always advertised at 40% off, results in a sale price of \$47.99. At this sale price, Michael's earns the planned \$8.00 profit.



Important Notes

You may ask, "If the product is always on sale, what is the importance of establishing the regular price?" While this textbook does not seek to explain the law in depth, it is worth mentioning that pricing decisions in Canada are regulated by the

Competition Act. With respect to the discussion of never-ending sales, the Act does require that the product be sold at a regular selling price for a reasonable period of time or in reasonable quantity before it can be advertised as a sale price.

If you revisit the Michael's example, note in the discussion that the village initially needs to be listed at the regular selling price before being lowered to the sale price.



Give It Some Thought

1. If a product has a markup on cost of 40% and a markdown of 40%, will it sell above or below cost?
2. What happens to the profit if a product that is always on sale actually sells at the regular selling price?
3. Under normal circumstances, arrange from smallest to largest: regular selling price, cost, and sale price.

Solutions:
 1. Below cost, since the 40% markdown is off of the selling price, which is a larger value.
 2. The profit will be increased by the markdown amount.
 3. Cost, sale price, regular selling price.

Example 6.3C: Setting the Price in a Never-Ending Sale

An electronics retailer has USB sticks on sale at 50% off. It initially priced these USB sticks for a short period of time at regular price, but it planned at the outset to sell them at the sale price. The company plans on earning a profit of 20% of the cost when the product is on sale. The unit cost of the USB stick is \$22.21, and expenses are 15% of the cost.

- a. At what price will the retailer sell the USB stick when it is on sale?
- b. To place the USB stick on sale, it must have a regular selling price. Calculate this price.
- c. If the USB stick is purchased at the regular selling price during the initial time period, how much profit is earned?

Plan	a. This company plans on always having the product on sale, so the pricing needs to be set for the sale price, or S_{onsale} . b. You need the regular selling price, or S . c. Solve for the profit at the regular selling price, or P .	
Understand	What You Already Know Step 1: You know the unit cost, the retailer's associated expenses, its planned profit at the sale price, and the markdown rate: $d = 0.50$ $C = \$22.21$ $E = 0.15C$ $P_{\text{onsale}} = 0.2C$	How You Will Get There Step 2: To solve part (a), apply the adapted version of Formula 6.5: $S_{\text{onsale}} = C + E + P_{\text{onsale}}$ Step 3: After you know the sale price, solve part (b) by applying Formula 6.10, rearranging for S . Step 4: Solving part (c) requires applying Formula 6.5.
Perform	Step 2: $S_{\text{onsale}} = \$22.21 + 0.15(\$22.21) + 0.2(\$22.21)$ $\quad = \$22.21 + \$3.33 + \$4.44 = \$29.98$ Step 3: $\$29.98 = S \times (1 - 0.5)$ $\quad \$29.98 = S \times 0.5$ $\quad \$59.96 = S$	Step 4: $\$59.96 = \$22.21 + \$3.33 + P$ $\quad \$59.96 = \$25.54 + P$ $\quad \$34.42 = P$
Present	The USB stick is on sale for \$29.98, letting the company achieve its profit of \$4.44 per unit. During the initial pricing period, the USB stick sells for \$59.96 (its regular selling price). If a consumer actually purchases a USB stick during the initial pricing period, the electronics store earns a profit of \$34.42 per unit (which is a total of the \$4.44 planned profit plus the planned markdown amount of \$29.96).	



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6.4: Merchandising

(How Does It All Come Together?)

Running a business requires you to integrate all of the concepts in this chapter. From discounts to markups and markdowns, all the numbers must fit together for you to earn profits in the long run. It is critical to understand how pricing decisions affect the various financial aspects of your business and to stay on top of your numbers.

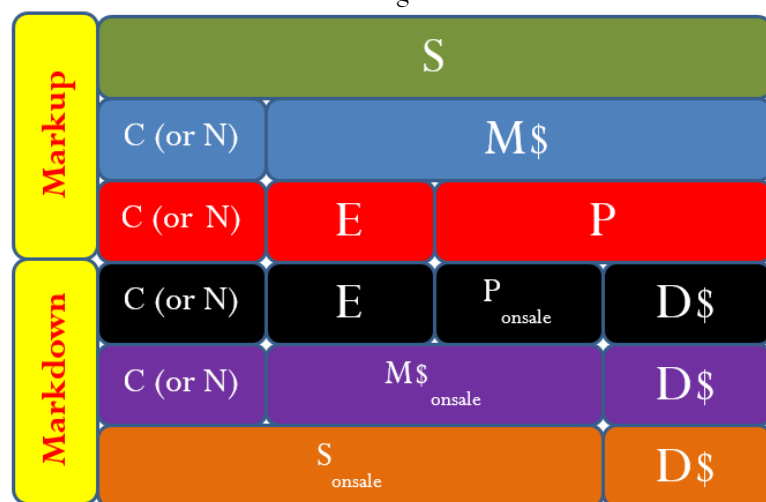
Merchandising does not involve difficult concepts, but to do a good job of it you need to keep track of many variables and observe how they relate to one another. Merchandising situations in this section will apply all of the previously discussed pricing concepts. Next, the concept of maintained markup helps you understand the combined effect of markup and markdown decisions on list profitability. Finally, you will see that coupons and rebates influence expenses and profitability in several ways beyond the face value of the discounts offered.

The Complete Pricing Picture

Up to this point, you have examined the various components of merchandising as separate topics. Your study of basic product pricing has included the following:

1. Taking an MSRP or list price and applying discounts to arrive at a product's cost
2. Marking up a product by adding expenses and profits to the cost to arrive at a regular selling price
3. Marking down a product by applying a discount and arriving at a sale price
4. Working with various percentages in both markup and markdown situations to either simplify calculations or present a clearer pricing picture

Now is the time to tie all of these merchandising concepts together. Section 6.2 introduced the concept of a markup chart to understand the relationship between various markup components. The figure below extends this figure to include all of the core elements of merchandising.



The figure reveals the following characteristics:

1. You calculate the cost of a product by applying a discount formula. The net price, N, is synonymous with cost, C.
2. When you mark a product down, not only is the profit reduced but the markup dollars are also reduced. The following relationships exist when a product is on sale:

$$M\$_{\text{onsale}} = E + P_{\text{onsale}} \quad \text{OR} \quad M\$_{\text{onsale}} = M\$ - D\$$$

$$P_{\text{onsale}} = P - D\$$$

3. At break-even (remember, P or P_{onsale} is zero) a sale product has the following relationships:

$$M\$_{\text{onsale}} = E$$

$$S_{\text{onsale}} = C + E$$

$$D\$ = P$$

None of these relationships represent new formulas. Instead, these reflect a deeper understanding of the relationships between the markup and markdown formulas.



How It Works

Follow these steps to solve complete merchandising scenarios:

Step 1: It is critically important to correctly identify both the known and unknown merchandising variables that you are asked to calculate.

Step 2: For each of the unknown variables in step 1, examine Formulas 6.1 through 6.12. Locate which formulas contain the unknown variable. You must solve one of these formulas to arrive at the answer. Based on the information provided, examine these formulas to determine which formula may be solvable. Write out this formula, identifying which components you know and which components remain unknown. For example, assume that in step 1 you were asked to solve for the expenses, or E. In looking at the formulas, you find this variable appears only in Formulas 6.5 and 6.6, meaning that you must use one of these two formulas to calculate the expenses. Suppose that, in reviewing the known variables from step 1, you already have the markup amount and the profit. In this case, Formula 6.6 would be the right formula to calculate expenses with.

Step 3: Note the unknown variables among all the formulas written in step 2. Are there common unknown variables among these formulas? These common variables are critical variables. Solving for these common unknowns is the key to completing the question. Note that these unknown variables may not directly point to the information you were requested to calculate, and they do not resolve the merchandising scenario in and of themselves. However, without these variables you cannot solve the scenario. For example, perhaps in step 1 you were requested to calculate both the markup on cost percent (MoC%) and the markup on selling price percent (MoS%). In step 2, noting that these variables are found only in Formulas 6.8 and 6.9, you wrote them down. Examination of these formulas revealed that you know the cost and selling price variables. However, in both formulas the markup dollars remain unknown. Therefore, markup dollars is the critical unknown variable. You must solve for markup dollars using Formula 6.7, which will then allow you to calculate the markup percentages.

Step 4: Apply any of the merchandising formulas from Formulas 6.1 to 6.12 to calculate the unknown variables required to solve the formulas. Your goal is to identify all required variables and then solve the original unknown variables from step 1.



Things To Watch Out For

Before proceeding, take a few moments to review the various concepts and formulas covered earlier in this chapter. A critical and difficult skill is now at hand. As evident in steps 2 through 4 of the solving process, you must use your problem-solving skills to figure out which formulas to use and in what order. Here are three suggestions to help you on your way:

1. Analyze the question systematically. Conscientiously use the Plan, Understand, Perform, and Present model.
2. If you are unsure of how the pieces of the puzzle fit together, try substituting your known variables into the various formulas. You are looking for
 - a. Any solvable formulas with only one unknown variable or
 - b. Any pair of formulas with the same two unknowns, since you can solve this system using your algebraic skills of solving two linear equations with two unknowns (covered in Section 2.5).
3. Merchandising has multiple steps. Think through the process. When you solve one equation for an unknown variable, determine how knowing that variable affects your ability to solve another formula. As mentioned in step 3, there is usually a critical unknown variable. Once you determine the value of this variable, a domino effect allows you to solve any other remaining formulas.

Example 6.4A: A Merchandising Situation

A skateboard shop stocks a premium skateboard. The MSRP for this board is \$82. If the shop is going to stock the board, it must order at least 25 units. The known information is as follows:

- The supplier offers a retail trade discount of 37% with a quantity discount of 12% for orders that exceed 15 units.
- Typical expenses associated with selling this premium skateboard amount to 31% of the regular selling price.
- The skateboard shop wants each of its products to have profitability of 13% of the regular selling price.
- In the event that the skateboards do not sell by the end of the season, the skateboard shop always holds an end-of-season clearance sale where it marks down its products to the break-even selling price.

Using this information about the skateboard, determine the following:

- a. Cost
- b. Regular unit selling price
- c. Markup on cost percentage
- d. Markup on selling price percentage
- e. Sale price
- f. Markdown percentage

Plan	In order, you are looking for C, S, MoC%, MoS%, S_{onsale} , and d(markdown).							
Understand	What You Already Know Step 1: The list price, two discounts (note the store qualifies for both since it meets the minimum quantity criteria), expense and profit information, as well as information on the sale price are known: $L = \$82.00$ $d_1 = 0.37$ $d_2 = 0.12$ $E = 0.31S$ $P = 0.13S$ $S_{\text{onsale}} = \text{break-even price}$ The unknown variables are: C, S, MoC%, MoS%, S_{onsale}, and d(markdown).	How You Will Get There Step 2: - Cost and net price are synonymous. Net price appears in Formulas 6.1, 6.2b, and 6.3. Since you know list price and two discounts, calculate cost through Formula 6.3. - Selling price appears in Formulas 6.5, 6.7, 6.9, 6.10, 6.11a, and 6.11b. Known variables include expenses and profit. You will know the cost once you have solved part (a). Choose Formula 6.5. - Markup on cost percent appears only in Formula 6.8. You will know the cost once you have solved part (a). M\$ is an unknown variable. - Markup on selling price percent appears only in Formula 6.9. You will know the selling price once you have solved parts (a) and (b). M\$ is an unknown variable. - A break-even price uses an adapted Formula 6.5. You will know the cost once you have solved part (a) and determined the expenses based on the selling price in part (b). - Markdown percent appears in Formulas 6.10, 6.11a, and 6.12. You will know the regular price once you have solved part (b). You will know the sale price once you have solved part (e). Choose Formula 6.10. Step 3: From the above rationalization, the only unknown variables are cost and markup dollars. Once you know these two variables you can complete all other formulas. Calculate cost as described in step 2a. To calculate markup dollars, use Formulas 6.6, 6.7, 6.8, and 6.9. Once you know the cost and selling price from parts 2(a) and 2(b), you can apply Formula 6.7, rearranging for markup dollars.						
Perform	Step 4: $C = N = \\$82 \times (1 - 0.37) \times (1 - 0.12) = \\$45.46$ $S = \\$45.46 + 0.31S + 0.13S$ $0.56S = \\$45.46$ $S = \\$81.18$ - As determined in step 3, you need markup dollars first. $\\$81.18 = \\$45.46 + M\\$ $\\$35.72 = M\\$ Now solve: $\text{MoC}\% = \frac{\\$35.72}{\\$45.46} \times 100 = 78.5746\%$ $\text{MoS}\% = \frac{\\$35.72}{\\$81.18} \times 100 = 44\%$ $S_{\text{onsale}} = S_{\text{BE}} = \\$45.46 + 0.31(\\$81.18) = \\$70.63$ $\\$70.63 = \\$81.18 \times (1 - d)$ $0.87 = 1 - d$ $0.13 = d$	Calculator Instructions For part (d) only, use 2nd Profit <table border="1"> <tr> <td>CST</td><td>SEL</td><td>MAR</td></tr> <tr> <td>45.46</td><td>81.18</td><td>Answer: 44</td></tr> </table>	CST	SEL	MAR	45.46	81.18	Answer: 44
CST	SEL	MAR						
45.46	81.18	Answer: 44						
Present	The cost of the skateboard is \$45.46. The regular selling price for the skateboard is \$81.18. This is a markup on cost percent of 78.5746% and a markup on selling price percent of 44%. To clear out the skateboards, the sale price is \$70.63, representing a markdown of 13%.							

Maintained Markup

A price change results in a markup change. Whenever a selling price is reduced to a sale price, it has been demonstrated that the amount of the markup decreases (from M\$ to M_{onsale}) by the amount of the discount (D\$). This means

that some of the products are sold at full markup and some at a partial markup. For businesses, the strategic marking up of a product must factor in the reality that they do not sell every product at the same level of markup. Therefore, if a business must achieve a specified level of markup on its products, it must set its price such that the average of all markups realized equals its markup objective. The average markup that is realized in selling a product at different price points is known as the **maintained markup**.



The Formula

To understand the formula, assume a company has 100 units for sale. From past experience, it knows that 90% of the products sell for full price and 10% sell at a sale price because of discounts, damaged goods, and so on. At full price, the product has a markup of \$10, and 90 units are sold at this level, equalling a total markup of \$900. On sale, the product is discounted \$5, resulting in a markup of \$5 with 10 units sold, equalling a total markup of \$50. The combined total markup is \$950. Therefore, the maintained markup is $\$950 \div 100 = \$9.50$ per unit. If the company's goal is to achieve a \$10 maintained markup, this pricing structure will not accomplish that goal. Formula 6.13 summarizes the calculations involved with maintained markup.

MM is Maintained Markup: The maintained markup is the average markup in dollars that is realized after factoring in the quantity of units sold and differing levels of markup. In its simplest form, this formula is an applied version of Formula 3.4 on weighted average discussed in Chapter 3.

M\$ is Markup Amount: The markup amount represents the amount of markup realized at the regular selling price for the product. If you do not know this amount directly, calculate it using any of the markup formula (Formulas 6.6 through 6.9).

Formula 6.13 – Maintained Markup:
$$MM = \frac{M\$(n_1) + (M\$ - D\$(n_2))}{n_1 + n_2}$$

n_1 is Full Markup Output: This variable represents the number of units sold at full markup (the regular selling price). It could be an actual or forecasted number of units sold. This variable acts as a weight: If you do not know actual unit information, then you can instead assign a forecasted weight or percentage.

n_2 is Partial Markup Output: This variable represents the number of units sold at the reduced markup (the sale price). If you do not know unit information, use a forecasted weight or percentage instead.

D\$ is Markdown Amount: The markdown amount represents the amount by which the regular selling price has been reduced to arrive at the sale price. If you do not know this amount directly, calculate it using any of the markdown formulas (Formulas 6.11a, 6.11b, or 6.12).



Important Notes

Formula 6.13 assumes that only one price reduction applies, implying that there is only one regular price and one sale price. In the event that a product has more than one sale price, you can expand the formula to include more terms in the numerator. To expand the formula for each additional sale price required:

1. Extend the numerator by adding another $(M\$ - D\$(n_x))$, where the D\$ always represents the total difference between the selling price and the corresponding sale price and n_x is the quantity of units sold at that level of markup.
2. Extend the denominator by adding another n_x as needed.

For example, if you had the regular selling price and two different sale prices, Formula 6.13 appears as:

$$MM = \frac{M\$(n_1) + (M\$ - D\$_1)(n_2) + (M\$ - D\$_2)(n_3)}{n_1 + n_2 + n_3}$$



How It Works

Follow these steps to calculate the maintained markup:

Step 1: Identify information about the merchandising scenario. You must either know markup dollars at each pricing level or identify all merchandising information such as costs, expenses, profits, and markdown information. Additionally, you must know the number of units sold at each price level. If unit information is not available, you will identify a weighting based on percentages.

Step 2: If step 1 already identifies the markup dollars and discount dollars, skip this step. Otherwise, you must use any combination of merchandising formulas (Formulas 6.1 to 6.12) to calculate markup dollars and discount dollars.

Step 3: Calculate maintained markup by applying Formula 6.13.



Give It Some Thought

A company's pricing objective is to maintain a markup of \$5. It knows that some of the products sell at regular price and some of the products sell at a sale price. When setting the regular selling price, will the markup amount be more than, equal to, or less than \$5?

Solutions: More than \$5. In order to average to \$5, the markup at regular price must be higher than \$5 and the markup at sale price must be lower than \$5.

Example 6.4B: What Level of Markup Is Maintained?

A grocery store purchases a 10 kg bag of flour for \$3.99 and resells it for \$8.99. Occasionally the bag of flour is on sale for \$6.99. For every 1,000 units sold, 850 are estimated to sell at the regular price and 150 at the sale price. What is the maintained markup on the flour?

Plan	You are looking to calculate the maintained markup (MM).	
Understand	What You Already Know Step 1: The cost, selling price, and sale price information along with the units sold at each price are known: $C = \$3.99$ $S = \$8.99$ $S_{\text{onsale}} = \$6.99$ $n_1 = 850$ $n_2 = 150$	How You Will Get There Step 2: To calculate markup dollars, apply Formula 6.7, rearranging for M\$. To calculate markdown dollars, apply Formula 6.11b. Step 3: Calculate maintained markup by applying Formula 6.13.
Perform	Step 2: Markup amount: $\$8.99 = \$3.99 + C$ $C = \$5.00$ Markdown amount: $D\$ = \$8.99 - \$6.99 = \$2.00$ Step 3: $MM = \frac{\$5(850) + (\$5 - \$2)(150)}{850 + 150} = \$4.70$	
Present	On average, the grocery store will maintain a \$4.70 markup on the flour.	

Example 6.4C: Setting Prices to Achieve the Maintained Markup

Maplewood Golf Club has set a pricing objective to maintain a markup of \$41.50 on all 18-hole golf and cart rentals. The cost of providing the golf and rental is \$10. Every day, golfers before 3 p.m. pay full price while “twilight” golfers after 3 p.m. receive a discounted price equalling \$30 off. Typically, 25% of the club’s golfers are “twilight” golfers. What regular price and sale price should the club set to meet its pricing objective?

Plan	The golf club needs to calculate the regular selling price (S) and the sale price (S_{onsale}).	
Understand	What You Already Know Step 1: The cost information along with the maintained markup, markdown amount, and weighting of customers in each price category are known: $C = \$10$ $MM = \$41.50$ $D\$ = \$30$ $n_1 = 100\% - 25\% = 75\%$ $n_2 = 25\%$	How You Will Get There Reverse steps 2 and 3 since you must solve Formula 6.13 first and use this information to calculate the selling price and sale price. Step 3: Apply Formula 6.13, rearranging for M\$. Step 2: Calculate the selling price by applying Formula 6.7. Step 2 (continued): Calculate the sale price by applying Formula 6.11b, rearranging for S_{onsale} .
Perform	Step 3: $\$41.50 = \frac{M\$(0.75) + (M\$ - \$30)(0.25)}{0.75 + 0.25}$ $\$41.50 = 0.75M\$ + 0.25M\$ - \$7.50$ $\$49 = M\$$ Step 2: $S = \$10 + \$49 = \$59$ Step 2 (continued): $\$30 = \$59 - S_{\text{onsale}}$ $S_{\text{onsale}} = \$29$	
Present	In order to achieve its pricing objective, Maplewood Golf Club should set a regular price of \$59 and a “twilight” sale price of \$29.	

The Pricing Impact of Coupons and Mail-in Rebates

Consumers sometimes need a nudge to help them decide on a purchase. Have you ever been thinking about buying a product but put it off for a variety of reasons? Then, while browsing online one day, you suddenly come across a coupon. You print it off and hurry down to your local store, because the coupon expires in the next few days. You just participated in a price adjustment.

A **marketing price adjustment** is any marketing activity executed by a member of the distribution channel for the purpose of altering a product’s price. This includes an alteration to the regular unit selling price or a sale price. For example, Sony could distribute a rebate giving its consumers \$50 off the price of one of its televisions. Alternatively, Superstore could run a special promotion where customers receive \$30 off if they purchase at least \$250 in merchandise.

When making price adjustments, businesses must understand the full impact on the bottom line. Many inexperienced businesses distribute coupons for \$1 off, assuming that the only cost to be accounted for is the \$1. This is untrue for most price adjustments.

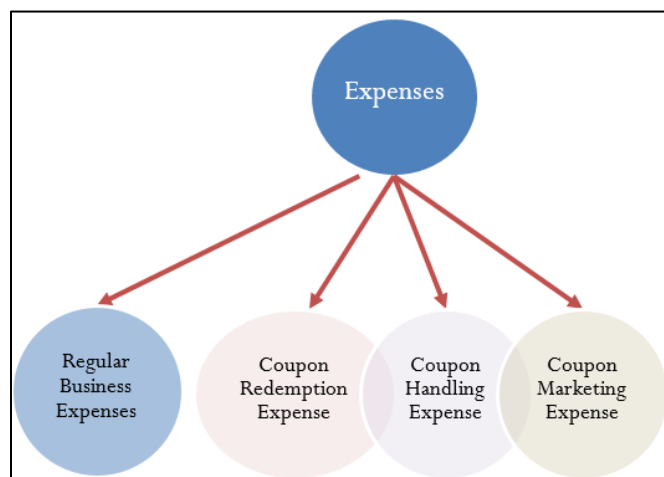
These marketing price adjustments reduce the prices that you pay as a consumer. Certainly, some price adjustments are more advantageous than others. Although they come in many types, this section focuses on two of the most widely used price adjustments: manufacturer coupons and mail-in rebates.

Manufacturer Coupons

A **coupon** is a promotion that entitles a consumer to receive a certain benefit, most commonly in the form of a price reduction. Coupons are typically offered by manufacturers to consumers.

The decision of a manufacturer to issue a coupon affects the expenses associated with the product in three ways.

1. **Coupon Redemption Expense.** The **coupon redemption expense** is the dollar amount price reduction that the coupon offers, also known as the *face value of the coupon*. The manufacturer must pay out this amount to the reseller as a reimbursement for redeeming the coupon, increasing the expenses accordingly. For example, if a coupon is for \$3 off, then the coupon redemption expense is \$3.
2. **Coupon Handling Expense.** Most commonly, retailers are the resellers that redeem manufacturer coupons. The resulting **coupon handling expense** for the manufacturer is known as a handling fee or handling charge, payable to the retailer. The manufacturer offers financial compensation to the retailer for redeeming the coupon, filling out the paperwork, and submitting it to manufacturer for reimbursement. Ultimately, the retailer should not be out any money for participating in the promotion. For example, in the small print a coupon may indicate that there is a \$0.09 handling fee. This amount is payable to the retailer upon the manufacturer's receipt of the redeemed coupon.
3. **Coupon Marketing Expense.** Coupons do not just magically appear in consumer hands. Companies must pay for the creation, distribution, and redemption of the coupons, which is known as the **coupon marketing expense**:
 - Coupons are designed, turned into print-ready artwork, and printed. This incurs design, labour, and material costs.
 - Coupons are commonly distributed separately from the product in media such as magazines, flyers, and newspapers. This requires the purchase of ad space.
 - Coupons are commonly redeemed through coupon clearinghouses that handle coupon redemption requests. Manufacturers must pay the fees charged by the clearinghouse.



The manufacturer issuing the coupon estimates how many coupons it expects consumers will redeem and assigns the average of the coupon marketing costs to each unit based on the forecasted volume.

It should be evident that the decision to issue coupons results in expenses much higher than the face value of the coupon. All three of the expenses listed above are added to the product's regular expenses. The end result is that the unit expenses increase while unit profits decrease.

As a consumer, you can appreciate the immediacy of the price reduction a coupon offers you at the cash register. Unlike with mail-in rebates, there is nothing further you need to do.

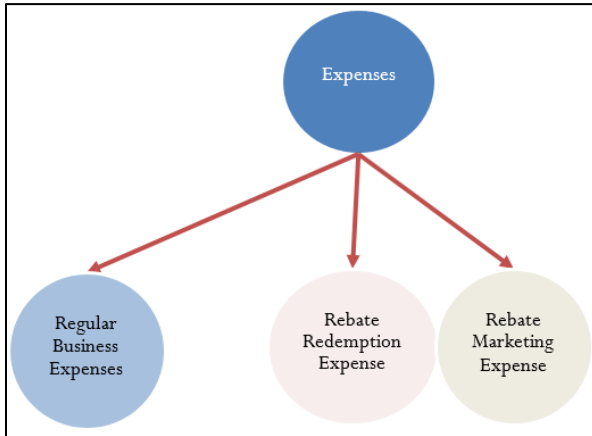
Mail-In Rebates

Have you ever purchased an item that came with a mail-in rebate but not submitted the rebate because it was too much hassle? Was your rebate refused because you failed to meet the stringent requirements? If you answered yes to either of these questions, then you have joined the ranks of many other consumers. In fact, marketers know you will either not mail in your rebates (some estimates are that 50% to 98% of mail-in rebates go unredeemed), or you will fail to send the required supporting materials. From a pricing point of view, this means that more product is sold at the regular price! That is why marketers use mail-in rebates.

Mail-in rebates are similar to coupons; however, there is a distinct difference in how the price adjustment works. A **mail-in rebate** is a refund at a later date after the purchase has already been made, whereas coupons instantly reward consumers with a price reduction at the cash register. Mail-in rebates commonly require consumers to fill in an information card as well as meet a set of stringent requirements (such as cutting out UPC codes and providing original receipts) that must be mailed in to the manufacturer at the consumer's expense (see the photo for an example). After a handling time of perhaps 8 to 12 weeks and if the consumer has met all requirements, the manufacturer issues a cheque in the amount of the rebate and sends it to the consumer.

With coupons there are three associated expenses: coupon redemption, handling, and marketing. In the case of rebates, since the participants in the channel are not involved with the rebate redemption, no handling expenses are paid out to the retailers.

1. **Rebate Redemption Expense.** The **rebate redemption expense** is the dollar amount price reduction that the rebate offers, also known as the *face value of the rebate*. This amount must be paid out to the consumer, commonly in the form of a cheque. For example, if a rebate is for \$50, then this is the amount sent to the consumer. This increases expenses, but often not by the full \$50. Recall that though many consumers purchase the product because the rebate is offered, not all of them redeem the rebate. To calculate the unit expense many companies average the rebate redemption amount based on the number of rebates they expect to fulfill. Take, for example, a company that issues a \$50 rebate on the purchase of a surround sound system. From past experience, it knows that only 5% of the rebates are actually redeemed properly and thus paid out. Therefore, the increase in expenses is not \$50, but 5% of \$50, or \$2.50 per unit sold under the rebate program.



2. **Rebate Marketing Expense.** Many mail-in rebates today are distributed either on or in the packaging of the product, or they can be located online. This tends to minimize (but not eliminate) the marketing expenses associated with creation, distribution, and redemption. The **rebate marketing expense** is commonly averaged across the increase in sales that results from the rebate

issuance and not the number of rebates redeemed. Therefore, if a company has \$100,000 in rebate marketing expenses, a direct increase in sales by 20,000 units, and only 1,000 rebates redeemed, the rebate marketing expense is $\$100,000 \div 20,000 = \$5.00$ per unit under the rebate program (note that the rebates redeemed is irrelevant to this calculation).



The Formula

With coupons and rebates, the focus is on how these marketing tools affect expenses and profit. You do not require any new formula; instead, Formula 6.5 is adapted to reflect the pricing impact of these tactics.

S is Selling Price: The manufacturer's regular selling price of the product is unchanged for both rebates and coupons. Mail-in rebates do not change the manufacturer's selling price to the wholesaler. Coupons lower the retail price and not the manufacturer's price.

C is Cost: The cost of the product to the manufacturer remains constant when marketers use coupons and rebates.

Formula 6.5: The Selling Price of a Product Under a Coupon or Rebate:

$$S = C + E(\text{adjusted}) + P_{\text{onsale}}$$

E(adjusted) is Expenses: When coupons and mail-in rebates are used, the manufacturer not only incurs regular operating expenses but must adjust this number upward to reflect the redemption, handling, and marketing expenses associated with these price adjustments. Therefore, products sold under a coupon program or rebate program always have higher expenses.

P_{onsale} is Profit: If the price and cost remain unchanged while the expenses have increased, then the profit per unit must be reduced. Using a coupon or rebate is essentially another form of putting the product on sale. You know that if a product is on sale, then the profits are reduced. The amount of this reduction will be equal to the amount by which the expenses increase.



How It Works

Follow these steps to calculate the selling price or any of its components when working with coupons or rebates:

Step 1: Identify the known components involved in the selling price, such as the cost, unit expense, profit, or selling price. Use any formulas needed to calculate any of these components. Also identify information about the coupon or rebate, including redemption, handling, and marketing expenses.

Step 2: Calculate the unit expense associated with the marketing pricing tactic.

- a. **Coupons.** The unit redemption expense is equal to the face value of the coupon. The unit handling expense is the promised handling fee of the coupon. The unit marketing expense is a simple average of the total marketing expense divided by the expected number of coupon redemptions. Calculate the total coupon expenses by summing all three elements.
- b. **Rebates.** The unit redemption expense is equal to the face value of the coupon multiplied by the redemption rate of the rebate. The unit marketing expense is equal to a simple average of the total marketing expense divided by the expected increase in sales as a result of the rebate. Calculate the total rebate expenses by summing both elements.

Step 3: Apply the required merchandising formulas to calculate the unknown variable(s). The modified Formula 6.5 presented earlier will most likely play prominently in these calculations.

Quick Coupon Example

Assume a manufacturer coupon is issued with a face value of \$5. The handling charge is \$0.15 per coupon. The marketing expenses are estimated at \$150,000, and the company forecasts that 100,000 coupons will be redeemed. The regular profit on the product is \$20. How much will the unit expenses of the product increase, and what is the unit profit under the coupon program?

Step 1: You know the following information: $P = \$20$, Face Value = \$5, Handling Charge = \$0.15; Total Marketing Expenses = \$150,000, and Forecasted Coupon Redemption = 100,000.

Step 2: The increase in expenses is a sum of the redemption, handling, and marketing expenses. The redemption expense is \$5. The handling expense is \$0.15. To get the marketing expense per unit, take the total expense and divide it by the number of redemptions expected. Therefore $\$150,000 \div 100,000 = \$1.50$. The total increase in expenses is then $\$5.00 + \$0.15 + \$1.50 = \$6.65$.

Step 3: If the expenses rise by \$6.65 with the same cost and selling price, then the profits fall by the same amount. The unit profit on any product sold using the coupon is $\$20.00 - \$6.65 = \$13.35$.

Quick Rebate Example

Assume a manufacturer mail-in rebate is issued with a face value of \$20. The marketing expenses are estimated at \$300,000 and the company plans on 10% of the mail-in rebates being redeemed. It is forecasted that sales will rise by 37,500 units as a result of the mail-in rebate. The regular profit on the product is \$25. How much will the unit expenses of the product increase, and what is the unit profit under the rebate program?

Step 1: You know the following information: $P = \$25$, Face Value = \$20, Total Marketing Expenses = \$300,000, Rebate Redemption Rate = 10%, and Increase in Sales = 37,500.

Step 2: The increase in expenses is a sum of the redemption and marketing expenses. The redemption expense is \$20.00 multiplied by the 10% redemption rate, or \$2. To calculate the marketing expense per unit, take the total expense and divide it by the increase in sales. Therefore $\$300,000 \div 37,500 = \$8$. The total increase in expenses is then $\$2 + \$8 = \$10$.

Step 3: If the expenses rise by \$10 with the same cost and selling price, then the profits fall by the same amount. The unit profit on any product sold using the coupon is $\$25 - \$10 = \$15$.



Things To Watch Out For

When calculating the per-unit expense, be careful to distinguish how to handle the various associated expenses for both manufacturer coupons and mail-in rebates, as summarized in the table below.

Marketing Tactic	Redemption Expense	Handling Expense	Marketing Expense
Coupon	Face value of the coupon	A flat rate per coupon	The total expense divided by the number of coupons redeemed
Mail-In Rebate	Face value of the mail-in rebate multiplied by the redemption rate	N/A	The total expense divided by the increase in sales attributed to the mail-in rebate

**TOP
SECRET**

Paths To Success

It pays to be smart when you deal with coupons and mail-in rebates as a consumer. Two important considerations are timing and hassle.

1. **Timing.** Coupon amounts are usually smaller than rebate amounts. You may be offered a \$35 coupon (sometimes called an “instant rebate,” but should not be confused with rebates) at the cash register, or a \$50 mail-in rebate. If you choose the former, you get the price savings immediately. If you choose the latter, you must spend the money up front and then wait 8 to 12 weeks (or more) to receive your rebate cheque.
2. **Hassle.** Mail-in rebates are subject to very strict rules for redemption. Most require original proofs of purchase such as your original cash register receipts, UPC codes, model numbers, and serial numbers (which are difficult to find sometimes). You must photocopy these for your records in the event there is a rebate problem. Specific dates of purchase and deadlines for redemption must be met. Some rebates, particularly computer software upgrades, even require you to send your old software in with your purchase, substantially raising your postage and envelope costs. Does all that extra time and cost justify the rebate? For example, a cough syrup ran a sales promotion offering an instant coupon for \$1; alternatively you could purchase two products and mail-in two UPC codes for a \$3 rebate. When factoring in costs for postage, envelope, photocopying, and time, the benefit from the mail-in rebate is marginal at best.



Give It Some Thought

What happens to the unit profit when a coupon or rebate is issued under the following scenarios:

1. The cost and selling price remain unchanged
2. The cost increases and the selling price remains unchanged
3. The cost remains the same and the selling price is reduced
4. The cost remains the same and the selling price increases

Solutions:

1. Profit decreases by the total unit expenses associated with the coupon or rebate.
2. Profit decreases by the sum of the total unit expenses associated with the coupon or rebate and the amount of the increased cost together.
3. Profit decreases by the sum of the total unit expenses associated with the coupon or rebate and the amount of the price decrease together.
4. Profit decreases by the total unit expenses associated with the coupon or rebate and increases by the amount of the price increase.

Example 6.4D: The Impact of a Coupon

When the DVD edition of the popular kids movie was released in retail stores, DreamWorks Animation executed a coupon marketing program. Assume the unit cost of the DVD is \$2.50, the normal expenses associated with the DVD amount to \$1.25, and that DreamWorks sells it directly to retailers for \$10. The coupon offered a \$3 savings on the DVD to consumers and promised the retailer an \$0.08 handling fee per coupon redeemed. The estimated costs of creating, distributing, and redeeming the coupons are \$285,000, and the projected number of coupons redeemed is 300,000.

- What is the unit profit of the DVD when it sold without the coupon?
- What unit profit did DreamWorks realize for those DVDs sold under its coupon promotion?

Plan	You need to first calculate the profit (P) when no coupon was available. Second, you will calculate the impact of the coupon on the profit when the DVD is discounted under the coupon program (P_{onsale}).	
Understand	What You Already Know Step 1: The unit cost of the DVD along with its normal associated expenses and regular unit selling price are known. The coupon details are: $C = \$2.50$ $E = \$1.25$ $S = \$10.00$ Face Value = \$3 Handling Fee = \$0.08 Total Marketing Expense = \$285,000 Number of Redeemed Coupons = 300,000	How You Will Get There Step 2: When the coupon is issued, the expenses associated with the DVD rise to accommodate the coupon redemption expense, the handling expense, and the marketing expenses. Step 3: Calculate the profit normally realized on the DVD by applying Formula 6.5, rearranging for P. Step 3 (continued): Recalculate the profit under the coupon program by adapting Formula 6.5, rearranging for P_{onsale} .
Perform	Step 2: Unit coupon expense = $\$3 + \$0.08 + (\$285,000 \div 300,000) = \$4.03$ Step 3: $\$10.00 = \$2.50 + \$1.25 + P$ $\$6.25 = P$ Step 3 (continued): $\$10.00 = \$2.50 + (\$1.25 + \$4.03) + P_{\text{onsale}}$ $\$10.00 = \$7.78 + P_{\text{onsale}}$ $\$2.22 = P_{\text{onsale}}$	
Present	Without the coupon promotion the kids movie DVD generates \$6.25 in profit. Under the coupon promotion, DreamWorks achieves a reduced profit per DVD of \$2.22.	

Example 6.4E: The Impact of a Mail-In Rebate

Computer manufacturers commonly issue rebates on items such as printers and monitors. Assume LG decided to distribute its new widescreen monitor (MSRP: \$329.99) with a prepackaged mail-in rebate included inside the box. The rebate is \$75. The company predicts a 25% rebate redemption rate. Its unit cost to produce the monitor is \$133.75, with expenses equalling 35% of cost. It sells the monitor directly to retailers for a regular unit selling price of \$225. The increased marketing expenses for promotion and rebate redemption equal \$8.20 per monitor. What is the profit for a monitor sold through the rebate program?

Plan	You are looking for the profit (P_{onsale}) for a monitor sold under the rebate program. This profit must factor in the rebate face value, redemption rate, and marketing expenses.	
Understand	What You Already Know Step 1: The unit cost, unit expenses, and regular unit selling price are known. The rebate details are: $C = \$133.75$ $E = 0.35C$ $S = \$225.00$ Face Value = \$75 Redemption rate = 25% Unit Marketing Expenses = \$8.20	How You Will Get There Step 2: When the rebate is issued, the expenses associated with the monitor rise to accommodate the rebate redemption expenses and the marketing expenses. Step 3: Apply an adjusted Formula 6.5, rearranging for P_{onsale} .

Perform	Step 2: Unit rebate expense = $\\$75 \times 25\% + \\$8.20 = \\$18.75 + 8.20 = \\$26.95$ Step 3: $\\$225.00 = \\$133.75 + (\\$46.81 + \\$26.95) + P_{\text{onsale}}$ $\\$225.00 = \\$207.51 + P_{\text{onsale}}$ $\\$17.49 = P_{\text{onsale}}$
Present	The rebate redemption expense for the LG monitor is \$18.75. LG averages a profit of \$17.49 per monitor sold under its mail-in rebate program.



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Chapter 6 Summary

Key Concepts Summary

Section 6.1: Figuring Out the Cost: Discounts (How Much?)

- The relationship between distribution and pricing
- Some of the types of discounts available to businesses and consumers
- How to calculate the net price when only one discount is involved
- How to calculate the net price when multiple discounts are involved
- Converting multiple discounts into single discounts

Section 6.2: Markup: Setting the Regular Price (Need to Stay in Business)

- The three components that compose a selling price
- Calculating the markup in dollars and the relationship to pricing components
- Calculating the markup as a percentage under two different approaches
- Determining the price point where all costs are paid but no profits are earned—the break-even point

Section 6.3: Markdown: Setting the Sale Price (Everybody Loves a Sale)

- How to take a regular selling price and make it into a sale price by applying markdowns
- How to plan the merchandising of products that are always on sale

Section 6.4: Merchandising (How Does It All Come Together?)

- Combining discounts, markup, and markdown into complete merchandising situations
- Calculating the maintained markup when a product was sold at both a regular selling price and a sale price
- How coupons and mail-in rebates impact product pricing

The Language of Business Mathematics

cost An outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services.

coupon A promotion that entitles a consumer to receive certain benefits, usually in the form of a reduction of the selling price for a product.

coupon handling expense A handling charge that is paid to channel members for redeeming a coupon.

coupon marketing expense The expense associated with creating, distributing, and redeeming a coupon.

coupon redemption expense The face value price reduction offered by a coupon.

discount A reduction in the price of a product.

expenses A business's financial outlays incurred in the selling of a product.

list price A price for a product that has been published or advertised in some way.

loyalty discount A discount given from a seller to a purchaser for repeat business.

mail-in rebate A refund that occurs after a product has been purchased.

maintained markup The average level of markup that is maintained across all units sold at various price levels including the selling price and the sale price(s).

manufacturer's suggested retail price (MSRP) A recommended product retail price that a manufacturer sets for a retailer based on market research.

markdown A reduction from the regular selling price of a product resulting in a new lower sale price.

marketing price adjustment Any marketing activity executed by a member of the distribution channel for the purposes of altering a product's price.

markup The process of taking a product's cost and increasing it by a certain amount to arrive at a selling price.

markup amount The dollar amount of the expenses and profit combined together into a single number; it represents the difference between the price and cost in dollars.

markup on cost percentage The markup dollars expressed as a rate using cost as the base.

markup on selling price percentage The markup dollars expressed as a rate using the regular selling price as the base.

net price The price of the product after a discount is removed from the list price.

profit The amount of money that remains after a business pays all of its costs and expenses.

quantity discount A discount for purchasing larger quantities of a certain product.

rebate marketing expense The expense associated with creating, distributing, and redeeming a rebate.

rebate redemption expense The face value amount that a consumer will receive as a refund if a submitted rebate fulfills all rebate conditions.

sale discount A temporary discount lowering the price from a product's regular selling price.

sale price A price for a product after a markdown that is lower than its regular selling price.

seasonal discount A discount offered to consumers and businesses for purchasing products out of season.

single equivalent discount A single discount rate that is equal to a series of multiple rate discounts.

trade discount A discount offered to businesses only based on the type of business and its location in the distribution system.

The Formulas You Need to Know

Symbols Used

C = cost

d = discount rate or markdown rate

$d_1, d_2, \dots, d_n$ = multiple discount rates, where the subscript represents each discount up to a count of n discounts

D\$ = discount amount or markdown amount

d_{equiv} = single equivalent discount rate

E = expenses

L = list price

M\$ = markup amount

MM = maintained markup

MoC% = markup on cost percentage

MoS% = markup on selling price percentage

n = number of pieces of data, which in this chapter is the level of output

N = net price

P = profit

P_{onsale} = planned profit amount when a product is sold at the sale price

S = regular selling price

S_{BE} = the selling price at the break-even point

S_{onsale} = sale price

Formulas Introduced

Formula 6.1 Single Discount: $N = L \times (1 - d)$ (Section 6.1)

Formula 6.2a Discount Amount: $D\$ = L \times d$ (Section 6.1)

Formula 6.2b Discount Amount: $D\$ = L - N$ (Section 6.1)

Formula 6.3 Multiple Discounts: $N = L \times (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$ (Section 6.1)

Formula 6.4 Single Equivalent Discount: $d_{\text{equiv}} = 1 - (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$ (Section 6.1)

Markup Formulas

Formula 6.5 The Selling Price of a Product: $S = C + E + P$ (Section 6.2)

Formula 6.6 Markup Amount: $M\$ = E + P$ (Section 6.2)

Formula 6.7 The Selling Price of a Product Using the Markup Amount: $S = C + M\$$ (Section 6.2)

Formula 6.8 Markup on Cost Percentage: $\text{MoC}\% = \frac{M\$}{C} \times 100$ (Section 6.2)

Formula 6.9 Markup on Selling Price Percentage: $\text{MoS}\% = \frac{\text{M}\$}{S} \times 100$ (Section 6.2)

Formula 6.13 Maintained Markup: $\text{MM} = \frac{\text{M}\$(n_1) + (\text{M}\$ - \text{D}\$(n_2))}{n_1 + n_2}$ (Section 6.4)

Markdown Formulas

Formula 6.10 The Sale Price of a Product: $S_{\text{onsale}} = S \times (1 - d)$ (Section 6.3)

Formula 6.11a Markdown Amount: $\text{D}\$ = S \times d$ (Section 6.3)

Formula 6.11b Markdown Amount: $\text{D}\$ = S - S_{\text{onsale}}$ (Section 6.3)

Formula 6.12 Markdown Percentage: $d = \frac{\text{D}\$}{S} \times 100$ (Section 6.3)

Technology

Calculator

The following calculator functions were introduced in this chapter:

Markup on Selling Price Percentage

2nd Profit to access this feature.

Enter two of the three variables by pressing Enter after each input and using $\uparrow$ and $\downarrow$ to scroll through the display. The variables are:

CST = The cost of the item

SEL = The selling price of the item

MAR = The markup on selling price percentage (in % format)

Press CPT on the unknown (when it is on the screen display) to compute.



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Chapter 7: Accounting Applications

(Bills, Bills, and More Bills)

The price tag on the product is not necessarily what you pay for it. In 1789, Benjamin Franklin might have said it best when he wrote, "In this world nothing can be said to be certain, except death and taxes."¹ Almost everything is taxed, whether a physical good, a service, or even property.

For instance, while shopping on eBay, you find a product listed in an eBay store located in your home province. Another product you need is listed by a seller in the United States. You need both products and the prices including shipping costs are reasonable, so for each you click the "Buy It Now" option. The Canadian invoice lists the product's pre-tax price along with two more lines indicating GST and PST amounts, all totalled at the bottom. The US invoice lists the price of the item in US currency. When paying this seller, PayPal converts the price from US currency to Canadian currency using an exchange rate that does not quite match the rate you have seen published in the newspaper, and you wonder why. If you go with this US seller, you know that the Canada Border Services Agency (customs) will charge for GST.

This scenario highlights three of this chapter's key topics: taxes, currency exchange, and invoicing.

1. If there is one thing you encounter in all aspects of your life, it is sales tax, which provides revenue to governments in Canada and abroad. In addition, anyone involved in the ownership of real estate must deal with property taxes. Corporate offices, production facilities, and warehouses sit on business-owned real estate, which is charged property taxes. Residential homeowners pay annual taxes to their municipalities. Even if you just rent property like an apartment, the apartment building owner is charged property taxes, which are passed on to tenants within the rental charges.
2. Canadian businesses and consumers operate in a global economy in which they commonly complete international transactions. Any payment will incorporate an exchange rate ensuring that the amount paid in Canadian currency is equivalent to the amount charged in the foreign currency.
3. Many transactions are completed through an invoicing procedure. Along with a list of the items purchased and their respective prices, invoices show terms of payment, consequences of failing to make the payment, and additional charges such as taxes and shipping.

There is no avoiding sales taxes, property taxes, currency exchange, and invoicing. As a consumer, you generally need to pay these costs. As a business, you not only pay these costs but you need to charge your customers appropriately and collect taxes on behalf of the government. This chapter explores the mathematics behind these concepts.

Outline of Chapter Topics

- 7.1: Sales Taxes (Everybody Wants a Piece of My Pie)
- 7.2: Property Taxes (I Owe, I Owe)
- 7.3: Exchange Rates and Currency Exchange (It Is a Global World)
- 7.4: Invoicing: Terms of Payment and Cash Discounts (Make Sure You Bill Them)

¹ Benjamin Franklin. Letter to Jean-Baptiste Leroy dated November 13, 1789.

7.1: Sales Taxes

(Everybody Wants a Piece of My Pie)

On your recent cross-Canada road trip, you purchased from many different Tim Hortons' stores. At each store, your products retailed for \$6.99. When you review your credit card receipts after returning home from your trip, you notice that you paid different totals everywhere. In Alberta, they only added GST and your combo cost \$7.34. In British Columbia, they added both PST and GST, resulting in total cost of \$7.83. In Ontario, they added something called HST, resulting in a total cost of \$7.90. You find it interesting that the same combo came to different totals as you travelled across Canada.

Three Sales Taxes

A **sales tax** is a percent fee levied by a government on the supply of products. In Canada, there are three types of sales taxes: the goods and services tax (GST), provincial sales tax (PST), and the harmonized sales tax (HST). In this section you will learn the characteristics of each of these taxes and then the mathematics for calculating any sales tax.

Goods & Services Tax (GST)

The goods and services tax, better known as **GST**, is a national federal tax of 5% that applies to the purchase of most goods and services in Canada. Every province and territory has GST. The consumer ultimately bears the burden of this sales tax.

Businesses must collect GST on most of their sales and pay GST on most purchases in the daily course of operations. However, when remitting these taxes, businesses claim a credit with the federal government to recover the GST they paid on resold purchases. The net result is that businesses do not pay the GST on these resold purchases. While this may outrage some people, the logic is simple. If a business pays the GST, it becomes a cost of the business, which is then passed on to consumers as it is incorporated into retail prices. When the consumer purchases the product, the consumer would be charged the GST again! In essence, a consumer would be double-taxed on all purchases if businesses paid the GST.

Some goods and services are exempt from GST. While there are many complexities and nuances to the exemptions, generally items that are deemed necessities (such as basic groceries), essential services (such as health, legal aid, and childcare), and charitable activities are nontaxable. You can find a complete listing of exemptions on the Canada Revenue Agency website at www.cra.gc.ca.

Provincial Sales Tax (PST)

Provincial sales taxes, or **PST**, are provincially administered sales taxes that are determined by each individual provincial or territorial government in Canada. The table here lists the current PST rates in Canada.

Similar to GST, PST applies to the purchase of most goods and services in the province, and consumers bear the burden. For the same reasons as with GST, businesses typically pay the PST on purchases for non-resale items (such as equipment and machinery) and do not pay the PST on resale items. Businesses are responsible for collecting PST on sales and remitting the tax to the provincial government. Individual provincial websites list the items and services that are exempt from PST.

Province/Territory	PST
British Columbia	7%
Alberta	0%
Saskatchewan	6%
Manitoba	7%
Quebec	9.975%
Northwest Territories	0%
Yukon Territory	0%
Nunavut	0%

Harmonized Sales Tax (HST)

The harmonized sales tax, or HST, is a combination of GST and PST into a single number. Since most goods and services are subjected to both taxes anyway, HST offers a simpler method of collecting and remitting the sales tax—a business has to collect and remit only one tax instead of two. Because there are pros and cons to HST, not all provinces use this method of collection, as summarized in the next table.

Pros	Cons
- Items that are previously PST payable to a business are now refunded, lowering input costs and lowering consumer prices - Results in overall lower corporate taxes paid - Increases the competitiveness of businesses and results in job creation - Businesses only remit one tax and not two, resulting in financial and auditing savings	- Many items such as utilities, services, and children's clothing that are ineligible for PST become taxed at the full HST rate - Consumer cost of living increases - Tax-exempt items see prices rise because HST is being applied to services and goods such as transportation and gasoline

Province/Territory	HST
Ontario	13%
New Brunswick	15%
Nova Scotia	15%
Newfoundland and Labrador	15%
Prince Edward Island	15%

To understand HST you can separate it into its GST and PST components. For example, in the table to the left Ontario has an HST of 13%. If GST is 5% and the HST is 13%, then it is clear that the province has a PST of 8%. HST operates mostly in the same manner as GST, in that consumers generally bear the burden for paying this tax, and businesses both collect and pay the HST but have the HST reimbursed on eligible purchases when remitting taxes to the government.



The Formula

With respect to sales taxes, you usually calculate two things:

1. The dollar amount of the sales tax.
2. The price of a product including the sales tax.

Calculating the Sales Tax Amount

A sales tax is a percent rate calculated on the base selling price of the product. Therefore, if you are interested solely in the amount of the sales tax (the portion owing), apply Formula 2.2 on Rate, Portion, Base:

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}} \text{ becomes } \text{Tax Rate} = \frac{\text{Tax Amount}}{\text{Price before Taxes}}$$

Rearranging this formula to solve for the tax amount gives the following:

$$\text{Tax Amount} = \text{Tax Rate} \times \text{Price before Taxes (which is the same as Portion} = \text{Rate} \times \text{Base)}$$

This is not a new formula. It applies the existing formula from Chapter 2.

Calculating a Price Including Tax

When calculating a selling price including the tax, you take the regular selling price and increase it by the sales tax percentage. This is a percent change calculation using Formula 3.1 from Chapter 3:

$$\Delta\% = \frac{\text{New} - \text{Old}}{\text{Old}} \times 100$$

The Old price is the price before taxes, the percent change is the sales tax percentage, and you need to calculate the New price including the tax. Rearranging the formula for New, you have

$$\text{New} = \text{Old} + \text{Old} \left(\frac{\Delta\%}{100} \right)$$

which you factor and rewrite as

$$\text{New} = \text{Old} \times \left(1 + \frac{\Delta\%}{100} \right) \quad \text{or} \quad \text{New} = \text{Old} + \left(\text{Old} \times \frac{\Delta\%}{100} \right)$$

This is a widespread application of the percent change formula. Since it can be tedious to keep rearranging Formula 3.1, Formula 7.1 expresses this relationship.

S_{tax} is Selling Price Including Tax: A subscript is added to the regular symbol of S for selling price to denote that this is the price that sums the selling price and sales tax amount together.

S is Selling Price Without Tax: The regular selling price before taxes.

Formula 7.1 - Selling Price Including Tax: $S_{\text{tax}} = S + (S \times \text{Rate})$

Rate is Tax Rate: This is the sales tax in any form of GST, HST, or PST expressed in its decimal format. Round the result of $(S \times \text{Rate})$ to two decimals. If two taxes are involved in the tax-inclusive price (such as GST and PST), you cannot combine the rates together into a single rate. For example, Manitoba has 5% GST and 7% PST. This is not necessarily equivalent to 12% tax since each tax is rounded to two decimals separately and then summed. If you use a single rate of 12%, you may miscalculate by a penny. Instead, expand the formula to include two separate $(S \times \text{Rate})$ calculations:

$$S_{\text{tax}} = S + (S \times \text{GST Rate}) + (S \times \text{PST Rate})$$

Ensure that you round off each calculation in brackets to two decimals before adding.



How It Works

Follow these steps to perform calculations involving sales taxes:

Step 1: Identify the pricing information. In particular, pay careful attention to distinguish whether the price is before taxes (S) or inclusive of taxes (S_{tax}). Also identify all applicable sales taxes, including GST, PST, and HST.

Step 2: Apply Formula 7.1 to solve for the unknown variable.

Step 3: If you need to find sales tax amounts, apply Formula 2.2 and rearrange for portion. Ensure that for the base you use the price before taxes.

Assume a \$549.99 product is sold in British Columbia. Calculate the amount of the sales taxes and the price including the sales taxes.

Step 1: The price before taxes is $S = \$549.99$. In British Columbia, GST is 5% and PST is 7% (from the PST Table).

Step 2: To calculate the price including the sales taxes, apply Formula 7.1:

$$S_{\text{tax}} = \$549.99 + (\$549.99 \times 5\%) + (\$549.99 \times 7\%) = \$615.99$$

Step 3: Applying the rearranged Formula 2.2 for the GST, you calculate:

$$\text{GST Tax Amount} = 5\% \times \$549.99 = \$27.50$$

Applying the same formula for the PST, you calculate:

$$\text{PST Tax Amount} = 7\% \times \$549.99 = \$38.50.$$

(Note: You may notice that you could just pull these amounts from the interim calculations in step 2). Therefore, on a \$549.99 item in British Columbia, \$27.50 in GST and \$38.50 in PST are owing, resulting in a price including sales taxes of \$615.99.



Important Notes

To calculate a price including a single sales tax rate, use the percent change function ($\Delta\%$) on the calculator. You can review the full instructions for this function at the end of Chapter 3. When using this function, OLD is the price before taxes, NEW is the price after taxes, and %CH is the single tax rate in its percentage format. You do not use the #PD variable, which therefore defaults to 1.

Note that if more than one tax rate applies on the same base, sum the tax rates together and enter as the percent change (or use as Rate in the formula). However, note that due to the penny rounding on the individual taxes, an error margin of a single penny may occur when trying to make S plus the taxes equal to S_{tax} . This may be an unavoidable error and it would remain impossible to exactly equal the price including the sales tax.

**TOP
SECRET**

Paths To Success

You will often need to manipulate Formula 7.1. Most of the time, prices are advertised without taxes and you need to calculate the price including the taxes. However, sometimes prices are advertised including the taxes and you must calculate the original price of the product before taxes. When only one tax is involved, this poses no problem, but when two taxes are involved (GST and PST), combine the taxes into a single amount before you solve for S .



Give It Some Thought

On any given product selling for the same price, put the following provinces in order from highest price to lowest price including taxes (GST and PST, or HST): Alberta, Saskatchewan, British Columbia, Ontario, Prince Edward Island.

Solutions:
PEI (15% HST), Ontario (13% HST), British Columbia (5% GST + 7% PST), Saskatchewan (5% GST + 6% PST), Alberta (5% GST, no PST)

Example 7.1A: Calculating Sales Taxes across Canada

Dell Canada lists a complete computer system on its Canadian website for \$1,999.99. Calculate the price including taxes if the Canadian buyer is located in:

- Alberta
- Ontario
- Quebec
- British Columbia (BC)

Plan	Four answers are required. For each of the provinces listed, calculate the appropriate GST/PST or HST to add onto the price and arrive at the selling price including taxes (S_{tax}).																							
Understand	What You Already Know Step 1: The price of the computer and tax rates are known: $S = \$1,999.99$ Alberta sales tax = 5% GST Ontario sales tax = 13% HST Quebec sales tax = 5% GST & 9.975% PST BC sales tax = 5% GST & 7% PST		How You Will Get There Step 2: For all provinces, apply Formula 7.1. a. For Alberta, the Rate is the 5% GST. Substitute the S and Rate to solve for S_{tax} . b. For Ontario, the Rate is the 13% HST. Substitute the S and Rate to solve for S_{tax} . c. For Quebec, both the 5% GST and 9.975% PST are based on the S . Expand the formula for each tax. Substitute these Rates with the S to arrive at S_{tax} . d. For British Columbia, both the 5% GST and 7% PST are based on the S . Expand the formula for each tax. Substitute these Rates with the S to arrive at S_{tax} .																					
Perform	Step 2: a. Alberta: $S_{\text{tax}} = \$1,999.99 + (\$1,999.99 \times 5\%) = \$2,099.99$ b. Ontario: $S_{\text{tax}} = \$1,999.99 + (\$1,999.99 \times 13\%) = \$2,259.99$ c. Quebec: $S_{\text{tax}} = \$1,999.99 + (\$1,999.99 \times 5\%) + (\$1,999.99 \times 9.975\%) = \$2,299.49$ d. British Columbia: $S_{\text{tax}} = \$1,999.99 + (\$1,999.99 \times 5\%) + (\$1,999.99 \times 7\%) = \$2,239.99$		Calculator Instructions All calculations use the $\Delta\%$ function. <table border="1"> <thead> <tr> <th>Province</th><th>OLD</th><th>NEW</th><th>%CH</th><th>#PD</th></tr> </thead> <tbody> <tr> <td>Alberta</td><td>1999.99</td><td>Answer: 2,099.9895</td><td>5</td><td>1</td></tr> <tr> <td>Ontario</td><td>1999.99</td><td>Answer: 2,259.9887</td><td>13</td><td>1</td></tr> <tr> <td>Quebec & BC</td><td colspan="4">Not recommended to use a calculator since two taxes on the same base are involved and a rounding error could result.</td></tr> </tbody> </table>		Province	OLD	NEW	%CH	#PD	Alberta	1999.99	Answer: 2,099.9895	5	1	Ontario	1999.99	Answer: 2,259.9887	13	1	Quebec & BC	Not recommended to use a calculator since two taxes on the same base are involved and a rounding error could result.			
Province	OLD	NEW	%CH	#PD																				
Alberta	1999.99	Answer: 2,099.9895	5	1																				
Ontario	1999.99	Answer: 2,259.9887	13	1																				
Quebec & BC	Not recommended to use a calculator since two taxes on the same base are involved and a rounding error could result.																							

Present

Depending on which province you reside in, the price for the computer from lowest to highest is Alberta at \$2,099.99, British Columbia at \$2,239.99, Ontario at \$2,259.99, and Quebec at \$2,299.49.

Tax-Inclusive Price of the Computer in Various Provinces

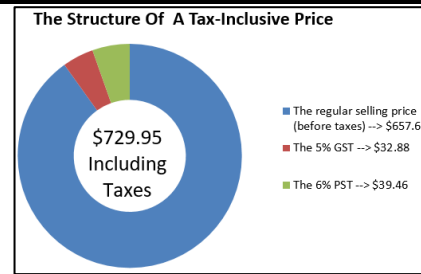
Province	MSRP	GST	PST/HST	Total Price
Alberta	\$1,999.99	\$100.00	\$0.00	\$2,099.99
Ontario	\$1,999.99	\$100.00	\$160.00	\$2,259.99
Quebec	\$1,999.99	\$100.00	\$199.50	\$2,299.49
British Columbia	\$1,999.99	\$100.00	\$140.00	\$2,239.99

The chart is a stacked bar graph with the y-axis representing price in dollars, ranging from \$1,850 to \$2,350 in increments of \$50. The x-axis lists the four provinces. Each bar is composed of three segments: a blue base segment for MSRP (\$1,999.99), a red middle segment for GST (\$100.00), and a top segment for PST or HST. The total price for each province is labeled above the bar: Alberta (\$2,099.99), Ontario (\$2,259.99), Quebec (\$2,299.49), and British Columbia (\$2,239.99). A legend on the right identifies the colors: blue for MSRP, red for GST, and purple/green for PST/HST.

Example 7.1B: Calculating Taxes on a Tax-Inclusive Price

“The Brick is having its Midnight Madness sale! Pay no taxes on products purchased during this event!” While this is good marketing, it probably goes without saying that governments do not give up the sales taxes. Essentially The Brick is advertising a tax-inclusive price. Calculate GST and PST amounts for a product advertised at \$729.95, including GST and PST, in Saskatchewan.

Plan	To calculate the PST and GST for Saskatchewan, you must calculate the price before taxes (S). Then calculate each tax amount based on the S.									
Understand	What You Already Know Step 1: The price after taxes and the tax rates are as follows: $S_{\text{tax}} = \\$729.95$ Tax rates = 5% GST and 6% PST	How You Will Get There Step 2: Apply Formula 7.1 using the combined PST and GST as the Rate to calculate the S. Step 3: Apply Formula 2.2 rearranged for Portion to calculate the tax amounts.								
	Step 2: Remove the taxes: $\\$729.95 = S + (S \times 11\%)$ $\\$729.95 = 1.11S$ $S = \\$657.61$ Step 3: GST: $5\% = \text{GST Portion} \div \\$657.61$; GST Portion = \$32.88 PST: $6\% = \text{PST Portion} \div \\$657.61$; PST Portion = \$39.46	Calculator Instructions <table border="1"> <thead> <tr> <th>OLD</th><th>NEW</th><th>%CH</th><th>#PD</th></tr> </thead> <tbody> <tr> <td>Answer: 657.61</td><td>729.95</td><td>11</td><td>1</td></tr> </tbody> </table>		OLD	NEW	%CH	#PD	Answer: 657.61	729.95	11
OLD	NEW	%CH	#PD							
Answer: 657.61	729.95	11	1							
Present	When paying a tax-inclusive price of \$729.95 in Saskatchewan, the regular selling price of the item is \$657.61 with \$ 32.88 GST and \$39.46 PST included.									



The GST/HST Remittance

When a business collects sales taxes, it is a go-between in the transaction. These sales tax monies do not belong to the business. On a regular basis, the business must forward this money to the government. This payment is known as a **tax remittance**.



The Formula

Generally speaking, a business does not pay sales taxes. As a result, the government permits a business to take all eligible sales taxes that it paid through its acquisitions and net them against all sales taxes collected from sales. The end result is that the business is reimbursed for any eligible out-of-pocket sales tax that it paid. Formula 7.2 expresses this relationship.

Remit is GST/HST Remittance: This is the dollar amount of the remittance.

- If this amount is positive, it means that the business collected more tax than it paid out; the company must remit this balance to the government.
- If this amount is negative, it means that the business paid out more taxes than it collected; the government must refund this balance to the company.

Formula 7.2 – GST/HST Remittance: **Remit = Tax Collected – Tax Paid**

Tax Collected and Tax Paid: Both of these parts of the formula apply Formula 2.2, representing the total amount of sales tax (portion) from all taxable amounts (base) at the appropriate sales tax rate (rate). The taxes collected are based on total tax-eligible revenues. The taxes paid are based on the total tax-eligible acquisitions.



How It Works

Follow these steps to complete a GST/HST remittance:

Step 1: Identify the total amounts of tax-eligible revenues and acquisitions upon which the sales tax is collected or paid, respectively. Identify the applicable sales tax rate of the GST or HST.

Step 2: Calculate the total taxes collected by applying Formula 2.2, where the sales tax is the rate and the total revenue is the base. Solve for portion.

Step 3: Calculate the total taxes paid by applying Formula 2.2, where the sales tax is the rate and the total acquisitions are the base. Solve for portion.

Step 4: Apply Formula 7.2 to calculate the tax remittance.

Assume a business has paid GST on purchases of \$153,000. It has also collected GST on sales of \$358,440. Calculate the GST remittance.

Step 1: Identifying the variables, you have:

$$\text{Total Revenue} = \$358,440$$

$$\text{Total Acquisitions} = \$153,000$$

$$\text{GST Tax Rate} = 5\%$$

Step 2: Calculate taxes collected by applying Formula 2.2, where:

$$\text{GST collected} = 5\% \times \$358,440 = \$17,922.$$

Step 3: Calculate taxes paid by applying Formula 2.2, where:

$$\text{GST paid} = 5\% \times \$153,000 = \$7,650.$$

Step 4: To calculate the remittance, apply Formula 7.2 and calculate $\text{Remit} = \$17,922 - \$7,650 = \$10,272$. The business should remit a cheque for \$10,272 to the government.

**TOP
SECRET**

Paths To Success

A shortcut can help you calculate the GST/HST Remittance using Formula 7.2. If you do not need to know the actual amounts of the tax paid and collected, you can net GST/HST-eligible revenues minus acquisitions and multiply the difference by the tax rate:

$$\text{Remit} = (\text{Revenues} - \text{Acquisitions}) \times \text{Rate}$$

In the example above, $\text{Remit} = (\$358,440 - \$153,000) \times 5\% = \$10,272$. If this calculation produces a negative number, then the business receives a refund instead of making a remittance.

Example 7.1C: Calculating a GST/HST Remittance

An Albertan lumber company reported the following quarterly purchases and sales in its 2013 operating year:

Quarter	Purchases	Sales
Winter	\$1,316,000	\$911,500
Spring	\$1,266,200	\$1,473,300
Summer	\$1,053,700	\$1,381,700
Fall	\$878,100	\$1,124,600

Assuming all purchases and sales are eligible and subject to GST, calculate the GST remittance or refund for each quarter.

Plan

Calculate the GST tax remittance, or Tax Remit, for each of the quarters.

Understand**What You Already Know**

Step 1: From the information provided, the total purchases and sales for each quarter along with the GST = 5%, which is the Tax Rate, are known.

How You Will Get There

Step 2: For each quarter, calculate the GST collected by rearranging and applying Formula 2.2.
 Step 3: Calculate the tax paid.
 Step 4: Apply Formula 7.2 for each quarter.

Perform

Quarter	Step 2		Step 3		Step 4
	Sales	GST Collected (Sales $\times$ 5%)	Purchases	GST Paid (Purchases $\times$ 5%)	Remittance (Tax Collected – Tax Paid)
Winter	\$911,500	\$45,575	\$1,316,000	\$65,800	\$45,575 – \$65,800 = –\$20,225
Spring	\$1,473,300	\$73,650	\$1,266,200	\$63,310	\$73,650 – \$63,310 = \$10,340
Summer	\$1,381,700	\$69,085	\$1,053,700	\$52,685	\$69,085 – \$52,685 = \$16,400
Fall	\$1,124,600	\$56,230	\$878,100	\$43,905	\$56,230 – \$43,905 = \$12,325

Present

In the Winter quarter, the company gets a GST refund of \$20,225. In the other three quarters, the company remits to the government payments of \$10,340, \$16,400, and \$12,325, respectively.



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Lyryx also offers end-of-chapter exercises for Lyryx users.

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7.2: Property Taxes

(I Owe, I Owe)

As you drive through your neighbourhood, you pass a city crew repairing the potholes in the road. Hearing sirens, you check your rear-view mirror and pull to the side of the road as a police car and fire engine race by, heading toward some emergency. Pulling back out, you drive slowly through a public school zone, where you smile as you watch children playing on the gigantic play structure. A city worker mows the lawn.

Where does the municipality get the money to pay for all you have seen? No one owns the roads, schools are free, fire crews and police do not charge for their services, play structures have no admission, and parks are open to everyone. These are just some examples of what your municipality does with the money it raises through property taxes.

Property Taxation

Property taxes are annual taxes paid by real estate owners to local levying authorities to pay for services such as roads, water, sewers, public schools, policing, fire departments, and other community services. Every individual and every business pays property taxes. Even if you don't own property, you pay property taxes that are included in your rental and leasing rates from your landlord.

Property taxes are imposed on real estate owners by their municipal government along with any other bodies authorized to levy taxes. For example, in Manitoba each divisional school board is authorized to levy property taxes within its local school division boundaries.



The Formula

Since property taxes are administered at the municipal level and every municipality has different financial needs, there are a variety of ways to calculate a total property tax bill. Formula 7.3 (on next page) is designed to be flexible to meet the varying needs of municipal tax calculations throughout Canada.



How It Works

Follow these steps when working with calculations involving property taxes:

Step 1: Identify all known variables. This includes the market value, tax policy, assessed value, all property tax rates, and the total property taxes.

Step 2: If you know the assessed value, skip this step. Otherwise, calculate the assessed value of the property by multiplying the market value and the tax policy.

Step 3: Calculate either the tax amount for each property tax levy or the grand total of all property taxes by applying Formula 7.3.

Continuing with the Winnipeg example in which a home has a market value of \$200,000, the tax policy of Winnipeg is to tax 45% of the market value. A Winnipegger receives a property tax levy from both the City of Winnipeg itself and the local school board. The mill rates are set at 14.6 and 16.724, respectively. Calculate the total property tax bill.

Step 1: The known variables are market value = \$200,000, tax policy = 45%, City of Winnipeg mill rate = 14.6, and school board mill rate = 16.724.

Step 2: Calculate the assessed value by taking the market value of \$200,000 and multiplying by the tax policy of 45%, or Assessed Value = $\$200,000 \times 45\% = \$90,000$.

Step 3: To calculate each property tax, apply Formula 7.3. The City of Winnipeg Property Tax = $\$90,000 \times 14.6 \div 1,000 = \$1,314$. The school board Property Tax = $\$90,000 \times 16.724 \div 1,000 = \$1,505.16$. Add these separate taxes together to arrive at Total Property Tax = $\$1,314 + \$1,505.16 = \$2,819.16$.

Property Tax is Property Tax Amount: The amount of property taxes that are owing represents the total of all property taxes from all taxable services. For example, your property tax bill could consist of a municipal tax, a public school tax, a water tax, and a sewer/sanitation tax. Each of these taxes is levied at a specific rate. Therefore, the formula is designed to sum all applicable taxes, as represented by the summation symbol (Σ) on the right-hand side of the equation.

Formula 7.3 – Property Taxes: $\text{Property Tax} = \Sigma(\text{AV} \times \text{PTR})$

AV is Assessed Value of Property: Every piece of real estate has two valuations: the market value and the assessed value. Only the assessed value is relevant when computing property taxes.

1. The **market value** of a property is a snapshot of the estimated selling price of your property. It is what you might have been able to sell your property for at a certain time period. For example, the City of Winnipeg updates the market value of all property in the city every two years.
2. The **assessed value** of a property is the portion of the market value that is subjected to the property tax rate. It is calculated by taking the market value and multiplying it by a percentage determined by the municipality's **tax policy**:

$$\text{Market Value} \times \text{Tax Policy} = \text{Assessed Value}$$

In some municipalities, the tax policy is to tax 100% of the market value. In others, the tax policy can be substantially less. Continuing with the example, the tax policy for Winnipeg is to tax 45% of the market value. Therefore, a \$200,000 market value home in Winnipeg has a \$90,000 assessed value. This \$90,000 is the base for the tax.

PTR is Property Tax Rate: Two methods commonly express how assessed value is taxed, a tax rate and a mill rate.

1. A **tax rate** is a tax per \$100 of assessed value. Most municipalities in Ontario and further east use this system. The mathematical expression for the tax rate is

$$\frac{\text{Tax rate}}{100}$$

2. A **mill rate** is a tax per \$1,000 of assessed value. Most municipalities in Manitoba and further west use this system. The mathematical expression for the mill rate is

$$\frac{\text{Mill rate}}{1,000}$$



Important Notes

Mill rates are commonly expressed with four decimals and tax rates are expressed with six decimals. Although some municipalities use other standards, this text uses these common formats in its rounding rules. In addition, each property tax levied against the property owner is a separate tax. Therefore, you must round each property tax to two decimals before summing the grand total property tax.



Things To Watch Out For

The most common mistake is to use the wrong denominator in the tax calculation. Ensure that you read the question accurately, noting which term it uses, tax rate or mill rate. If neither appears, remember that Ontario eastward uses tax rates and Manitoba westward uses mill rates.

A second common mistake is to add multiple property tax rates together when the assessed value remains constant across all taxable elements. For example, if the assessed value of \$250,000 is used for two tax rates of 2.168975 and 1.015566, you may be tempted to sum the rates, which would yield a rate of 3.184541. This does not always work and may produce a small error (a penny or two) since each tax is itemized on a tax bill. You must round each individual tax to two decimals before summing to the total property tax.

Example 7.2A: Calculating a Property Tax

A residence has a market value of \$340,000. The municipality's tax policy is set at 70%. Real estate owners have to pay three separate taxes: the municipality tax, a library tax, and an education tax. The tax rates for each are set at 1.311666, 0.007383, and 0.842988, respectively. Calculate the total property tax bill for the residence.

Plan	Calculate the total property tax for the residence by summing the assessed value multiplied by the respective tax rates.	
Understand	What You Already Know Step 1: The house, tax policy, and tax rates are known: Market Value = \$340,000 Tax Policy = 70% Municipal PTR = 1.311666 Library PTR = 0.007383 Education PTR = 0.842988	How You Will Get There Step 2: Calculate the assessed value (AV) by taking the market value and multiplying by the tax policy. Step 3: Apply Formula 7.3. Note that this municipality uses a tax rate, so the PTR is divided by 100.
Perform	Step 2: $AV = \$340,000 \times 70\% = \$340,000 \times 0.7 = \$238,000$ Step 3: Municipal Tax = $\$238,000 \times \frac{1.311666}{100} = \$3,121.77$ Library Tax = $\$238,000 \times \frac{0.007383}{100} = \$17.57$ Education Tax = $\$238,000 \times \frac{0.842988}{100} = \$2,006.31$ Property Tax = $\$3,121.77 + \$17.57 + \$2,006.31 = \$5,145.65$	
Present	The property owner owes the municipality \$3,121.77, the library \$17.57, and education \$2,006.31 for a total property tax bill of \$5,145.65.	

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The collective property taxes paid by all of the property owners form either all or part of the operating budget for the municipality. Thus, if a municipality consisted of 1,000 real estate owners each paying \$2,000 in property tax, the municipality's operating income from property taxes is $\$2,000 \times 1,000 = \$2,000,000$. If the municipality needs a larger budget from property owners, either the assessed values, the mill/tax rate, or some combination of the two needs to increase.

Example 7.2B: Setting a New Mill Rate

A school board is determining next year's operating budget and calculates that it needs an additional \$5 million. Properties in its municipality have an assessed value of \$8.455 billion. The current mill rate for the school board is set at 6.1998. If the assessed property values are forecasted to rise by 3% next year, what mill rate should the school set?

Plan	Aim to calculate the new property tax rate (PTR) for next year's mill rate.	
Understand	What You Already Know Step 1 (Current Year): The information about the school board and its municipality are known: AV = \$8.455 billion PTR (current mill rate) = 6.1998 $\Delta\%$ (to AV next year) = 3% Property tax increase needed is \$5 million	How You Will Get There Step 2 (Current Year): Skip. Calculations are based on assessed values already. Step 3 (Current Year): Calculate the school board's current budget using Formula 7.3. The property tax collected is the board's operating budget. Step 1 (Next Year): Increase the budget, or Property Tax, for next year. Also increase the current assessed value base by applying Formula 3.1 (percent change), rearranging to solve for New. The current AV is the Old value. Step 2 (Next Year): The assessed values are known. Skip this step. Step 3 (Next Year): Recalculate the new mill rate using the new Property Tax and new Assessed Value from step 1 (Next Year). Apply Formula 7.3 and solve for the mill rate in the PTR.

Perform	Step 3 (Current Year):	Calculator Instructions For step 1 (Next Year) only, use the percent change function (Δ%) to calculate the New Assessed Value. <table><tr><td>OLD</td><td>NEW</td><td>%CH</td><td>#PD</td></tr><tr><td>8455000000</td><td>Answer: 8,708,650,000</td><td>3</td><td>1</td></tr></table>	OLD	NEW	%CH	#PD	8455000000	Answer: 8,708,650,000	3	1
	OLD		NEW	%CH	#PD					
	8455000000		Answer: 8,708,650,000	3	1					
	Property Tax = \$8,455,000,000 × $\frac{6.1998}{1,000}$ = \$52,419,309									
	Step 1 (Next Year):									
Next Year Property Tax = \$52,419,309 + \$5,000,000 = \$57,419,309										
$3\% = \frac{\text{New} - \$8,455,000,000}{\$8,455,000,000} \times 100$										
$\$253,650,000 = \text{New} - \$8,455,000,000$										
$\$8,708,650,000 = \text{New} = \text{Next Year Assessed Value}$										
Step 3 (Next Year): $\$57,419,309 = \$8,708,650,000 \times \frac{\text{Mill Rate}}{1,000}$										
$6.5934 = \text{Mill Rate}$										
Present	The current budget for the school board is \$52,419,309, which will be increased to \$57,419,309 next year. After the board adjusts for the increased assessed values of the properties, it needs to set the mill rate at 6.5934 next year, which is an increase of 0.4736 mills.									



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7.3: Exchange Rates and Currency Exchange

(It Is a Global World)

You have set aside \$6,000 in Canadian funds toward hostel costs during a long backpacking trip through the United States, Mexico, and Europe. After searching on the Internet, you decide to use Hotwire.com to reserve your hostel rooms. The website quotes you the following amounts for each country: 1,980 US dollars, 21,675 Mexican pesos, and 1,400 euros. Have you allocated enough money to cover these costs?

Whether you are a consumer backpacking around the world on distant vacations, investing in international securities, or shopping online at global Internet sites, you must pay for your purchases in local currencies out of your Canadian currency accounts. Businesses are no different as they export and import products to and from other countries. With outsourcing on the rise, it is also common for business services such as call centres and advertising agencies to be located abroad along with manufacturing facilities. Large-scale companies may have operations in several countries throughout the world.

All of these transactions and operations require the conversion of Canadian currency to a foreign currency or vice versa. This section shows you the basics of currency conversion rates and then explores finer details such as charges for currency conversion and what happens when one currency gets stronger or weaker relative to another.

Exchange Rates

An **exchange rate** between two currencies is defined as the number of units of a foreign currency that are bought with one unit of the domestic currency, or vice versa. Since two currencies are involved in every transaction, two published exchange rates are available. Let's use Canada and the United States to illustrate this concept.

- The first exchange rate indicates what one dollar of Canada's currency becomes in US currency.
- The second exchange rate indicates what one dollar of US currency becomes in Canadian currency.

These two exchange rates allow Canadians to determine how many US dollars their money can buy and vice versa. These currency rates have an inverse relationship to one another: if 1 Canadian dollar equals 0.80 US dollars, then 1 US dollar equals $\frac{1}{\$0.80} = 1.25$ Canadian dollars.

Most Canadian daily and business newspapers publish exchange rates in their financial sections. Although exchange rates are published in a variety of ways, a currency cross-rate table, like the table below, is most common. Note that exchange rates fluctuate all of the time as currencies are actively traded in exchange markets. Therefore, any published table needs to indicate the date and time at which the rates were determined². Also note that the cells where the same currency appears show no published rate as you never need to convert from Canadian dollars to Canadian dollars!

	Per CAD	Per USD	Per €	Per ¥	Per MXN	Per AUD
Canadian Dollar (CAD)		1.2318	1.4618	0.0111	0.0623	0.9273
US Dollar (USD)	0.8118		1.1872	0.0090	0.0506	0.7531
Euro (€)	0.6841	0.8423		0.0076	0.0426	0.6345
Japanese Yen (¥)	90.0901	111.1111	131.5789		5.6180	83.3333
Mexican Peso (MXN)	16.0514	19.7628	23.4742	0.1780		14.8810
Australian Dollar (AUD)	1.0784	1.3278	1.5760	0.0120	0.0672	

In the table, all exchange rates have been rounded to four decimals. In true exchange markets, most exchange rates are expressed in 10 decimals or more such that currency exchanges in larger denominations are precisely performed. For the purposes of this textbook, the Bank of Canada's four decimal standard for all exchange rates is applied to simplify the calculations while still illustrating the principles of currency exchange.

Technically, only half of the table is needed, since one side of the diagonal line is nothing more than the inverse of the other. For example, the euro is 0.6841 per CAD on the bottom of the diagonal. The inverse, or $\frac{1}{0.6841} = 1.4618$ per €, is what is listed on top of the diagonal (when rounded to four decimals).

² This currency cross-rate table was generated July 2, 2021 at approximately 1:30pm, Central Time Zone. Exchange rates are always changing and current rates may be significantly different from these numbers. The focus of this section is on the mathematics of exchange rates, not the exchange rate itself. For current rates, visit any exchange website.



The Formula

The formula is yet again another adaptation of Formula 2.2 on Rate, Portion, and Base. Formula 7.4 expresses this relationship in the language of currency exchange.

Desired Currency is what you are converting to: In Formula 2.2, the desired currency represents the portion. This is what your money is worth in the other currency.

Formula 7.4 – Currency Exchange: **Desired Currency = Exchange Rate × Current Currency**

Exchange Rate is per unit of Current Currency: In Formula 2.2, the exchange rate represents the rate, which is the current exchange rate per currency unit in which your money is currently valued. Looking at a Cross-Rate table, find the column that represents your existing currency and then cross-tabulate this column with the row for the desired currency (so you have desired currency per current currency). For example, if you have one euro that you want to convert into yen, locate the euro column and the yen row. The exchange rate is 131.5789. It is critical that both the exchange rate and current currency always be expressed in the same unit of currency type.

Current Currency is what you are converting from: In Formula 2.2, the current currency represents the base, which is the amount of money (number of units) in your existing currency.



Important Notes

Currencies, exchange rates, and currency cross-tables all raise issues regarding decimals and financial fees.

1. **Decimals in Currencies.** Not all currencies in the world have decimals. Here in North America, MXN, USD, and CAD have two decimals. Mexicans call those decimals *centavos* while Canadians and Americans call them *cents*. Australia and the European countries using the euro also have cents. However, the Japanese yen does not have any decimals in its currency. If you are unsure about the usage of decimals, perform a quick Internet search to clarify the issue.
2. **Financial Fees/Commissions.** Technically, the rates in a cross-rate table are known as mid-rates. A **mid-rate** is an exchange rate that does not involve or provide for any charges for currency conversion. When you convert currencies, you need to involve a financial organization, which will charge for its services.
 - a. **Sell Rates.** When you go to a bank and convert your domestic money to a foreign currency, the bank charges you a **sell rate**, which is the rate at which a foreign currency is sold. When you exchange your money, think of this much like a purchase at a store—the bank's product is the foreign currency and the price it charges is marked up to its selling price. The sell rate is always higher than the mid-rate in terms of CAD per unit of foreign currency and thus is always lower than the mid-rate in terms of foreign currency per unit of CAD. For example, the exchange rate of CAD per USD is 1.2318 (you always look up the “per currency” column that you are purchasing). This means it will cost you CAD\$1.2318 to purchase USD\$1.00. The bank, though, will sell you this money for a sell rate that is higher, say \$1.2564. This means it will cost you an extra CAD\$0.0246 per USD to exchange the money. That \$0.0246 difference is the fee/commission (equal to 2% commission) from the bank for its services, and it is how the bank makes a profit on the transaction.
 - b. **Buy Rates.** When you go to a bank and convert your foreign currency back into your domestic money, the bank charges you a **buy rate**, which is the rate at which a foreign currency is purchased. The buy rate is always lower than the mid-rate in terms of CAD per unit of foreign currency and thus is always higher than the mid-rate in terms of foreign currency per unit of CAD. Using the same example as above, if you want to take your USD\$1.00 to the bank and convert it back to Canadian funds, the bank provides you a buy rate that is lower, say \$1.2072. In other words,

you receive CAD\$0.0246 less per USD. Again, the \$0.0246 difference (equal to 2% commission) is the bank's fee/commission for making the currency exchange on your behalf.



How It Works

Follow these steps when performing a currency exchange:

Step 1: Identify all known variables. Specifically, identify the currency associated with any amounts. You also require the mid-rate. If buy rates and sell rates are involved, identify how these rates are calculated.

Step 2: If there are no buy or sell rates, skip this step. If buy and sell rates are involved, calculate these rates in the manner specified by the financial institution. Apply any appropriate per currency unit fee or commission to the rate, not the currency.

Step 3: Apply Formula 7.4 using the appropriate mid-rate, buy rate, or sell rate to convert currencies. If there is a final flat fee charged (not related to the rate), it is always charged to the local currency amount (Canadians are charged Canadian fees at a Canadian bank).

This section opened with your backpacking vacation to the United States, Mexico, and Europe, for which you were quoted prices of USD\$1,980, MXN\$21,675, and €1,400 for hostels. Assume all purchases are made with your credit card and that your credit card company charges 2.5% on all currency exchanges. Can your CAD\$6,000 budget cover these costs?

Step 1: There are three currency amounts: USD\$1,980, MXN\$21,675, and €1,400. Using the cross-rate table, the Canadian exchange mid-rate per unit of each of these currencies is 1.2318 USD, 0.0623 MXN, and 1.4618 €.

Step 2: Calculate the buy rates (since you are converting foreign currency into domestic currency) for each currency:

$$\text{USD} = 1.2318(1.025) = 1.2626$$

$$\text{MXN} = 0.0623(1.025) = 0.0639$$

$$\text{€} = 1.4618(1.025) = 1.4983$$

Step 3: Apply Formula 7.4 to each of these currencies:

$$\text{USD: Desired Currency} = \text{USD\$1,980} \times 1.2626 = \text{CAD\$2,499.95.}$$

$$\text{MXN: Desired Currency} = \text{MXN\$21,675} \times 0.0639 = \text{CAD\$1,385.03.}$$

$$\text{€: Desired Currency} = \text{€1,400} \times 1.4983 = \text{CAD\$2,097.62.}$$

Putting the three amounts together, your total hostel bill is:

$$\text{\$2,499.95} + \text{\$1,385.03} + \text{\$2,097.62} = \text{\$5,982.60}$$

Because this is under budget by \$17.40, all is well with your vacation plans.



Things To Watch Out For

When working with currency exchange, probably the trickiest element is that you have to choose one of two inverse exchange rates depending on which way the money conversion is taking place. In any currency situation, it is important that you take the time to understand the basis on which the currency rate is being expressed. Typically, exchange rates are expressed on a per-unit basis in the country's domestic currency. For example, Canadians express the US dollar exchange rate on a per CAD basis. From the cross-rate table, that exchange rate is USD0.8118. In contrast, Americans express the Canadian dollar exchange rate on a per USD basis, or CAD1.2318.

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Let the buyer beware when it comes to international transactions. If you have ever purchased and returned an item to an international seller, you may have noticed that you did not receive all of your money back. For most consumers, international purchases are made via credit cards. What most consumers do not know is that the credit card companies do in fact use buy and sell rates that typically charge 2.5% of the exchange rate when both buying and selling.

For example, if you purchase a USD\$2,000 item at the rates listed in the cross-rate table, your credit card provides a buy rate of $1.2318(1.025)=1.2626$ and is charged $\text{\$2,000} \times 1.2626 = \text{\$2,525.20}$. If you return an item you do not want, your credit card provides a sell rate of $1.2318(0.975)=1.2010$ and is refunded $\text{\$2,000} \times 1.2010 = \text{\$2,402}$. In other words, you are out $\text{\$2,525.20} - \text{\$2,402} = \text{\$123.20}$! This amount represents your credit card company's fees/commissions for the currency conversions—a whopping 6.16% of your purchase price!



Give It Some Thought

In each of the following situations and using the cross-rate table, determine on a strictly numerical basis whether you would have more or fewer units of the target currency than of the original currency. You want to convert:

1. CAD into USD
2. AUD into MXN
3. MXN into ¥
4. CAD into €

Solutions:
 1. Less, since CAD\$1 becomes USD\$0.8118
 2. More, since AUD\$1 becomes MXN\$14.8810
 3. More, since MXN\$1 becomes ¥5.6180
 4. Less, since CAD\$1 becomes €0.6841

Example 7.3A: A Straight Currency Conversion

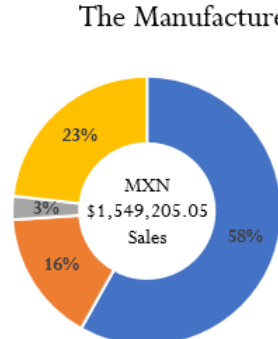
Strictly using the mid-rates from the cross-rate table presented earlier, if you wanted to convert CAD\$1,500 into MXN for your spring break vacation in Cancun, Mexico, how many Mexican pesos would you have?

Plan	Take Canadian currency and convert it into the desired Mexican currency.	
Understand	What You Already Know Step 1: The amount and the exchange rate are known: Current Currency = \$1,500 Exchange Rate = 16.0514	How You Will Get There Step 2: No buy or sell rates are involved. Skip this step. Step 3: Apply Formula 7.4.
Perform	Step 3: Desired Currency = $16.0514 \times \$1,500 = \$24,077.10$	
Present	You have \$24,077.10 Mexican pesos available for your spring break vacation.	

Example 7.3B: Calculating Profit Using Buy and Sell Rates

A Mexican manufacturer imports its parts from Canada and assembles its product in Mexico, then exports some of the finished product back to Canada to sell at retail. Suppose the total cost of the imported parts, purchased and paid for in Canadian funds, is CAD5.50 per unit, assembly costs are MXN\$24.54 per unit, and expenses are MXN\$4.38 per unit. If the product is exported back to Canada at a selling price of CAD\$9.95 (for which its Canadian customers pay in Canadian funds), what is the total profit in Mexican pesos on 10,000 units? The manufacturer's financial institution charges a sell rate 2% higher than the mid-rate and has a buy rate that is 3% lower than the mid-rate. Use the mid-rates from the cross-rate table.

Plan	Calculate the total profit (P) in MXN for the Mexican manufacturer. However, since money is being expressed in different currencies, you must first convert all amounts into Mexican pesos.

Understand	What You Already Know Step 1: You know the costs, expenses, price, and mid-rate: $C_{\text{parts}} = 10,000 \times \text{CAD}\$5.50 = \text{CAD}\$55,000$ $C_{\text{assembly}} = 10,000 \times \text{MXN}\$24.54 = \text{MXN}\$245,400$ $E = 10,000 \times \text{MXN}\$4.38 = \text{MXN}\$43,800$ $S = 10,000 \times \text{CAD}\$9.95 = \text{CAD}\$99,500$ Mid-Rate = 16.0514 per CAD Sell Rate = 2% higher Buy Rate = 3% lower Note the mid-rate used is “per CAD” since the manufacturer needs to purchase Canadian funds to pay its suppliers and also needs to sell its Canadian revenues to the bank.	How You Will Get There Step 2: Calculate the buy and sell rates. Step 3 (Parts): Convert the purchase of the parts in CAD into MXN using the sell rate. Apply and adapt Formula 7.4. Step 3 (Sale): Convert the sale of the product in CAD into MXN using the buy rate. Apply and adapt Formula 7.4. Step 4: All amounts are in MXN. Apply Formula 6.5: $S = C_{\text{parts}} + C_{\text{assembly}} + E + P$
Perform	Step 2: Sell Rate = 2% higher than mid-rate = $16.0514 (1 + 0.02) = 16.3724$ Buy Rate = 3% lower than mid-rate = $16.0514 (1 - 0.03) = 15.5699$ Step 3 (Parts): (C_{parts} in MXN) = $16.3724 \times \\$55,000 = \\$900,482$ Step 3 (Sale): (S in MXN) = $15.5699 \times \\$99,500 = \\$1,549,205.05$ Step 4: $\\$1,549,205.05 = \\$900,482 + \\$245,400 + \\$43,800 + P$ $\$1,549,205.05 = \$1,189,682 + P$ $\$359,523.05 = P$	
Present	After converting all Canadian currency into Mexican pesos and factoring in the financial institution's buy and sell rates, the manufacturer realizes a profit of \$359,523.05 in Mexican pesos.	The Manufacturer's Numbers  ■ Cost of Parts --> MXN\$900,482 ■ Cost of Assembly --> MXN\$245,400 ■ Expenses --> MXN\$43,800 ■ Profit --> MXN\$359,523.05

Currency Appreciation and Depreciation

An analyst on *Global News* is discussing how the Canadian dollar has strengthened against the US dollar. Your first reaction is that a strong Canadian dollar ought to be good thing, so hearing that this change might hurt Canada's exports, you wonder how that could be.

Currencies are actively traded in the international marketplace, which means that exchange rates are changing all the time. As such, exchange rates rise and decline. A currency **appreciates** (or strengthens) relative to another currency when it is able to purchase *more* of that other currency than it could previously. A currency **depreciates** (or weakens) relative to another currency when it is able to purchase *less* of that other currency than it could previously. Take a look at two examples illustrating these concepts:

- **EXAMPLE 1:** If CAD\$1 buys USD\$0.81 and the exchange rate rises to USD\$0.82, then your CAD\$1 purchases an additional penny of the US dollars. Therefore, the Canadian currency appreciates, or strengthens, relative to the US dollar.
- **EXAMPLE 2:** Similarly, if the exchange rate drops to USD\$0.80, then your CAD\$1 purchases one less penny of the US dollars. Therefore, the Canadian currency depreciates, or weakens, relative to the US dollar.

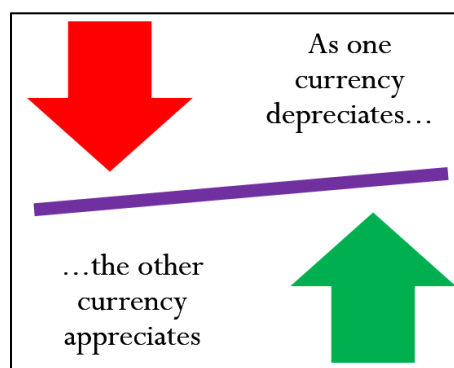
These concepts are particularly important to international business and global economies. Generally speaking, when a currency appreciates it has a positive effect on imports from the other country because it costs less money than it used to for domestic companies to purchase the same amount of products from the other country. However, the currency appreciation

tends to also have a negative effect on exports to other countries because it costs the foreign companies more money to purchase the same amount of products from the domestic companies.



Important Notes

It is critical to observe that if currency A appreciates relative to currency B, then the opposite is true for currency B relative to currency A (currency B depreciates relative to currency A). The figure to the right illustrates this concept. Recalling Example 1 above, the Canadian currency appreciated, so the US currency depreciated. In Example 2, the Canadian currency depreciated, so the US currency appreciated.



How It Works

When you work with currency appreciation or depreciation, you still use the same basic steps as before. The additional skill you require in the first step is to adjust an exchange rate appropriately based on how it has appreciated or depreciated.



Things To Watch Out For

If Currency A relative to one unit of Currency B...	Currency A	Currency B
Increases	Depreciates or weakens	Appreciates or strengthens
Decreases	Appreciates or strengthens	Depreciates or weakens

It is very easy to confuse the two relative currencies, their values, and the concepts of appreciation and depreciation. For example, if the exchange rate increases between currency A relative to a unit of currency B, which exchange rate appreciated? If currency A has increased per unit of currency B, then it takes more money of currency A to buy one unit of currency B. As a result, currency B appreciates because a single unit of

currency B can now buy more of currency A. For example, if the exchange rate is USD\$0.8118 relative to CAD\$1 and it increases to USD\$0.8218 relative to CAD\$1, then the Canadian dollar purchases one penny more. The figure helps you understand the relationships involved and provides a visual reminder of which direction everything is moving.



Give It Some Thought

1. If the exchange rate in terms of US dollars per unit of euros increases, which currency weakened?
2. If the Australian dollar weakens against the Canadian dollar, did the exchange rate increase or decrease in terms of Australian dollars per unit of Canadian dollars?
3. If the exchange rate in terms of yen per unit of the Mexican pesos decreases, which currency weakened?
4. If the British pound (£) appreciates against the US dollar, did the exchange rate increase or decrease in terms of pounds per unit of US dollars?

Solutions:
1. US dollars 2. Increase 3. Mexican pesos 4. Decrease

Example 7.3C: Effects of Currency Appreciation

A Canadian manufacturer requires parts from the United States. It purchases from its supplier in lots of 100,000 units at a price of USD\$7.25 per unit. Since the last time the manufacturer made a purchase, the Canadian dollar has appreciated 0.0178 from the previous mid-rate of USD\$0.8118 per CAD. If the sell rate is 1.5% above the mid-rate, how have the manufacturer's costs changed?

Plan	Calculate the cost of the product both before and after the currency appreciation. The difference between the two numbers is the change in the manufacturer's costs.	
Understand	What You Already Know Step 1: The following purchase and exchange rates are known: Current Currency = Total Purchase = $100,000 \times \$7.25 = \text{USD}\$725,000$ Mid-Rate = 0.8118USD per CAD Sell Rate = 1.5% higher Canadian appreciation = 0.0178	How You Will Get There Step 2: Calculate the old and new sell rates, factoring in the currency appreciation. Notice that the Current Currency is in US dollars but the sell rates are per Canadian dollar. You will need to invert the rates so that the exchange rate is expressed per US dollar to match the Current Currency. Step 3: Apply Formula 7.4 for each transaction. Step 4: Calculate the difference between the two numbers to determine the change in cost.
Perform	Step 2: Previous Sell Rate = $\frac{1}{0.8118} = \text{CAD}\$1.2318 \text{ per USD}$ Previous Sell Rate = $1.2318 \times (1 + 0.015) = \text{CAD}\$1.2503 \text{ per USD}$ If the Canadian dollar has appreciated, it buys more US dollars per CAD. Therefore, New Sell Rate = $\frac{1}{(0.8118+0.0178)} = \text{CAD}\$1.2054 \text{ per USD}$ New Sell Rate = $1.2054 \times (1 + 0.015) = \text{CAD}\$1.2235 \text{ per USD}$ Step 3: Previous CAD = $1.2503 \times \$725,000 = \$906,467.50$ New CAD = $1.2235 \times \$725,000 = \$887,037.50$ Step 4: Change in Cost = Previous CAD – New CAD = $\$906,467.50 - \$887,037.50 = \$19,430$	
Present	The manufacturer has its input costs decrease by \$19,430 since the CAD appreciated.	



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7.4: Invoicing: Terms of Payment and Cash Discounts

(Make Sure You Bill Them)

What a way to start your Monday morning! With dread you face a great stack of envelopes in your in-basket. The first envelope contains an invoice from your major supplier indicating an outstanding balance of \$3,600 with terms of 2/10, 1/20, net 30. In the second envelope, an office supplies bill totals \$500 with terms of 3/15, net 45 EOM. The third envelope holds yet another invoice from your transportation company indicating an outstanding balance of \$21,000 with terms of 2/15, 1/25, net 60 ROG. It also indicates that you have an overdue balance of \$4,000 subject to a 3% monthly penalty. Before opening any more envelopes that probably contain still more bills, you pour yourself another cup of coffee and settle down to figuring out how much you need to pay and when.

In the world of business, most purchases are not paid for in cash. Instead, businesses tend to work through an invoicing system in which they send out bills to their clients on a regular basis. In accounting, this means that the purchase is placed into accounts receivables until such time as a cheque arrives and the purchase is converted to cash. Invoices provide detailed transaction information, listing the amount owed and also indicating the terms on which payment is expected. Companies may offer what are known as cash discounts as incentives for early invoice payment. The rationale for these discounts is simple—a sale is not a sale until you have the cash in hand. The longer an invoice remains in accounts receivable, the less likely that it will be paid. Thus, it might turn into bad debt, which the creditor cannot collect at all.

Invoicing is less common in consumer purchases because of the sheer volume of transactions involved and the higher risk of nonpayment. Imagine purchasing items at Walmart and receiving an invoice to pay your bill next month instead of paying cash. It is hard to fathom the number of invoices Walmart would have to distribute monthly. How much would it cost to collect those debts? How many of those invoices would go unpaid? That is why consumer purchases typically do not involve invoicing.

Invoicing does commonly occur at a consumer level on credit card transactions along with many services where the business may not be able to assess the exact amount of the bill at the time of the transaction or until the service is delivered. Two examples illustrate this point:

1. Think of your MasterCard bill. You are able to make purchases, say, from March 9 to April 8. Then on April 9 the company sends out a statement saying you have until April 29 to pay your bill. If you do not, interest and late payment penalties are involved.
2. You have a dental visit for a regular cleaning. Before charging you, the dentist's office needs to determine how much your insurance will cover. It may not find out the answer for a day or two, so it sends you an invoice at a later date once it hears from the insurance company. The invoice terms indicate that payment is due upon receipt and you will incur late penalties if payment is not forthcoming.

This section explores the most common aspects of invoicing, including terms of payments and cash discounts. You will learn to calculate the amount required to pay invoices and how to reduce the outstanding balance of an invoice if a partial payment is received. Additionally, the calculation of late payments and penalties is introduced.

Invoice Terms and Invoice Dating

You must know how to read a business invoice. The figure on the next page³ provides a sample invoice and highlights the following areas:

1. **Invoicing Company.** The invoice must identify who is sending and issuing the invoice.
2. **Invoice Date.** The date on which the invoice was printed, along with the invoice tracking number (in this case, the order date and shipping date are equivalent to the invoice date). When an invoice is paid, the cheque must reference the invoice number so that the invoicing company can identify which invoice to credit the payment to.
3. **Transaction Details.** The details of the transaction might include number of units, unit prices, and any discounts for which the item is eligible.
4. **Invoice Total.** The total amount owing is indicated, including any taxes or additional charges.

³ Invoice provided by and courtesy of Kerry Mitchell Pharmacy Ltd. And Shoppers Drug Mart

5. **Terms of Payment.** The terms of payment include any cash discounts and due dates. The date of commencement (as discussed below) is determined from this part of the invoice.
6. **Late Penalty.** A penalty, if any, for late payments is indicated on the invoice. Whether or not a company enforces these late penalties is up to the invoice-issuing organization.

QR G.S.T. REGISTRATION # R104265889

INVOICE/FACTURE NO. 163737

ORDER DATE/DATE DE LA COMMANDE: Feb 4 10 SHIP DATE/DATE DE L'EXPÉDITION: Feb 4 10

POOR RICHARD'S Distributing Corp.
Divisions: BIO-CARE LABORATORIES • ZEN ORGANIC FOODS
FLASH 5 ENERGY FOODS • NUTRI-NATION FUNCTIONAL FOODS
2820 Huntington Place, Port Coquitlam, B.C. Canada V3C 4T3
PHONE/TÉLÉPHONE: (604) 464-4511 FAX: (604) 464-4513

SHOPPERS DRUG #2428
K MITCHELL PHCY LTD
710 ST ANNE'S RD
WINNIPEG MB
R2N 0A2

ACCOUNT NO. 2358 SALESMAN 41 CUSTOMER ORDER NO. 71777 SHIPPED VIA CANPAR
PROVINCIAL TAX NO. TERMS 1% 10 NET 30

PRODUCT NO. NO PRODUIT	DESCRIPTION	ORDERED COMMANDE	B/O À VENIR	SHIPPED EXPÉDIE	PRICE PRIX	* PER CH	AMOUNT MONTANT
07-00192	SPRING 2010 NEW STYLE PK	1	0	1	355.40	PK	355.40
07-0314	H-COMFORT ZONE 314			3		EA	
07-0319	C-COMFORT ZONE 319			3		EA	
07-0504	I-COMFORT ZONE 504			3		EA	
07-0317	G-COMFORT ZONE 317			3		EA	
07-0318	G-COMFORT ZONE 318			3		EA	
07-0315	H-COMFORT ZONE 315			3		EA	
07-0320	G-COMFORT ZONE 320			3		EA	
07-0316	H-COMFORT ZONE 316			3		EA	
07-0321	F-COMFORT ZONE 321			3		EA	
07-0602	P-COMFORT STUD 602			4		EA	
07-0692	T-COMFORT STUD 692			4		EA	
07-0683	S-COMFORT STUD 683			4		EA	
07-0684	S-COMFORT STUD 684			4		EA	
07-0690	T-COMFORT STUD 690			4		EA	
07-0627	P-COMFORT STUD 627			4		EA	
07-0672	P-COMFORT STUD 672			4		EA	
07-0630	P-COMFORT STUD 630			4		EA	

REMARKS/REMARQUES: ** SPRING 2010 NEW STYLE PREPACK **

SUB TOTAL/SOUS TOTAL: 355.40
SHIPPING/EXPÉDITION: 0.00
G.S.T./T.P.S.: 17.77
PROV. TAX PROV. RETAIL ITEMS: 0.00
** PROV. TAX PROV. **

TOTAL: 373.17

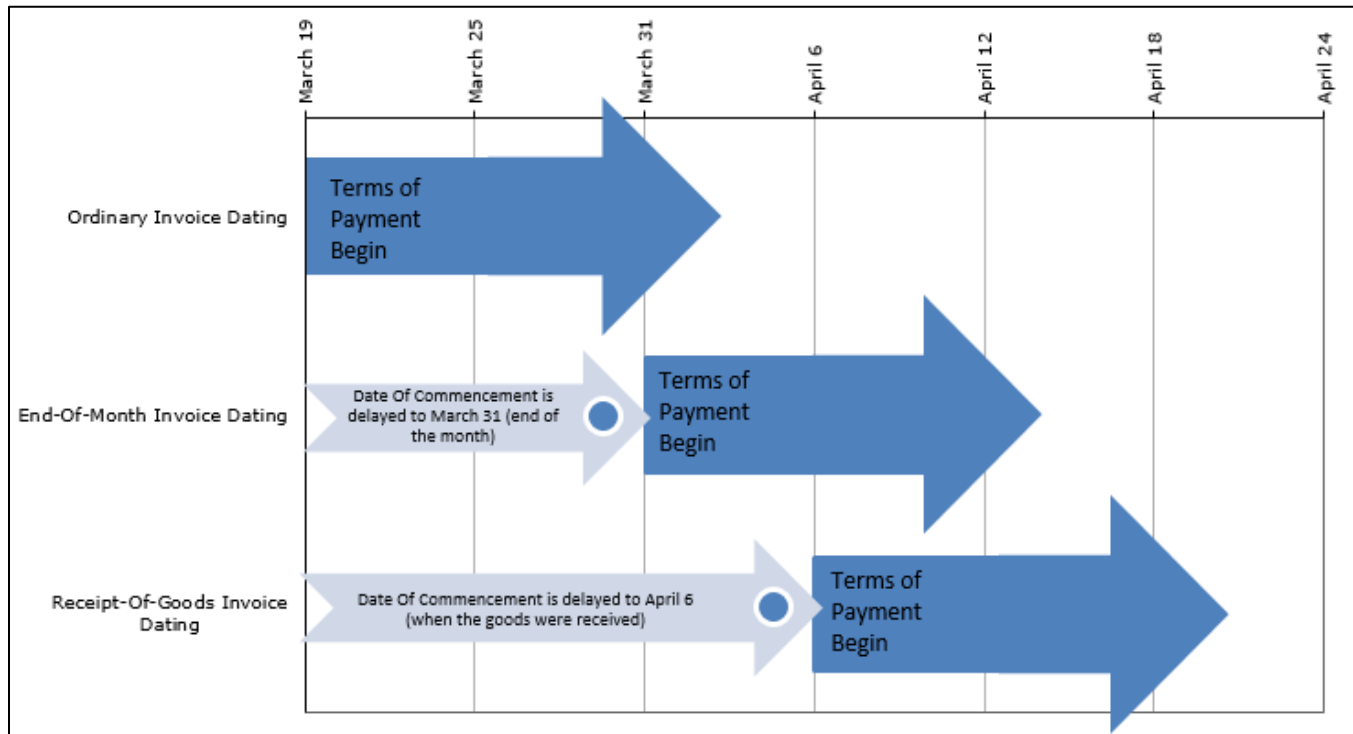
CONDITIONS OF SALE: CHECK ALL GOODS UPON RECEIPT. CLAIMS OR RETURNS NOT ACCEPTED AFTER 5 DAYS FROM RECEIPT OF INVOICE. NO CASH REFUND. NO RETURNS WITHOUT OUR AUTHORIZATION. RETURNS ARE SUBJECT TO A RE-STOCKING CHARGE. OVERDUE ACCOUNTS - A SERVICE CHARGE OF AT LEAST 2% PER MONTH (24% PER ANNUM) ADDED, BASED ON PREVIOUS MONTHS UNPAID BALANCE.

CONDITIONS DE VENTE: VÉRIFIEZ TOUTE LA MARCHANDISE À LA RÉCEPTION. TOUTE RÉVENDICATION OU RETOUR EFFECTUÉS 5 JOURS APRÈS RÉCEPTION DE LA FACTURE SERONT REFUSÉS. AUCUN REMBOURSEMENT COMPTANT AUCUN RETOUR SANS NOTRE AUTORISATION. COMPTES NON PAYÉS - FRAIS DE SERVICE DE 2% PAR MOIS (24% PAR ANNÉE) AJOUTÉS D'AUTRES LA BALANCE NON PAYÉE.

* CODES: EA - EACH DZ - DOZEN PK - PREPACK * SPECIAL Pick: GE Pack: GE Check: ☒ E & O.E. / SAUF E. ET O.

Three Dates of Commencement

All invoice terms are affected by what is known as the **date of commencement**, which is the first day from which all due dates stated on the invoice will stem. The date of commencement is determined in one of three ways as illustrated below.



1. **Ordinary Invoice Dating.** In **ordinary invoice dating**, or just invoice dating for short, the date of commencement is the same date as the invoice date. Therefore, if the invoice is printed on March 19, then all terms of payment commence on March 19. This is the default manner in which most companies issue their invoices, so if an invoice does not specify any other date of commencement you can safely assume it is using ordinary dating.
2. **End-of-Month Invoice Dating.** **End-of-month invoice dating** applies when the terms of payment include the wording "end-of-month" or the abbreviation "EOM" appears after the terms of payment. In end-of-month dating, the date of commencement is the last day of the same month as indicated by the invoice date. Therefore, if the invoice is printed on March 19, then all terms of payment commence on the last day of March, or March 31. Many companies use this method of dating to simplify and standardize all of their due dates, in that if the date of commencement is the same for all invoices, then any terms of payment will also share the same dates.
3. **Receipt-Of-Goods Invoice Dating.** **Receipt-of-goods invoice dating** applies when the terms of payment include the wording "receipt-of-goods" or the abbreviation "ROG" appears after the terms of payment. In receipt-of-goods dating, the date of commencement is the day on which the customer physically receives the goods. Therefore, if the invoice is printed on March 19 but the goods are not physically received until April 6, then all terms of payment commence on April 6. Companies with long shipping times involved in product distribution or long lead times in production commonly use this method of dating.

Terms of Payment

The most common format for the terms of paying an invoice is shown on the next page and illustrated by an example.

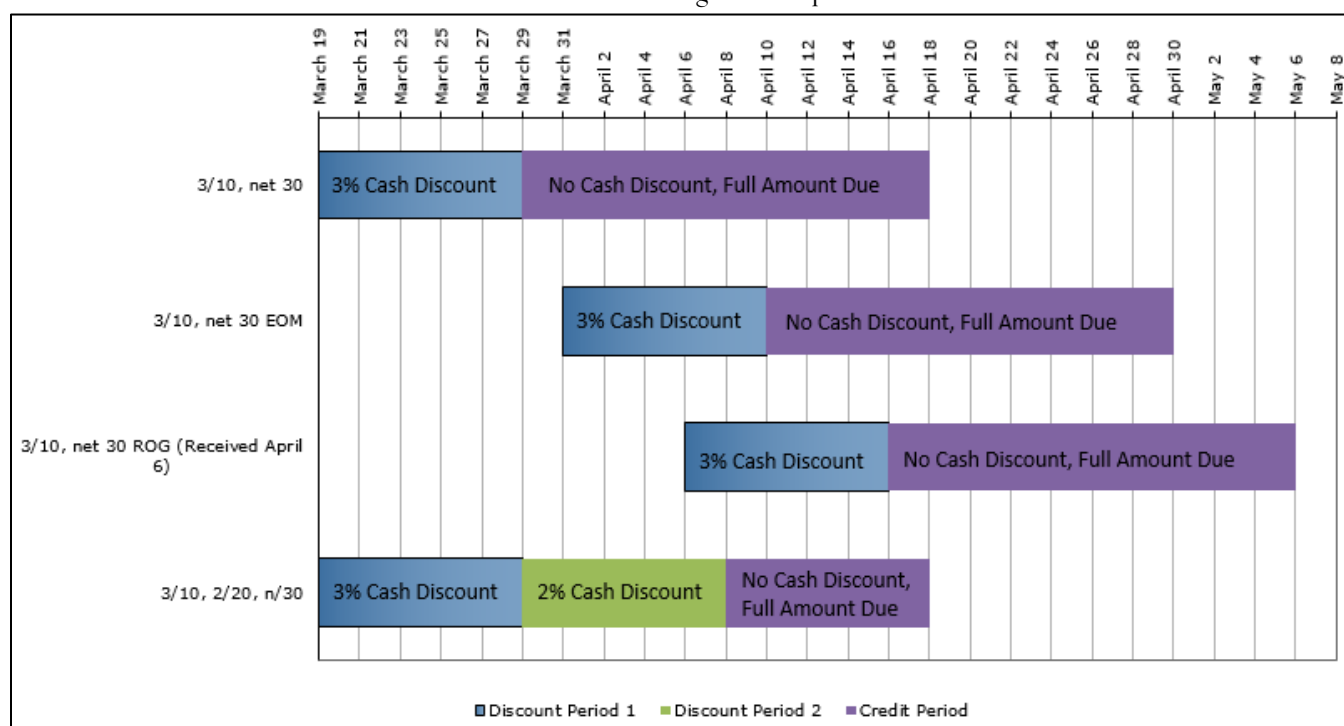
3 is Cash Discount: A **cash discount** is the percentage of the balance owing on an invoice that can be deducted for payment received either in full or in part during the discount period (see below). In this case, 3% is deducted from the total invoice amount.

3/10, net 30

10 is Discount Period: The **discount period** is the number of days *from the date of commencement* in which the customer is eligible to take advantage of the cash discount. For this example, there is a period of 10 days during which a payment received in full or any portion thereof is eligible for the 3% cash discount.

net 30 is Credit Period: The **credit period** is the number of interest-free days *from the date of commencement* that the customer has to pay the invoice in full before the creditor applies any penalties to the invoice.

The figure below plots the invoice term of "3/10, net 30" along a time diagram to illustrate the terms of payment. This term is combined with various dates of commencement using the examples from the "Date of Commencement" section.



Note the following observations:

- In all four scenarios, the total length of all bars is the same, since the credit period is 30 days.
- Although all four invoices have the same invoice date of March 19, the date of commencement is modified in the ROG and EOM scenarios, which shift the discount and credit periods into the future. In these cases, any payment before the end of the discount period qualifies for the discount. For example, in the EOM scenario a company could pay its bill early on March 27 and qualify for the 3% cash discount.

You may see invoice terms displayed in several other common formats:

1. 3/10, n/30: In this case, the word "net" has been abbreviated to "n/". This is illustrated in the fourth scenario in the figure.

2. $3/10, 2/20, n/30$. In this case, multiple cash discounts are being offered, meaning 3% within 10 days from the date of commencement and 2% from the 11th to the 20th day from the date of commencement. This is also illustrated in the fourth scenario of the figure.
3. $n/30$. In this case, no cash discount is offered and only the credit period is identified.

Rules for Invoice Terms

Some common business practices are implemented across most industries. Note that you must always check with the invoicing organization to ensure it is applying these practices.

1. **No Net Figure.** If no net figure is stated in the terms of payment, you should assume that the credit period is 20 days after the last discount period. If there are no discount periods, then the credit period is 20 days from the date of commencement. For example, " $3/10, 2/15$ " means that the credit period ends 20 days after the second discount period of 15 days. Hence, the credit period is 35 days after the date of commencement.
2. **No Cash Discount.** Not every invoice receives a cash discount. If no terms indicate a cash discount, then the invoicing company seeks full payment only. For example, " $n/30$ " means that no cash discount applies and the credit period ends 30 days from the date of commencement.
3. **Nonbusiness Days.** Most businesses operate Monday through Friday and are closed on weekends and holidays. As such, any date falling on a nonworking day is moved to the next business day. For example, if an invoice is ordinary dated December 22 and lists a cash discount of $2/10$, the end of the discount period falls on January 1. Since this is New Year's Day, the discount period extends to January 2. In this textbook, you should apply this practice to any of the five known statutory holidays discussed in Chapter 4 (New Year's Day, Good Friday or Easter Monday in Quebec only, Canada Day, Labour Day, and Christmas Day).

Three Types of Payments

When businesses pay invoices, three situations can occur:

1. **Full Payment.** A full payment means that the company wants to pay its invoice in full and reduce its balance owing to zero dollars. This is the most common practice.
2. **Partial Payment.** A partial payment means that the company wants to lower its balance owing but will not reduce that balance to zero. A company will generally employ this method when it wants to either lower its accounts payable or demonstrate good faith in paying its invoices. Partial payment may also occur if the company wants to take advantage of a cash discount but lacks the funds to clear the invoice in its entirety.
3. **Late Payment.** A late payment occurs when the company pays its invoice either in full or partially after the credit period has elapsed. Late payments occur for a variety of reasons, but the most common are either insufficient funds to pay the invoice or a simple administrative oversight.

The rest of this section shows you how to handle each of these types of payments mathematically, and it also explores the implications of cash discounts.



Give It Some Thought

In each of the following cases, determine which term of payment results in the longest credit period extending from the invoice date.

1. $2/10, n/30$ or $3/15$
2. $3/15, n/45$ or $2/10, 1/20$
3. $2/10, n/60$ or $3/10, n/30$ EOM on an invoice dated January 7
4. $3/15, n/45$ or $2/10, n/35$ ROG, or $2/20$ EOM on an invoice dated September 22 and goods received on September 29

Solutions:

1. 30 days or 35 days
 2. 45 days or 40 days
 3. 60 days or 54 days (24 days left in January plus 30 more)
 4. 45 days or 42 days (7 days until received plus 35) or 48 days (8 left in month plus 40 more)

Full Payments

In the opening scenario in this section, the first invoice on your desk was for \$3,600 with terms of 2/10, 1/20, net 30. Suppose that invoice was dated March 19. If you wanted to take advantage of the 2% cash discount, when is the last day that payment could be received, and in what amount would you need to write the cheque? In this section, we will look at how to clear an invoice in its entirety.



The Formula

The good news is that a cash discount is just another type of discount similar to what you encountered in Section 6.1, and you do the calculations using the exact same formula. Apply Formula 6.1 on single discounts, which is reprinted below.

N is Net Payment Amount: The net price is the amount after the discount has been applied. In other words, it is the dollar amount of the actual payment. If there is no cash discount on the payment, then the net price would be the same as the gross price.

Formula 6.1 – Single Discount:
$$N = L \times (1 - d)$$

L is Invoice Balance: The “list” price is the amount of the invoice before any discounts are applied. In other words, it is the total dollar amount of the invoice and represents the balance owing.

d is Cash Discount Rate: This discount rate is the cash discount (in decimal format) that is deducted from the total invoice (the gross price) if payment is received within the discount period. This cash discount needs to be clearly stated in the terms of payment.



How It Works

Follow these steps when working with payments and invoices:

Step 1: Look for and identify key information such as the invoice date, date of commencement modifiers such as EOM or ROG, terms of payment, late penalty, and the invoice amount.

Step 2: Draw a timeline similar to the figure on the next page. On this timeline, chronologically arrange from left to right the invoice date and amount, the date of commencement, the end of any discount (if any) or credit periods, cash discounts (if any) that are being offered, and penalties (if any).

Step 3: Determine when payments are being made and in what amount. Locate them on the timeline.

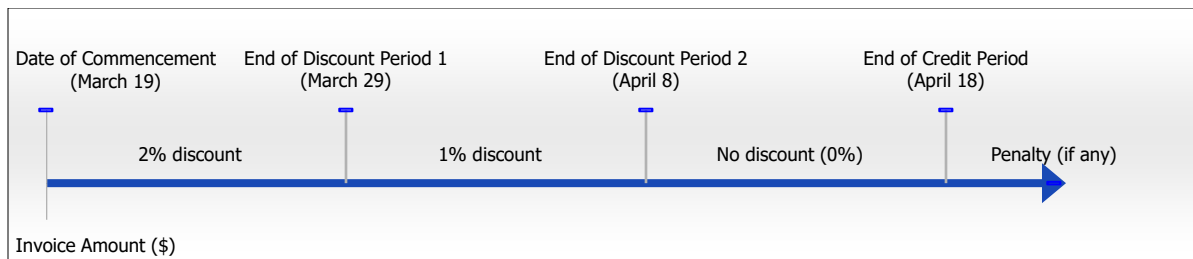
Step 4: Apply the correct calculations for the payment, depending on whether the payment is a full payment, partial payment, or late payment.

Let's continue with the example of the first invoice on your desk. You mail in a cheque to be received by March 29.

What amount is the cheque?

Step 1: The invoice amount is $L = \$3,600$, invoice date is March 19, and terms of payment are 2/10, 1/20, net 30.

Step 2: The figure on the next page displays the invoice timeline.



Step 3: Note on the timeline that a payment on March 29 is the last day of the 2% discount period.

Step 4: According to Formula 6.1, the amount to pay is $N = \$3,600 \times (1 - 0.02) = \$3,600 \times .98 = \$3,528$. A cheque for \$3,528 pays your invoice in full. By taking advantage of the cash discount, you reduce your payment by \$72.



Things To Watch Out For

If there is one area of invoicing that causes the most confusion, it is assigning information to the net price and invoice amount variables. These are commonly assigned backwards. Remember these rules so that you always get the correct answer:

1. The invoice amount or any reference to the balance owing on an invoice is *always* a list price.

$$\text{List Price} = \text{Invoice Amount}$$

2. The payment of an invoice, whether in full, partial, or late, is *always* a net price.

$$\text{Net Price} = \text{Payment Amount}$$

**TOP
SECRET**

Paths To Success

Who cares about 1% or 2% discounts when paying bills? While these percentages may not sound like a lot, remember that these discounts occur over a very short time frame. For example, assume you just received an invoice for \$102.04 with terms of 2/10, n/30. When taking advantage of the 2% cash discount, you must pay the bill 20 days early, resulting in a \$100 payment. That is a \$2.04 savings over the course of 20 days. To understand the significance of that discount, imagine that you had \$100 sitting in a savings account at your bank. Your savings account must have a balance of \$102.04 twenty days later. This requires your savings account to earn an annual interest rate of 44.56%! Therefore, a 20-day 2% discount is the same thing as earning interest at 44.56%. Outrageously good!



Give It Some Thought

In each of the following cases, determine the cash discount or penalty for which the payment qualifies.

Invoice Date	Terms of Payment	Payment Received on
1. April 14	2/10, n/30	April 24
2. July 7	3/10, 2/20, n/30 EOM, 1% per month penalty	August 12
3. November 12 (goods received November 28)	2/20 ROG, 2% per month penalty	December 29
4. February 27 (non-leap year)	4/10, 2/15, 1/25 EOM	March 25

Solutions:

1. 2% discount 2. 2% discount 3. no discount (0%) or penalty 4. 1% discount

Example 7.4A: Paying Your Bills in Full

You receive an invoice for \$35,545.50 dated August 14. The terms of payment are listed as 3/10, 2/20, net 45 EOM. The accounting department is considering paying this debt on one of three days. Determine the full payment required if payment is received by the invoicing company on each of the following dates:

- September 3
- September 19
- September 30

Plan

In each case, calculate the full payment (N) after applying any cash discount.

Understand**What You Already Know**

Step 1: The invoice amount, terms of payment, and any payments are known:

$L = \$35,545.50$ Invoice date = August 14

Terms of payment = 3/10, 2/20, net 45 EOM

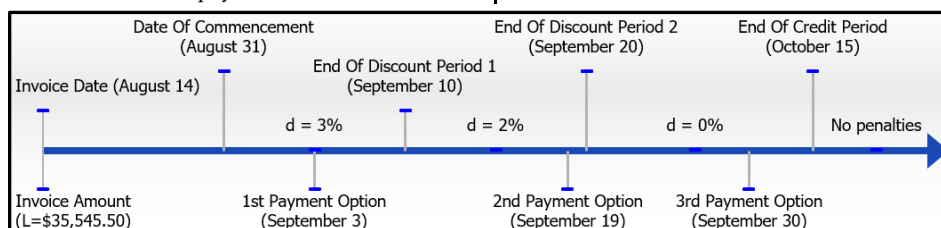
Three payment date options are September 3, September 19, and September 30.

Steps 2 & 3: The figure below illustrates the timeline for the invoice and identification of payments.

How You Will Get There

Step 4: Calculate the payment required by applying Formula 6.1. As noted on the timeline:

- The first payment option qualifies for $d = 3\%$.
- The second payment option qualifies for $d = 2\%$.
- The third payment option qualifies for $d = 0\%$ (no cash discount).

**Perform**

Step 4:

a. $N = \$35,545.50 \times (1 - 0.03) = \$35,545.50 \times 0.97 = \$34,479.14$

b. $N = \$35,545.50 \times (1 - 0.02) = \$35,545.50 \times 0.98 = \$34,834.59$

c. $N = \$35,545.50 \times (1 - 0) = \$35,545.50 \times 1 = \$35,545.50$

(Note that since there is no discount in part (c), you could have concluded that $N = L$)

Present

If the invoicing company receives payment on September 3, a 3% discount is allowed and \$34,479.14 clears the invoice. If the payment is received on September 19, a 2% discount is allowed and \$34,834.59 pays it off. Finally, if the payment arrives on September 30, there is no discount so the full invoice amount of \$35,545.50 is due.

Partial Payments

In the section opener, your third invoice is for \$21,000 with terms of 2/15, 1/25, net 60 ROG. After you paid the \$3,528 to clear the first invoice, you realize that your company has insufficient funds to take full advantage of the 2% cash discount being offered by the transportation company. Not wanting to lose out entirely, you decide to submit a partial payment of \$10,000 before the first discount period elapses. What is the balance remaining on the invoice?

Unless you pay attention to invoicing concepts, it is easy to get confused. Perhaps you think that \$10,000 should be removed from the \$21,000 invoice total, thereby leaving a balance owing of \$11,000. Or maybe you think the payment should receive the discount of 2%, which would be \$200. Are you credited with \$9,800 off of your balance? Or maybe \$10,200 off of your balance? In all of these scenarios, you would be committing a serious mistake and miscalculating your balance owing. Let's look at the correct way of handling this payment.



The Formula

Recall that invoice amounts are gross amounts (amounts before discounts), or G, and that payment amounts are net amounts (amounts after discounts), or N. Also recall that an algebraic equation requires all terms to be expressed in the same unit.

In the case of invoice payments, you can think of the balance owing as being in the unit of "pre-discount" and any payment being in the unit of "post-discount." Therefore, a payment cannot be directly deducted from the invoice balance since it is in the wrong unit. You must convert the payment from a "post-discount" amount into a "pre-discount" amount using a rearranged version of Formula 6.1. You can then deduct it from the invoice total to calculate any balance remaining.

N is Net Payment Amount: This is the actual payment amount made that must be converted into its value toward the invoice total. Since the payment represents an amount after the discount has been removed, you need to find out what it is worth before the cash discount was removed.

Formula 6.1 – Single Discount Rearranged:
$$L = \frac{N}{(1-d)}$$

L is Invoice Amount: This is the amount deducted from the invoice balance as a result of the payment. It is what the payment is worth once the cash discount has been factored into the calculation.

d is Cash Discount Rate: Any payment received within any discount period, whether in full or partial, is eligible for the specified cash discount indicated in the terms of payment.



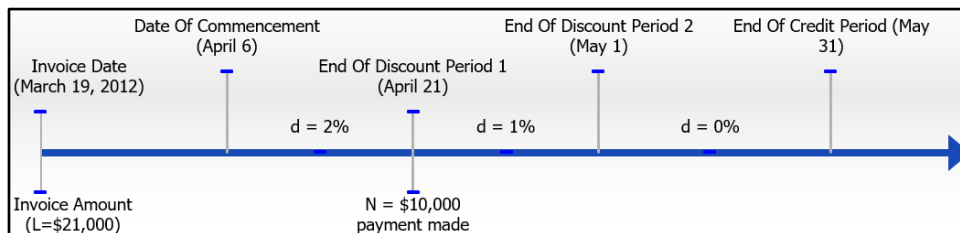
How It Works

Follow the same invoice payment steps even when working with partial payments. Commonly, once you calculate the gross amount of the payment in step 4 you will also have to calculate the new invoice balance by deducting the gross payment amount.

Let's continue working with the \$21,000 invoice with terms of 2/15, 1/25, net 60 ROG. Assume the invoice is dated March 19 and the goods are received on April 6. If a \$10,000 payment is made on April 21, what balance remains on the invoice?

Step 1: The invoice amount is \$21,000, the invoice date is March 19, goods are received on April 6, and the terms of payment are 2/15, 1/25, net 60. A payment of \$10,000 is made on April 21.

Step 2: The figure below shows the invoice timeline.



Step 3: The payment of \$10,000 on April 21 falls at the end of the first discount period and qualifies for a 2% discount.

Step 4: To credit the invoice, apply the rearranged Formula 6.1:

$$L = \frac{\$10,000}{(100\% - 2\%)} = \frac{\$10,000}{98\%} = \frac{\$10,000}{0.98} = \$10,204.08$$

This means that before any cash discounts, the payment is worth \$10,204.08 toward the invoice total. The balance remaining is \$21,000.00 - \$10,204.08 = \$10,795.92.



Important Notes

In the case where a payment falls within a cash discount period, the amount credited toward an invoice total is always larger than the actual payment amount. If the payment does not fall within any discount period, then the amount credited toward an invoice total is equal to the actual payment amount (since there is no cash discount).

To help you understand why a partial payment works in this manner, assume an invoice is received in the amount of \$103.09 and the customer pays the invoice in full during a 3% cash discount period. What amount is paid? The answer is $N = \$103.09(100\% - 3\%) = \$100$. Therefore, any payment of \$100 made during a 3% cash discount period is always equivalent to a pre-discount invoice credit of \$103.09. If the balance owing is more than \$103.09, that does not change the fact that the \$100 payment during the discount period is worth \$103.09 toward the invoice balance.



Give It Some Thought

In each of the following situations, determine for the partial payment whether you would credit the invoice for an amount that is larger than, equal to, or less than the partial payment amount.

1. An invoice dated April 7 with terms 4/20, 2/40, n/60 EOM. The goods are received on April 9 and a partial payment is made on May 21.
2. An invoice dated July 26 with terms 3/30, n/45 ROG. The goods are received on August 2 and a partial payment is made on September 3.
3. An invoice dated January 3 with terms $2\frac{1}{2}/10$, 1/20. The goods are received on January 10 and a partial payment is made on January 22.

Solutions

1. Since the payment falls within a discount period, the credited amount is larger than the payment.
 2. Since the payment does not fall within a discount period, the credited amount is equal to the payment.
 3. Since the payment falls within a discount period, the credited amount is larger than the payment.

Example 7.4B: Applying Your Partial Payments toward Your Balance Owing

Heri just received an invoice from his supplier, R&B Foods. The invoice totalling \$68,435.27 is dated June 5 with terms of $2\frac{1}{2}/10$, 1/25, n/45. Heri sent two partial payments in the amounts of \$20,000 and \$30,000 that R&B Foods received on June 15 and June 29, respectively. He wants to clear his invoice by making a final payment to be received by R&B Foods on July 18. What is the amount of the final payment?

Plan	Determine the amount of the final payment (N) after the first two partial payments are credited toward the invoice total. To credit the two partial payments you must find the list amount before any cash discounts (L) such that the payment can be deducted from the invoice total.	
Understand	What You Already Know Step 1: The invoice amount, terms of payment, and payment amounts are known: $L = \\$68,435.27$ Invoice date = June 5 Terms of payment = $2\frac{1}{2}/10$, 1/25, n/45 $N_1 = \\$20,000$ on June 15 $N_2 = \\$30,000$ on June 29 $N_3 = ?$ on July 18 Steps 2 & 3: The figure illustrates the timeline for the invoice and identifies the payments.	How You Will Get There Step 4: The first two payments are partial. Apply the rearranged Formula 6.1. As noted on the timeline: a. The first partial payment qualifies for $d = 2\frac{1}{2}\%$. b. The second partial payment qualifies for $d = 1\%$. c. The last payment is a full payment, where $d = 0\%$ (no cash discount).

	Invoice Date & Date Of Commencement (June 5) End Of Discount Period 1 (June 15) End Of Discount Period 2 (June 30) End Of Credit Period (July 20) $d = 2\frac{1}{2}\%$ $d = 1\%$ $d = 0\%$ No penalties Invoice Amount ($L = \\$68,435.27$) 1st Payment $N = \\$20,000$ (June 15) 2nd Payment $N = \\$30,000$ (June 29) Final Payment $N = ?$ (July 18)
Perform	Step 4: a) $L = \frac{\\$20,000}{(1 - 0.025)} = \frac{\\$20,000}{0.975} = \\$20,512.82$ Balance Owing = $\\$68,435.27 - \\$20,512.82 = \\$47,922.45$ b) $L = \frac{\\$30,000}{(1 - 0.01)} = \frac{\\$30,000}{0.99} = \\$30,303.03$ Balance Owing = $\\$47,922.45 - \\$30,303.03 = \\$17,619.42$ c) Since there is no cash discount, $N_3 = L$. The balance owing is $L = \\$17,619.42$. Therefore, the payment is the same number and $N_3 = \\$17,619.42$.
Present	The first two payments receive a credit of \$20,512.82 and \$30,303.03 toward the invoice total, respectively. This leaves a balance on the invoice of \$17,619.42. Since the final payment is made during the credit period, but not within any cash discount period, the final payment is the exact amount of the balance owing and equals \$17,619.42.

Late Payment Penalties

Among all of the invoices that you were paying in the section opener, you realize upon opening that third envelope that you somehow missed an invoice last month and failed to pay the \$4,000 owing, which is now overdue. In looking at the invoice, you notice that it states on the bottom that late payments are subject to a 3% per month penalty. You urgently want to pay this invoice to maintain good relations with your supplier but wonder in what amount you must issue the cheque?

Why Do Late Penalties Exist?

Suppliers are doing their business customers a favour when they use invoicing to seek payment. In essence, suppliers provide the products to the customer but are not paid for those products up front. This means customers are granted a period during which they "borrow" the products for free without any associated interest costs that are usually tied to borrowing. If the invoice is not paid in full by the time the credit period elapses, then the supplier starts treating any remaining balance like a loan and charges interest, which is called a late payment penalty.



The Formula

When a payment arrives after the credit period has expired, you need to adjust the balance of the invoice upward to account for the penalty. Continue to use Formula 6.1 on Single Discounts for this calculation. In this unique case, the discount rate represents a penalty. Instead of deducting money from the balance, you add a penalty to the balance. As a result, the late penalty is a *negative* discount. For example, if the penalty is 3% then $d = -3\%$.



How It Works

Follow the same steps to solve the invoice payment once again when working with late payments. As an example, work with the \$4,000 overdue invoice that is subject to a 3% per month late penalty. If the invoice is paid within the first month of being overdue, what payment is made?

Step 1: The invoice balance is $L = \$4,000$, and the penalty percent is $d = -0.03$.

Steps 2 & 3: Since the focus is strictly on the late payment calculation, you do not need a timeline for the discount or credit periods. The late payment occurs within the first month past the expiry of the credit period.

Step 4: To determine the new invoice balance, apply Formula 6.1 to calculate $N = \$4,000 \times (1 - (-0.03)) = \$4,000 \times 1.03 = \$4,120$. Therefore, to clear the invoice you must issue a cheque in the amount of \$4,120. This covers the \$4,000 owing along with a late penalty of \$120.



Important Notes

Not all invoices have penalties, nor are they always enforced by the supplier. There are many reasons for this:

1. Most businesses pay their invoices in a timely manner, which minimizes the need to institute penalties.
2. Waiving a penalty maintains good customer relations and reinforces a positive, cooperative business partnership.
3. Strict application of late penalties results in difficult situations. For example, if a payment was dropped in the mail on August 13 from your best customer and it arrives one day after the credit period expires on August 15, do you penalize that customer? Is it worth the hassle or the risk of upsetting the customer?
4. The application of late penalties generally involves smaller sums of money that incur many administrative costs. The financial gains from applying penalties may be completely wiped out by the administrative expenses incurred.
5. Businesses have other means at their disposal for dealing with delinquent accounts. For example, if a customer regularly pays its invoices in an untimely manner, the supplier may just decide to withdraw the privilege of invoicing and have the customer always pay up front instead.

In this textbook, all penalties are strictly and rigidly applied. If a payment is late, the invoice balance has the appropriate late penalty applied.



Things To Watch Out For

When a late penalty is involved, you must resist the temptation to adjust the *payment* by the penalty percentage. The discussion and calculations in this section focus on adjusting the outstanding invoice *balance* to figure out the payment required. For example, assume there is a \$500 outstanding balance subject to a 2% penalty.

- This means that with the penalty, the invoice total is \$510, which determines that a payment of \$510 is required.
- If the customer makes a \$200 partial payment on this late debt, you cannot apply the partial payment procedure, which would give a credit of \$204.08 for the payment, resulting in a balance owing of \$295.92. If you make this mistake, you in fact reward the customer for being tardy!
- Nor can you deduct the 2% penalty from the payment and give credit for $\$200(1 - 0.02) = \$196$ (resulting in a balance owing of \$304) because the 2% penalty applies to the whole invoice amount and not just the partial payment.
- Instead, the \$200 payment must be deducted directly from the penalty-adjusted balance of \$510. In this case, the customer still owes \$310 to clear the invoice.

**TOP
SECRET**

Paths To Success

An alternative method for applying a late penalty is to treat the penalty like a positive percent change (see Section 3.1). In this case, the penalty is how much you want to increase the balance owing. Thus, if an invoice for \$500 is subject to a 2% penalty, the $\Delta\% = 2\%$ and Old = \$500. The goal is to look for the New balance in Formula 3.1:

$$\Delta\% = \frac{\text{New} - \text{Old}}{\text{Old}} \times 100 \qquad 2\% = \frac{\text{New} - \$500}{\$500} \times 100 \qquad \text{New} = \$510$$

Example 7.4C: Whoops! I Missed the Due Date!

You received an invoice dated December 17 in the amount of \$53,455.55 with terms of 4/15, 2/30, n/60 ROG, 2.75% penalty per month for late payments. The merchandise is received on January 24. You sent in a partial payment of \$40,000 on January 31 with intention to pay the remaining balance before the credit period expired. However, you forgot about the invoice and realized your mistake on March 30, when you submit payment in full for the invoice. What amount is the final payment? (Note: Assume this is not a leap year).

Plan	The final payment must cover the invoice balance owing on any remaining balance after the partial payment is credited and the outstanding balance is penalized as per the late penalty.
Understand	What You Already Know Step 1: The invoice amount, terms of payment, and payment information are known: $L = \\$53,455.55$ Invoice date = December 17 Goods received = January 24 Terms of payment = 4/15, 2/30, n/60 ROG, 2.75% penalty per month for late payments $N_1 = \\$40,000$ on January 31 $N_2 = ?$ on March 30 Steps 2 & 3: The figure on the next page illustrates the timeline for the invoice and identifies the payments. How You Will Get There Step 4 (Partial Payment): Credit the partial payment during the discount period ($d = 4\%$) by applying the rearranged Formula 6.1. Step 4 (New Invoice Balance): Deduct the partial payment from the invoice total. Step 4 (Late Payment): The second payment occurs after the credit period elapses. Therefore the outstanding invoice balance is penalized by $d = -0.0275$ as per the policy. Apply Formula 6.1. This calculates the outstanding balance including the penalty, which equals the final payment.
Perform	Step 4 (Partial Payment): $L = \frac{\\$40,000}{(1 - 0.04)} = \frac{\\$40,000}{0.96} = \\$41,666.67$ Step 4 (New Invoice Balance): Balance Owing = $\\$53,455.55 - \\$41,666.67 = \\$11,788.88$ Step 4 (Late Payment): $N = \\$11,788.88 \times (1 - (-0.0275))$ $N = \\$11,788.88 \times 1.0275 = \\$12,113.07$.
Present	The first payment resulted in a \$41,666.67 deduction from the invoice. The overdue remaining balance of \$11,788.88 has a penalty of \$324.19 added to it, resulting in a final clearing payment of \$12,113.07.



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Chapter 7 Summary

Key Concepts Summary

Section 7.1: Sales Taxes (Everybody Wants a Piece of My Pie)

- The three sales taxes, current rates, and how to calculate prices including taxes
- How businesses complete GST/HST remittances

Section 7.2: Property Taxes (I Owe, I Owe)

- How municipalities levy mill rates and tax rates against real estate owners

Section 7.3: Exchange Rates and Currency Exchange (It Is a Global World)

- Converting currencies through exchange rates
- The rise and decline of exchange rates: currency appreciation and depreciation

Section 7.4: Invoicing: Terms of Payment and Cash Discounts (Make Sure You Bill Them)

- How invoicing works: understanding invoice terms and invoice dating
- The calculations involved when a full payment amount is made
- The calculations involved when a partial payment amount is made
- The calculations involved when a late payment amount is made

The Language of Business Mathematics

assessed value The portion of the market value of a property that is subject to municipal taxes.

buy rate The rate at which a foreign currency is bought; it will always be lower than the mid-rate in terms of money per unit of foreign currency.

cash discount A percentage of the balance owing on an invoice that can be deducted for payment received either in full or in part during the discount period.

credit period The number of interest-free days from the date of commencement before full payment of the invoice is required.

currency appreciation When one currency strengthens relative to another currency, resulting in an ability to purchase more of that other currency than it could previously.

currency depreciation When one currency weakens relative to another currency, resulting in an ability to purchase less of that other currency than it could previously.

date of commencement The first day from which the invoice terms extend forward in time and from which all due dates are established.

discount period The number of days from the date of commencement for which a cash discount is offered.

end-of-month invoice dating A term of payment where the date of commencement is the last day of the same month as indicated by the invoice date.

exchange rate The number of units of a foreign currency that are bought with one unit of the domestic currency, or vice versa.

GST The goods and services tax is a Canadian federal sales tax on most products purchased by businesses and consumers alike.

HST The harmonized sales tax is a tax that combines the GST and PST into a single tax.

market value A snapshot of the estimated selling price of a property at some specified point in time.

mid-rate An exchange rate that does not involve or provide for any charges for currency conversion.

mill rate A tax per \$1,000 of assessed value to determine property taxes.

ordinary invoice dating A term of payment where the date of commencement is the same date as the invoice date.

property taxes Annual taxes paid by the owners of real estate to local levying authorities to pay for services such as roads, water, sewers, public schools, policing, fire departments, and other community services.

PST The provincial sales tax is a consumer sales tax administered by Canadian provinces and territories.

receipt-of-goods invoice dating A term of payment where the date of commencement is the day on which the customer physically receives the goods.

sales tax A percent fee levied by a government on the supply of products.

sell rate The rate at which a foreign currency is sold. It is always higher than the mid-rate in terms of money per unit of foreign currency.

tax policy A municipality-based percentage of the market value of a property that is used to convert the market value into an assessed value.

tax rate A tax per \$100 of assessed value to determine property taxes.

tax remittance The fulfillment of a tax obligation.

The Formulas You Need to Know

Symbols Used

AV = assessed value of a property

Current Currency = currency you are converting from

Desired Currency = currency you are converting to

Exchange Rate = per-unit conversion rate for current currency

Property Tax = property tax amount

PTR = property tax rate, usually set on a per \$100 (tax rate) or per \$1,000 (mill rate) basis

Rate = sales tax

Remit = tax remittance

S = selling price

S_{tax} = selling price including taxes

Tax Collected = total tax collected through sales

Tax Paid = total tax paid through purchases

Formulas Introduced

Formula 7.1 Selling Price Including Tax: $S_{\text{tax}} = S + (S \times \text{Rate})$ (Section 7.1)

Formula 7.2 GST/HST Remittance: $\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$ (Section 7.1)

Formula 7.3 Property Taxes: $\text{Property Tax} = \sum(\text{AV} \times \text{PTR})$ (Section 7.2)

Formula 7.4 Currency Exchange: $\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$ (Section 7.3)

Technology

Calculator

The Percent Change function is used in this chapter. See the end of Chapter 3 for a full discussion on this calculator function.



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Chapter 8: Simple Interest: Working with Single Payments and Applications

(Money Is Not Free)

If you have a lot of surplus cash, it would hardly make sense to hide it under your mattress or bury it in the backyard of your townhouse. As any investment specialist will tell you, you want your money to work for you instead of just sitting idle. So you can invest and be rewarded by earning interest.

Do you know if your account at the Royal Bank of Canada (RBC) has paid you the right amount of interest? You might assume that because everything at a bank is fully automated it is guaranteed to be correct. However, ultimately all automation at the Royal Bank relies on input by an employee. What if your interest rate is keyed in wrong? Maybe the computer programmer erred in a line of computer code and the amount of interest is miscalculated. What if RBC had a computer glitch? These pitfalls highlight the importance of always checking your account statements.

In business, it would be rare for an organization to borrow or lend money for free. That is not to say that businesses are ruthless with each other; in the previous chapter, you saw that most business transactions are completed through interest-free credit transactions involving invoicing. However, this generosity does not extend indefinitely. After the credit period elapses, the business essentially treats an unpaid invoice as a loan and starts charging an interest penalty.

When businesses need to borrow money, promissory notes, demand loans, and commercial papers are just three of the possibilities discussed later in this chapter.

Even governments need to borrow money. Have you ever thought about how the government of Canada or your provincial government goes about doing that? While a government has many methods at its disposal, a common method is to use treasury bills—in essence, borrowing the money from investors.

On a personal level, almost all people invest and borrow money at some time. This chapter examines consumer lines of credit and student loans, both of which carry simple interest rates. On the earnings side, it is common for people to have savings accounts that earn low rates of interest. Other short-term investments include treasury bills, commercial papers, and short-term guaranteed investment certificates (GICs).

The bottom line is that money does not come free. In this chapter, you will explore the concept of simple interest, learn how to calculate it, and then apply simple interest to various financial tools.

Outline of Chapter Topics

- 8.1: Principal, Rate, Time (How Does Interest Work?)
- 8.2: Moving Money Involving Simple Interest (Move and Nobody Gets Hurt)
- 8.3: Application: Savings Accounts and Short-Term GICs (Safe and Secure)
- 8.4: Application: Promissory Notes (A Promise Is a Promise)
- 8.5: Application: Loans (The Bank Comes Knocking)
- 8.6 Application: Treasury Bills and Commercial Papers (When Governments and Businesses Borrow)

8.1: Principal, Rate, Time

(How Does Interest Work?)

Your investments may be at risk if stock and bond markets slump, as a story in the *Globe and Mail* predicts. You wonder if you should shift your money into relatively secure short-term investments until the market booms again. You consider your high-interest savings account, but realize that only the first \$60,000 of your savings account is insured. Perhaps you should put some of that money into treasury bills instead.

Looking ahead, what income will you live on once you are no longer working? As your career develops, you need to save money to fund your lifestyle in retirement. Some day you will have \$100,000 or more that you must invest and reinvest to reach your financial retirement goals.

To make such decisions, you must first understand how to calculate simple interest. Second, you need to understand the characteristics of the various financial options that use simple interest. Armed with this knowledge, you can make smart financial decisions!

The world of finance calculates interest in two different ways:

1. **Simple Interest.** A **simple interest** system primarily applies to short-term financial transactions, with a time frame of less than one year. In this system, which is explored in this chapter, interest accrues but does not compound.
2. **Compound Interest.** A **compound interest** system primarily applies to long-term financial transactions, with a time frame of one year or more. In this system, which the next chapter explores, interest accrues and compounds upon previously earned interest.

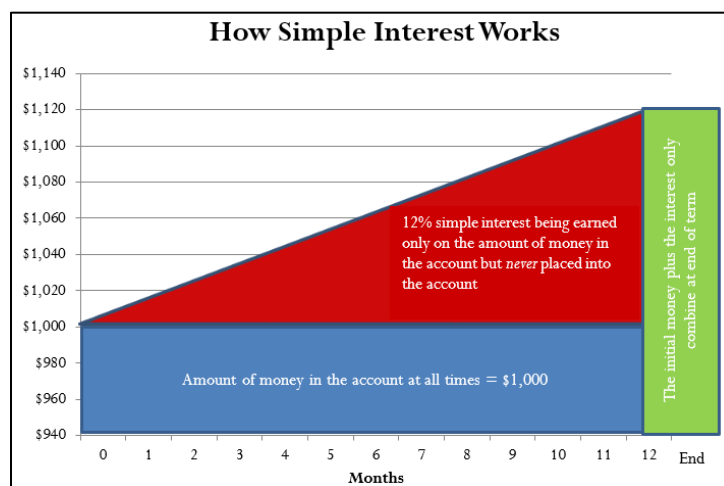
Simple Interest

In a simple interest environment, you calculate interest solely on the amount of money at the beginning of the transaction. When the term of the transaction ends, you add the amount of the simple interest to the initial amount. Therefore, throughout the entire transaction the amount of money placed into the account remains unchanged until the term expires. It is only on this date that the amount of money increases. Thus, an investor has more money or a borrower owes more money at the end.

The figure illustrates the concept of simple interest. In this example, assume \$1,000 is placed into an account with 12% simple interest for a period of 12 months. For the entire term of this transaction, the amount of money in the account always equals \$1,000. During this period, interest accrues at a rate of 12%, but the interest is never placed into the account. When the transaction ends after 12 months, the \$120 of interest and the initial \$1,000 are then combined to total \$1,120.

A loan or investment always involves two parties—one giving and one receiving. No matter which party you are in the transaction, the amount of interest remains unchanged. The only difference lies in whether you are earning or paying the interest.

- If you take out a personal loan from a bank, the bank gives you the money and you receive the money. In this situation, the bank earns the simple interest and you are being charged simple interest on your loan. In the figure, this means you must pay back not only the \$1,000 you borrowed initially but an additional \$120 in interest.
- If you place your money into an investment account at the bank, you have given the money and the bank has received the money. In this situation, you earn the simple interest on your money and the bank pays simple interest to your investment account. In the figure, this means the bank must give you back your initial \$1,000 at the end plus an additional \$120 of interest earned.





The Formula

The best way to understand how simple interest is calculated is to think of the following relationship:

Amount of simple interest = *How much at What simple interest rate for How long*

Notice that the key variables are the amount, the simple interest rate, and time. Formula 8.1 combines these elements into a formula for simple interest.

I is Interest Amount: The **interest amount** is the dollar amount of interest that is paid or earned. To interpret the interest amount properly, remember who you are in the transaction. If you are borrowing the money, this is the interest amount charged to you. If you are investing the money, this is the interest amount you earn.

P is Present Value or Principal: The amount of money at the beginning of the time period being analyzed is known as the **present value**, or **P**. If this is in fact the amount at the start of the financial transaction, it is also called the **principal**, which is the original amount of money that is borrowed or invested. Or it can simply be the amount at some earlier time before the future value was known. In any case, the amount excludes the interest.

Formula 8.1 – Simple Interest: **$I = Prt$**

r is Interest Rate: The **interest rate** is the rate of interest that is charged or earned during a specified time period. It is usually expressed in percent format. Unless noted otherwise, interest rates are expressed on an annual basis. An interest rate is the result of Formula 2.2, where

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

In this case, you calculate an annual interest rate in its decimal format as follows:

$$\text{Annual Interest Rate} = \frac{\text{Annual Interest Amount}}{\text{Principal}}$$

Thus, if you are earning \$3 of interest annually on a \$100 principal, then your annual interest rate is $3/100 = 0.03$ or 3%.

t is Time: The **time period** or term is the length of the financial transaction for which interest is charged or earned.



How It Works

Follow these steps when you calculate the amount of simple interest:

Step 1: Formula 8.1 has four variables, and you need to identify three for any calculation involving simple interest. If necessary, draw a timeline to illustrate how the money is being moved over time.

Step 2: Ensure that the simple interest rate and the time period are expressed with a common unit. If they are not already, you need to convert one of the two variables to the same units as the other.

Step 3: Apply Formula 8.1 and solve for the unknown variable. Use algebra to manipulate the formula if necessary.

Assume you have \$500 earning 3% simple interest for a period of nine months. How much interest do you earn?

Step 1: Note that your principal is \$500, or $P = \$500$. The interest rate is assumed to be annual, so $r = 3\%$ per year. The time period is nine months.

Step 2: Convert the time period from months to years: $t = \frac{9}{12}$.

Step 3: According to Formula 8.1, $I = \$500 \times 3\% \times \frac{9}{12} = \$11.25$. Therefore, the amount of interest you earn on the \$500 investment over the course of nine months is \$11.25.



Important Notes

Recall that algebraic equations require all terms to be expressed with a common unit. This principle remains true for Formula 8.1, particularly with regard to the interest rate and the time period. For example, if you have a 3% annual interest rate for nine months, then either

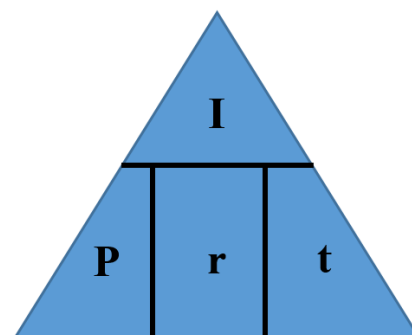
- The time needs to be expressed annually as $9/12$ of a year to match the yearly interest rate, or
- The interest rate needs to be expressed monthly as $3\%/12 = 0.25\%$ per month to match the number of months.

It does not matter which you do so long as you express both interest rate and time in the same unit. If one of these two variables is your algebraic unknown, the unit of the known variable determines the unit of the unknown variable. For example, assume that you are solving Formula 8.1 for the time period. If the interest rate used in the formula is annual, then the time period is expressed in number of years.



Paths To Success

Four variables are involved in the simple interest formula, which means that any three can be known, requiring you to solve for the fourth missing variable. To reduce formula clutter, the triangle technique illustrated here helps you remember how to rearrange the formula as needed.



Give It Some Thought

1. If you have money in your savings account (or any other investment) and it earns simple interest over a period of time, would you have more or less money in your account in the future?
2. If you have a debt for which you haven't made any payments, yet it is being charged simple interest on the principal, would you owe more or less money in the future?

Solutions:

1. More, because the interest is earned and therefore is added to your savings account.
 2. More, because you owe the principal and you owe the interest, which increases your total amount owing.

Example 8.1A: How Much Interest Is Owed?

Julio borrowed \$1,100 from Maria five months ago. When he first borrowed the money, they agreed that he would pay Maria 5% simple interest. If Julio pays her back today, how much interest does he owe her?

Plan	Calculate the amount of interest that Julio owes Maria (I).	
Understand	What You Already Know	How You Will Get There
	Step 1: The terms of their agreement are as follows: $P = \$1,100$ $r = 5\%$ $t = 5$ months	Step 2: The rate is annual, and the time is in months. Convert the time to an annual number. Step 3: Apply Formula 8.1.
	Start $P = \\$1,100$ $r = 5\%$ annually $t = 5$ months $I = \\$?$	

Perform	Step 2: Five months out of 12 months in a year is $\frac{5}{12}$ of a year, or $t = \frac{5}{12}$. Step 3: $I = \\$1,100 \times 5\% \times \frac{5}{12} = \\$1,100 \times 0.05 \times 0.41\bar{6} = \\$22.92$
Present	For Julio to pay back Maria, he must reimburse her for the \$1,100 principal borrowed plus an additional \$22.92 of simple interest as per their agreement.

Example 8.1B: Do You Know Your Interest Rate?

A \$3,500 investment earned \$70 of interest over the course of six months. What annual rate of simple interest did the investment earn?

Plan	Calculate the annual interest rate (r).	
Understand	What You Already Know Step 1: The principal, interest amount, and time are known: $P = \\$3,500$ $I = \\$70$ $t = 6$ months	How You Will Get There Step 2: The computed interest rate needs to be annual, so you must express the time period annually as well. Step 3: Apply Formula 8.1, rearranging for r.
Perform	Step 2: Six months out of 12 months in a year is $\frac{6}{12}$ of a year, or $t = \frac{6}{12}$. Step 3: $\\$70 = \\$3,500 \times r \times \frac{6}{12}$ $r = \frac{\\$70}{\\$3,500 \times \frac{6}{12}} = \frac{\\$70}{\\$1,750} = 0.04$ or 4%	
Present	For \$3,500 to earn \$70 simple interest over the course of six months, the annual simple interest rate must be 4%.	

Example 8.1C: What Did You Start with?

What amount of money invested at 6% annual simple interest for 11 months earns \$2,035 of interest?

Plan	Calculate the amount of money originally invested, which is known as the present value or principal, symbolized by P .	
Understand	What You Already Know Step 1: The interest rate, time, and interest earned are known: $r = 6\%$ $t = 11$ months $I = \\$2,035$	How You Will Get There Step 2: Convert the time from months to an annual basis to match the interest rate. Step 3: Apply Formula 8.1, rearranging for P.
Perform	Step 2: Eleven months out of 12 months in a year is $\frac{11}{12}$ of a year, or $t = \frac{11}{12}$. Step 3: $\\$2,035 = P \times 6\% \times \frac{11}{12}$ $P = \frac{\\$2,035}{6\% \times \frac{11}{12}} = \frac{\\$2,035}{0.06 \times 0.91\bar{6}} = \\$37,000$	

Present	To generate \$2,035 of simple interest at 6% over a time frame of 11 months, \$37,000 must be invested.

Example 8.1D: How Long?	
For how many months must \$95,000 be invested to earn \$1,187.50 of simple interest at an interest rate of 5%?	
Plan	Calculate the length of time in months (t) that it takes the money to acquire the interest.
Understand	What You Already Know Step 1: The amount of money invested, interest earned, and interest rate are known: $P = \\$95,000.00$ $I = \\$1,187.50$ $r = 5\%$ How You Will Get There Step 2: Express the time in months. Convert the interest rate to a "per month" format. Step 3: Apply Formula 8.1, rearranging for t.
Perform	Step 2: 5% per year converted into a monthly rate is $r = \frac{0.05}{12}$ Step 3: $\\$1,187.50 = \\$95,000 \times \frac{0.05}{12} \times t$ $t = \frac{\\$1,187.50}{\\$95,000 \times \frac{0.05}{12}} = \frac{\\$1,187.50}{\\$95,000 \times 0.0041\bar{6}} = 3 \text{ months}$
Present	For \$95,000 to earn \$1,187.50 at 5% simple interest, it must be invested for a three-month period.

Time and Dates

In the examples of simple interest so far, the time period equalled an exact number of months. While this is convenient in many situations, financial institutions and organizations calculate interest based on the exact number of days in the transaction, which changes the interest amount.

To illustrate this, assume you had money saved for the entire months of July and August, where $t = \frac{2}{12}$ or $t = 0.1\bar{6}$ of a year. However, if you use the exact number of days, the 31 days in July and 31 days in August total 62 days. In a 365-day year that is $t = \frac{62}{365}$ or $t = 0.169863$ of a year. Notice a difference of 0.003196 occurs. Therefore, to be precise in performing simple interest calculations, you must calculate the exact number of days involved in the transaction.

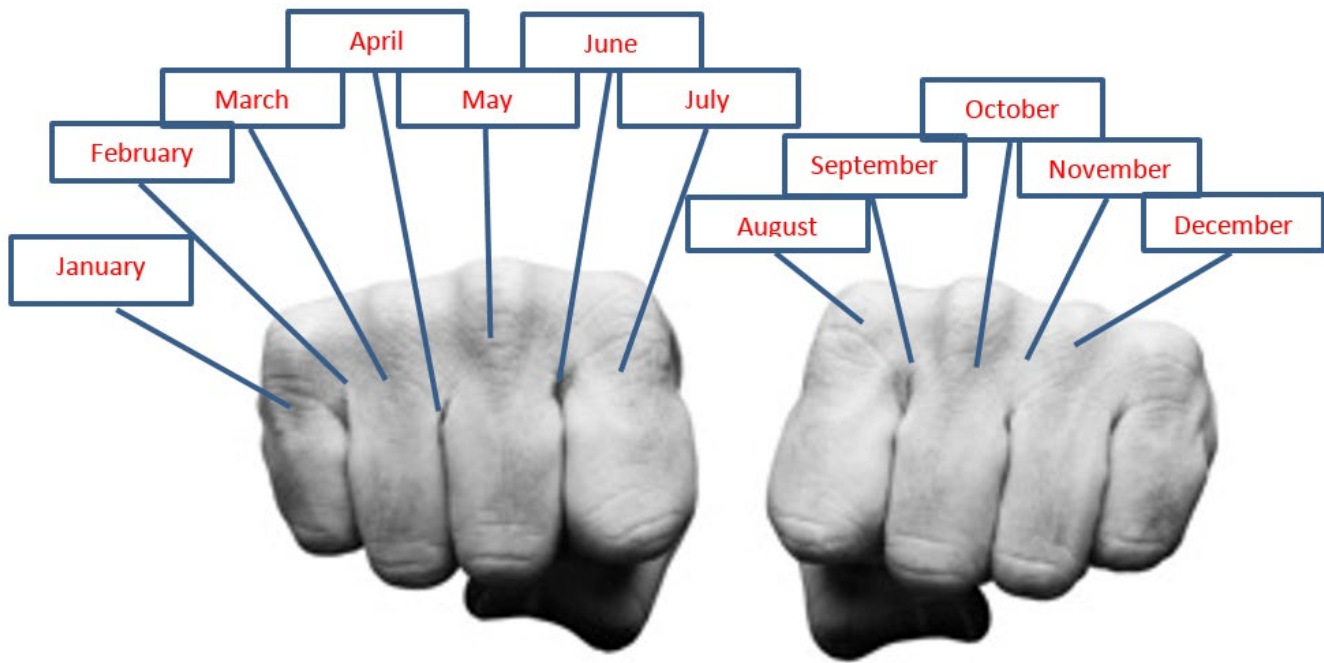
Figuring Out the Days

In practice, when counting the number of days in a transaction you include the first day but not the last day. This is because interest is calculated on the daily closing balance, not on interim balances throughout a day. For example, if you borrowed \$500 on June 3 and paid it back with interest on June 5, the closing balance on June 3 and June 4 is \$500. However, on June 5 a zero balance is restored. Therefore, you count June 3 and 4 as days of interest but not June 5, so you will owe two days of interest, not three.

There are three common ways to calculate the number of days involved in a transaction.

Method 1: Use Your Knuckles

You can use your knuckles as an aid to remembering which months of the year have more or fewer days.



Put your knuckles side by side and start on your outermost knuckle. Every knuckle represents a month with 31 days. Every valley between the knuckles represents a month that does not have 31 days (30 in all except February, which has 28 or 29). Now start counting months. Your first knuckle (pinkie finger) is January, which has 31 days. Next is your first valley, which is February and it does not have 31 days. Your next knuckle (ring finger) is March, which has 31 days ... and so on. When you get to your last knuckle (your index finger), move to the other hand's first knuckle (an index finger again). Notice that July and August are the two months back to back with 31 days.

If you are trying to calculate the number of days between March 20 and May 4, use your knuckles to recall that March has 31 days and April has 30 days. The calculations are illustrated in the table here.

Start Date	End Date	Days between Dates
March 20	March 31	$31 - 20 = 11$ days
March 31 (which precedes April 1, so it is day "0" for April)	April 30	$30 - 0 = 30$ days
April 30 (which precedes May 1, so it is day "0" for May)	May 4	$4 - 0 = 4$ days
TOTAL		$11 + 30 + 4 = 45$ days

Why does the end date of one line become the start date of the next line? Recall that you count the first day, but not the last day. Therefore, on the first line March 31 has not been counted yet and must be counted on the second line. Ultimately a transaction extending from March 20 to May 4 involves 45 days as the time period.

Method 2: Use A Table of Serial Numbers

In the table on the next page, the days of the year are assigned a serial number. The number of days between any two dates in the same calendar year is simply the difference between the serial numbers for the dates.

Using the example in Method 1, when the dates are within the same calendar year, find the serial number for the later date, May 4 (Day 124) and the serial number for the earlier date, March 20 (Day 79). Then, find the difference between the two serial numbers ($124 - 79 = 45$ days).

Day of Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Day of Month
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29	*	88	119	149	180	210	241	272	302	333	363	29
30	30		89	120	150	181	211	242	273	303	334	364	30
31	31		90		151		212	243		304		365	31

*For Leap years, February 29 becomes Day 60 and the serial number for each subsequent day is increased by 1.

Method 3: Use The BAIL Plus Date Function

Your BAIL Plus calculator is programmed with the ability to calculate the number of days involved in a transaction according to the principle of counting the first day but not the last day. The function "Date" is located on the second shelf above the number one. To use this function, open the window by pressing 2nd 1.

- Scroll amongst the four display lines by using your $\uparrow$ and $\downarrow$ arrows. There are four variables:
 1. DT1 is the starting date of the transaction.
 2. DT2 is the ending date of the transaction.
 3. DBD is the days between dates, which is your t in Formula 8.1.

4. ACT or 360 is the method of calculating the days between dates. In ACT (for “actual”), the calculator calculates the DBD based on the correct number of days in the month. In 360, the DBD is based on treating every month as having 30 days ($30 \times 12 = 360$ days). For all calculations, set the calculator to the ACT mode. Toggle this setting by pressing 2nd SET.

- To enter a date into the calculator, the calculator uses the format of MM.DDYY where:
 - MM is the numerical value of the month; you do not need to enter the leading zero on months with only one digit.
 - DD is the day of the month; you must always enter this value with two digits using a leading zero for days 1 through 9 in a month.
 - YY is the last two digits of the year; you must always enter this with two digits. If your question does not involve a year, select an appropriate year of your choice.
- As long as you know any two of the DT1, DT2, or DBD variables, you can calculate the third. Enter any two of the three variables, pressing ENTER after each and ensuring the window shows the output variable you are seeking, and then press CPT.

Using the example in Method 1, to calculate the time from March 20 to May 4 you first need to choose a year arbitrarily, perhaps 2011. Therefore, perform the following sequence:

2nd DATE
 2nd CLR Work (to erase previous calculations)
 3.2011 ENTER ↓ (this is March 20, 2011)
 5.0411 ENTER ↓ (this is May 4, 2011)
 CPT
 Answer: 45

Note that the answer is the same as the one you arrived at manually using your knuckles.



Important Notes

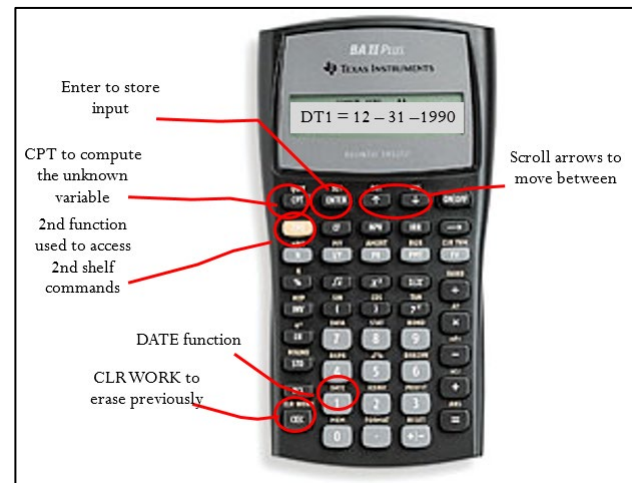
When solving for t , decimals may appear in your solution. For example, if calculating t in days, the answer may show up as 45.9978 or 46.0023 days; however, interest is calculated only on complete days. This occurs because the interest amount (I) used in the calculation has been rounded off to two decimals. Since the interest amount is imprecise, the calculation of t is imprecise. When this occurs, round t off to the nearest integer.



Things To Watch Out For

Miscalculating the number of days in February has to be one of the most common errors. To help you remember how many days are in February, recall your knowledge of leap years.

- February has 28 days except for leap years, in which it has 29 days.
- Leap years are those years that are evenly divisible by four or 400. However, those years that are evenly divisible by 100 (excepting years that are evenly divisible by 400) are not leap years. Here are some examples to illustrate this rule:
 - The year 1900 was not a leap year, since it is evenly divisible by 100 and not by 400.
 - The year 1996 was a leap year, since it is evenly divisible by four and not by 100.
 - The year 2000 was one of those exception years, and it was a leap year since it is evenly divisible by 400.
 - The year 2004 was once again a leap year since it is evenly divisible by four and not by 100. Thus, in this century 2004, 2008, 2012, 2016, 2020, 2024, and so on by fours are all leap years with 29 days.
 - The year 2100 will not be a leap year, since it is evenly divisible by 100.



Note that when it comes to the total number of days in a year, simple interest calculations ignore the 366th day in a leap year. Therefore, assume a year has 365 days in all calculations.

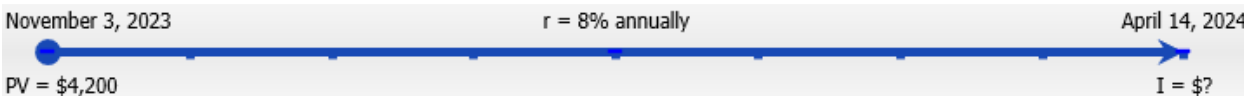
**TOP
SECRET**

Paths To Success

When entering two dates on the calculator, the order in which you key in the dates is not important. If you happen to put the last day of the transaction into the first field (DT1) and the start day of the transaction into the last field (DT2), then the number of days will compute as a negative number since the dates are reversed. Ignore the negative sign in these instances. Using the example, on your calculator if you had keyed May 4 into DT1 and March 20 into DT2, the DBD computes as -45 , in which you ignore the negative sign to determine the answer is 45 days.

Example 8.1E: Time Using Dates

Azul borrowed \$4,200 from Hannah on November 3, 2023. Their agreement required Azul to pay back the loan on April 14, 2024, with 8% simple interest. In addition to the principal, how much interest does Azul owe Hannah on April 14?

Plan	Calculate the amount of simple interest (I) that Azul owes Hannah on his loan.																																			
Understand	What You Already Know Step 1: The principal, simple interest rate, and dates are known: $P = \$4,200$ $r = 8\%$ Start Date = November 3, 2023 End Date = April 14, 2024		How You Will Get There Step 1 (continued): Calculate the number of days in the transaction. Note that this transaction involves February 2024, which is a leap year. Step 2: The rate is annual and the time is in days. Convert the time to an annual number. Step 3: Apply Formula 8.1.																																	
																																				
Perform	Step 1 (continued):		Calculator Instructions To get the number of days, use the DATE function:																																	
	<table><thead><tr><th>Start Date</th><th>End Date</th><th>Days between Dates</th></tr></thead><tbody><tr><td>November 3, 2023</td><td>November 30, 2023</td><td>$30 - 3 = 27$</td></tr><tr><td>November 30, 2023</td><td>December 31, 2023</td><td>$31 - 0 = 31$</td></tr><tr><td>December 31, 2023</td><td>January 31, 2024</td><td>$31 - 0 = 31$</td></tr><tr><td>January 31, 2024</td><td>February 29, 2024</td><td>$29 - 0 = 29$</td></tr><tr><td>February 29, 2024</td><td>March 31, 2024</td><td>$31 - 0 = 31$</td></tr><tr><td>March 31, 2024</td><td>April 14, 2024</td><td>$14 - 0 = 14$</td></tr><tr><td colspan="2">TOTAL</td><td>$27 + 31 + 31 + 29 + 31 + 14 = 163$ days</td></tr></tbody></table>		Start Date	End Date	Days between Dates	November 3, 2023	November 30, 2023	$30 - 3 = 27$	November 30, 2023	December 31, 2023	$31 - 0 = 31$	December 31, 2023	January 31, 2024	$31 - 0 = 31$	January 31, 2024	February 29, 2024	$29 - 0 = 29$	February 29, 2024	March 31, 2024	$31 - 0 = 31$	March 31, 2024	April 14, 2024	$14 - 0 = 14$	TOTAL		$27 + 31 + 31 + 29 + 31 + 14 = 163$ days	<table><tr><td>DT1</td><td>DT2</td><td>DBD</td><td>Mode</td></tr><tr><td>11.0323</td><td>4.1424</td><td>Answer: 163</td><td>ACT</td></tr></table>		DT1	DT2	DBD	Mode	11.0323	4.1424	Answer: 163	ACT
	Start Date	End Date	Days between Dates																																	
	November 3, 2023	November 30, 2023	$30 - 3 = 27$																																	
November 30, 2023	December 31, 2023	$31 - 0 = 31$																																		
December 31, 2023	January 31, 2024	$31 - 0 = 31$																																		
January 31, 2024	February 29, 2024	$29 - 0 = 29$																																		
February 29, 2024	March 31, 2024	$31 - 0 = 31$																																		
March 31, 2024	April 14, 2024	$14 - 0 = 14$																																		
TOTAL		$27 + 31 + 31 + 29 + 31 + 14 = 163$ days																																		
DT1	DT2	DBD	Mode																																	
11.0323	4.1424	Answer: 163	ACT																																	
Step 2: 163 days out of 365 days in a year is $t = \frac{163}{365}$																																				
Step 3: $I = \$4,200 \times 8\% \times \frac{163}{365} = \$4,200 \times 0.08 \times 0.446575 = \$150.05$																																				
Present	Azul borrowed the money for a period of 163 days and owes Hannah \$150.05 of simple interest on his loan, in addition to repaying the principal of \$4,200.																																			

Example 8.1F: Figuring Out Dates Using Time

On September 13, 2023, Aladdin decided to pay back the Genie on his loan of \$15,000 at 9% simple interest. If he paid the Genie the principal plus \$1,283.42 of interest, on what day did he borrow the money from the Genie?

Plan	To determine the starting date, the time involved in this transaction, or t , must be calculated.

Understand	What You Already Know Step 1: The principal, interest amount, simple interest rate, and end date are known: $P = \\$15,000.00$ $I = \\$1,283.42$ $r = 9\%$ End Date = September 13, 2023 Start Date Unknown $r = 9\%$ annually September 13, 2023 $PV = \\$15,000$ $I = \\$1,283.42$	How You Will Get There Step 2: The time is in days, but the rate is annual. Convert the rate to a daily rate. Step 3: Apply Formula 8.1, rearranging for t. Step 3 (continued): Using the End Date, use the t to determine the Start Date.							
	Step 2: Convert the interest rate to daily: $r = \frac{9\%}{365}$. Step 3: $\\$1,283.42 = \\$15,000 \times \frac{9\%}{365} \times t$ $t = \frac{\\$1,283.42}{\\$15,000 \times \frac{0.09}{365}} = 346.998741 \text{ days} \rightarrow 347 \text{ days}$ Step 3 (continued): The number of days is close to a year. Go back 365 days and start at September 13, 2022. Notice that 347 days is $365 - 347 = 18$ days short of a full year. Adding 18 days onto September 13, 2022, equals October 1, 2022. This is the start date.	Calculator Instructions To calculate the start date, use the DATE function: <table><tr><td>DT1</td><td>DT2</td><td>DBD</td><td>Mode</td></tr><tr><td>Answer: 10-01-2022</td><td>9.1323</td><td>347</td><td>ACT</td></tr></table>	DT1	DT2	DBD	Mode	Answer: 10-01-2022	9.1323	347
DT1	DT2	DBD	Mode						
Answer: 10-01-2022	9.1323	347	ACT						
Present	If Aladdin owed the Genie \$1,283.42 of simple interest at 9% on a principal of \$15,000, he must have borrowed the money 347 days earlier, which is October 1, 2022.								

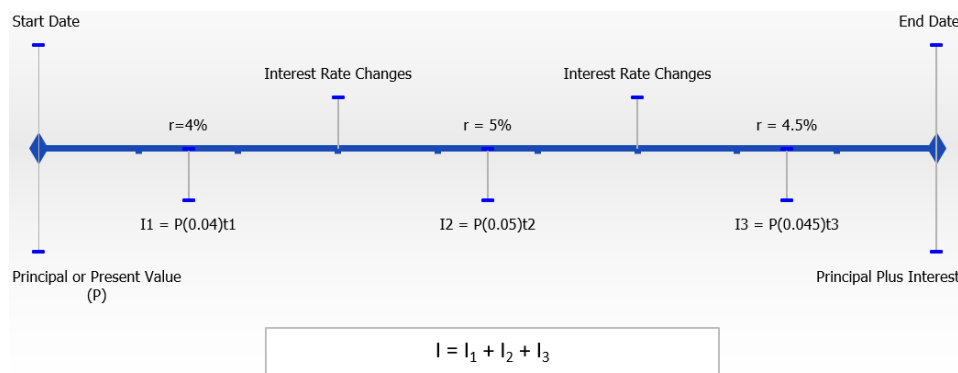
Variable Interest Rates

Not all interest rates remain constant. There are two types of interest rates:

1. **Fixed.** A **fixed interest rate** is an interest rate that is unchanged for the duration of the transaction.
2. **Variable.** A **variable interest rate** is an interest rate that is open to fluctuations over the duration of a transaction. This variable interest rate is usually tied to the **prime rate**, which is an interest rate set by the Bank of Canada that usually forms the lowest lending rate for the most secure loans. Most people and companies are usually charged the prime rate plus a percentage (typically 0.5% to 5%) to arrive at the variable interest rate. The Bank of Canada periodically adjusts this rate because of economic and financial considerations in Canada.

In all of the previous examples in this section, the interest rate (r) was fixed for the duration of the transaction. When r fluctuates, you must break the question into time periods or time fragments for each value of r . In each of these time fragments, you calculate the amount of simple interest earned or charged and then sum the interest amounts to form the total interest earned or charged.

This figure illustrates this process with a variable rate that changes twice in the course of the transaction. To calculate the total interest amount, you need to execute Formula 8.1 for each of the time fragments. Then total the interest amounts from each of the three time fragments to calculate the interest amount (I) for the entire transaction.



Example 8.1G: A Floating Rate

Julio borrowed \$15,000 on May 17 at a variable interest rate of prime + 3% when the prime rate was $2\frac{1}{4}\%$. On June 22 the prime rate increased by $\frac{1}{4}\%$, and on September 2 it increased again by $\frac{1}{2}\%$. How much interest will Julio pay when he repays his loan on October 4?

Plan

Calculate the amount of interest that Julio owes for the entire time frame of his loan, or I , from May 17 to October 4.

Understand**What You Already Know**

Step 1: There is an initial interest rate and then it changes twice over the course of the transaction. That means there are three time periods and three different interest rates. The principal is also known: $P = \$15,000$.

Date	Interest Rate
May 17 to June 22	$2\frac{1}{4}\% + 3\% = 5\frac{1}{4}\%$
June 22 to September 2	$5\frac{1}{4}\% + \frac{1}{4}\% = 5\frac{1}{2}\%$
September 2 to October 4	$5\frac{1}{2}\% + \frac{1}{2}\% = 6\%$

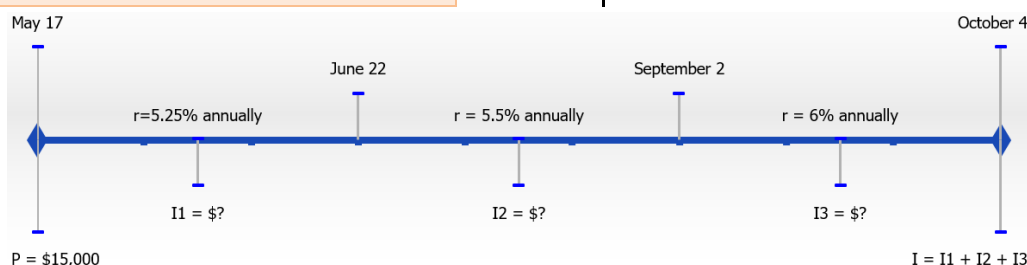
How You Will Get There

Step 1 (continued): Calculate the number of days for each of the three time periods.

Step 2: The rates are annual while each of the times are in days. Convert each of the times to an annual number.

Step 3: For each time period, apply Formula 8.1.

Step 3 (continued): Sum the three interest amounts to arrive at the total interest paid.

**Perform**

Step 1 (continued)			Step 2	Step 3
Dates	Calculations	Total # of Days	Convert t to annual	Calculate interest
May 17 to June 22	May 17 to May 31 = 14 days May 31 to June 22 = 22 days	14 + 22 = 36 days	$\frac{36}{365}$	$I_1 = \$15,000(0.0525)\left(\frac{36}{365}\right)$ $= \$77.671232$
June 22 to September 2	June 22 to June 30 = 8 days June 30 to July 31 = 31 days July 31 to August 31 = 31 days August 31 to Sept 2 = 2 days	8 + 31 + 31 + 2 = 72 days	$\frac{72}{365}$	$I_2 = \$15,000(0.055)\left(\frac{72}{365}\right)$ $= \$162.739726$
September 2 to October 4	Sept 2 to Sept 30 = 28 days Sept 30 to October 4 = 4 days	28 + 4 = 32 days	$\frac{32}{365}$	$I_3 = \$15,000(0.06)\left(\frac{32}{365}\right)$ $= \$78.904109$
Step 3 (continued): TOTAL				\$319.32

Calculator Instructions

Arbitrarily pick the year 2011 and use the DATE function:

Time Fragment	DT1	DT2	DBD	Mode
1	5.1711	6.2211	Answer: 36	ACT
2	6.2211	9.0211	Answer: 72	ACT
3	9.0211	10.0411	Answer: 32	ACT

Present

Julio owes \$319.32 of simple interest on his October 4 repayment date.



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8.2: Moving Money Involving Simple Interest

(Move and Nobody Gets Hurt)

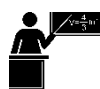
Can you calculate the amount of money required to meet a future goal? Assume you will graduate college with your business administration diploma in a few months and have already registered at your local university to continue with your studies toward a bachelor of commerce degree. You estimate your total tuition, fees, and textbooks at \$8,000. After investigating some short-term investments, you find the best simple rate of interest obtainable is 4.5%. How much money must you invest today for it to grow with interest to the needed tuition money?

In the previous section, you calculated the amount of interest earned or charged on an investment or loan. While this number is good to know, most of the time investors are interested solely in how much in total, including both principal and interest, is either owed or saved. Also, to calculate the interest amount in Formula 8.1, you must know the principal. When people plan for the future, they know the future amount of money that they need but do not know how much money they must invest today to arrive at that goal. This is the case in the opening example above, so you need a further technique for handling simple interest.

This section explores how to calculate the principal and interest together in a single calculation. It adds the flexibility of figuring out how much money there is at the beginning of the time period so long as you know the value at the end, or vice versa. In this way, you can solve almost any simple interest situation.

Maturity Value (or Future Value)

The **maturity value** of a transaction is the amount of money resulting at the end of a transaction, an amount that includes both the interest and the principal together. It is called a maturity value because in the financial world the termination of a financial transaction is known as the "maturing" of the transaction. The amount of principal with interest at some point in the future, but not necessarily the end of the transaction, is known as the **future value**.



The Formula

For any financial transaction involving simple interest, the following is true:

Amount of money at the end = Amount of money at the beginning + Interest

Applying algebra, you can summarize this expression by the following equation, where the future value or maturity value is commonly denoted by the symbol S :

$$S = P + I$$

Substituting in Formula 8.1, $I = Prt$, yields the equation

$$S = P + Prt$$

Simplify this equation by factoring P on the right-hand side to arrive at Formula 8.2.

S is Future Value or Maturity Value: This is the combined value of the principal and the simple interest together at a future point in time.

P is Present Value or Principal: The amount of money at the beginning of the time period being analyzed. If this is in fact the amount at the start of the financial transaction, it is the principal. Or it can simply be the amount at some earlier time before the future value was known. In any case, the amount excludes the interest.

Formula 8.2 – Simple Interest for Single Payments: **$S = P(1 + rt)$**

r is Interest Rate: The interest rate in Formula 8.2 operates in the same manner as in Formula 8.1. It is the simple rate of interest charged or earned during the specified time period. It is usually expressed annually and must share a common unit with the time period.

t is Time: This is the length of time for the financial transaction in which interest is charged or earned. It must be expressed in a common unit with the interest rate.

Depending on the financial scenario, what information you know, and what variable you need to calculate, you may need a second formula offering an alternative method of calculating the simple interest dollar amount.

I is Interest Amount: The interest amount is the dollar amount of interest earned or charged over the span of a financial transaction involving a single payment.

S is Future Value or Maturity Value: The amount of money in the future (or at maturity) including both the principal (or present value) and the interest together.

Formula 8.3 – Interest Amount for Single Payments: $I = S - P$

P is Present Value or Principal: The amount of money at the start of the transaction or at some other time before maturity that excludes the interest involved in the transaction.



How It Works

Follow these steps when working with single payments involving simple interest:

Step 1: Formula 8.2 has four variables, any three of which you must identify to work with a single payment involving simple interest.

- If either the P or S is missing but the I is known, apply Formula 8.3 to determine the unknown value.
- If only the interest amount I is known and neither P nor S is identified, recall that the interest amount is a portion and that the interest rate is a product of rate and time (e.g., 6% annually for nine months is an earned interest rate of 4.5%). Apply Formula 2.2 on rate, portion, and base to solve for base, which is the present value or P.
- If necessary, draw a timeline to illustrate how the money is being moved through time.

Step 2: Ensure that the simple interest rate and the time period are expressed with a common unit. If not, you need to convert one of the two variables to achieve the commonality.

Step 3: Apply Formula 8.2 and solve for the unknown variable, manipulating the formula as required.

Step 4: If you need to calculate the amount of interest, apply Formula 8.3.

Assume that today you have \$10,000 that you are going to invest at 7% simple interest for 11 months. How much money will you have in total at the end of the 11 months? How much interest do you earn?

Step 1: The principal is $P = \$10,000$, the simple interest rate is 7% per year, or $r = 0.07$, and the time is $t = 11$ months.

Step 2: The time is in months, but to match the rate it needs to be expressed annually as $t = \frac{11}{12}$.

Step 3: Applying Formula 8.2 to calculate the future value including interest,

$S = \$10,000 \times (1 + 0.07 \times \frac{11}{12}) = \$10,641.67$. This is the total amount after 11 months.

Step 4: Applying Formula 8.3 to calculate the interest earned, $I = \$10,641.67 - \$10,000.00 = \$641.67$. You could also apply Formula 8.1 to calculate the interest amount; however, Formula 8.3 is a lot easier to use. The \$10,000 earns \$641.67 in simple interest over the next 11 months, resulting in \$10,641.67 altogether.



Things To Watch Out For

As with Formula 8.1, the most common mistake in the application of Formula 8.2 is failing to ensure that the rate and time are expressed in the same units. Before you proceed, always check these two variables!

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Paths To Success

When solving Formula 8.2 for either rate or time, it is generally easier to use Formulas 8.3 and 8.1 instead. Follow these steps to solve for rate or time:

1. If you have been provided with both the present value and future value, apply Formula 8.3 to calculate the amount of interest (I).
2. Apply and rearrange Formula 8.1 to solve for either rate or time as required.



Give It Some Thought

In each of the following situations, determine which statement is correct.

1. If you have a future value of \$5,000 and interest is involved, then your present value is
 - a. More than \$5,000
 - b. The same
 - c. Less than \$5,000
2. If you had a principal of \$10,000 earning 10% simple interest for any period less than a year, you earn
 - a. More than \$1,000 interest
 - b. Exactly \$1,000 interest
 - c. Less than \$1,000 interest
3. If you took out a loan for \$2,500 at a 5% simple interest rate for six months and you made no payments during that time frame, you owe
 - a. More than \$2,500
 - b. Exactly \$2,500
 - c. Less than \$2,500

Solutions:

1. Less than \$5,000. When being charged or earning interest, the present value is always less than the future value.
2. Less than \$1,000 interest. The \$10,000 earns exactly \$1,000 interest per year, so any time frame less than a year produces a smaller interest amount.
3. More than \$2,500. When interest is being charged, the future value is always larger than the present value.

Example 8.2A: Saving for a Down Payment on a Home

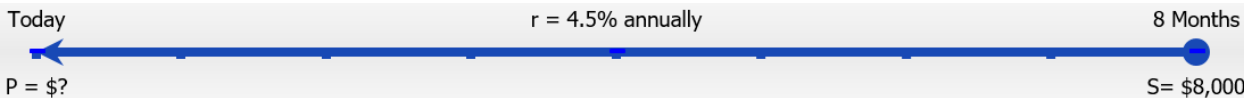
You just inherited \$35,000 from your uncle's estate and plan to purchase a house four months from today. If you use your inheritance as your down payment on the house, how much will you be able to put down if your money earns $4\frac{1}{4}\%$ simple interest? How much interest will you have earned?

Plan	Calculate the amount of money four months from now including both the principal and interest earned. This is the maturity value (S). Also calculate the interest earned (I).	
Understand	What You Already Know Step 1: The principal, interest rate, and time frame are known: $P = \$35,000$ $r = 4\frac{1}{4}\%$ per year $t = 4$ months	How You Will Get There Step 2: While the rate is annual, the time is in months. Convert the time to an annual number. Step 3: Apply Formula 8.2. Step 4: Apply Formula 8.3 to calculate the interest amount.
Perform	Step 2: Four months out of 12 months in a year is $t = \frac{4}{12}$ Step 3: $S = \\$35,000 \times (1 + 4\frac{1}{4}\% \times \frac{4}{12}) = \\$35,000 \times (1 + 0.0425 \times 0.\bar{3}) = \\$35,495.83$ Step 4: $I = \\$35,495.83 - \\$35,000.00 = \\$495.83$	

Present	Four months from now you will have \$35,495.83 as a down payment toward your house, which includes \$35,000 in principal and \$495.83 of interest.

Example 8.2B: Saving for Tuition

Recall the section opener, where you needed \$8,000 for tuition in the fall and the best simple interest rate you could find was 4.5%. Assume you have eight months before you need to pay your tuition. How much money do you need to invest today?

Plan	Calculate the principal amount of money today (P) that you must invest such that it will earn interest and end up at the \$8,000 required for the tuition.	
Understand	What You Already Know Step 1: The future value, interest rate, and time period are known: $S = \$8,000$ $r = 4.5\%$ per year $t = 8$ months 	How You Will Get There Step 2: While the rate is annual, the time is in months. Convert the time to an annual number. Step 3: Apply Formula 8.2, rearranging for P .
Perform	Step 2: Eight months out of 12 months in a year is $t = \frac{8}{12}$ Step 3: $\$8,000 = P(1 + 4.5\% \times \frac{8}{12})$ $P = \frac{\$8,000}{(1 + 0.045 \times 0.\bar{6})} = \$7,766.99$	
Present	If you place \$7,766.99 into the investment, it will grow to \$8,000 in the eight months.	

Example 8.2C: What Exactly Are You Being Offered?

You are sitting in an office at your local financial institution on August 4. The bank officer says to you, "We will make you a great deal. If we advance that line of credit and you borrow \$20,000 today, when you want to repay that balance on September 1 you will only have to pay us \$20,168.77, which is not much more!" Before answering, you decide to evaluate the statement. Calculate the simple interest rate that the bank officer used in her calculations.

Plan	Determine the rate of interest that you would be charged on your line of credit, or r .	
Understand	What You Already Know Step 1: The principal, maturity amount, and time frame are known: $P = \$20,000.00$ $S = \$20,168.77$ $t = \text{August 4 to September 1}$	How You Will Get There Step 1 (continued): Calculate the number of days in the transaction. Step 2: Since interest rates are usually expressed annually, convert the time from days to an annual number. Step 3: Use the combination of Formulas 8.3 and 8.1. First calculate the interest amount using Formula 8.3. Step 3 (continued): Then apply Formula 8.1, rearranging for r . 

Perform	Step 1 (continued):	Calculator Instructions			
		Assume the year 2011 and use the DATE function:			
		DT1	DT2	DBD	Mode
		08.0411	09.0111	Answer: 28	ACT
Present	The interest rate on the offered line of credit is 11.0002% (note that it is probably exactly 11%; the extra 0.0002% is most likely due to the rounded amount of interest used in the calculation).				

Equivalent Payments

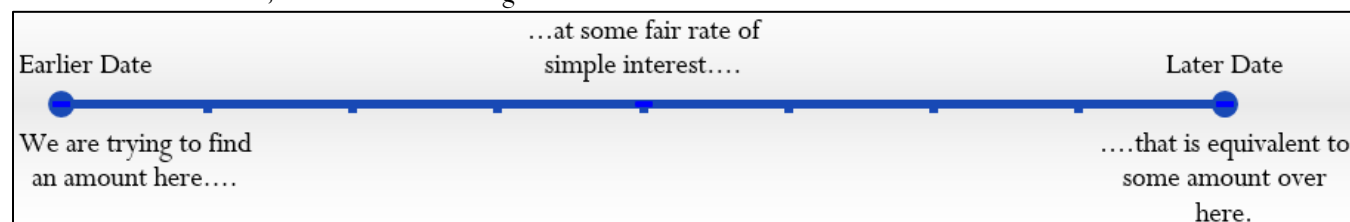
Life happens. Sometimes the best laid financial plans go unfulfilled. Perhaps you have lost your job. Maybe a reckless driver totalled your vehicle, which you now have to replace at an expense you must struggle to fit into your budget. Maybe a global pandemic struck. No matter the reason, you find yourself unable to make your debt payment as promised.

On the positive side, maybe you just received a large inheritance unexpectedly. What if you bought a scratch ticket and just won \$25,000? Now that you have the money, you might want to pay off that debt early. Can it be done?

Whether paying late or paying early, any amount paid must be equivalent to the original financial obligation. As you have learned, when you move money into the future it accumulates simple interest. When you move money into the past, simple interest must be removed from the money. This principle applies both to early and late payments:

- **Late Payments.** If a debt is paid late, then a financial penalty that is fair to both parties involved should be imposed. That penalty should reflect a current rate of interest and be added to the original payment. Assume you owe \$100 to your friend and that a fair current rate of simple interest is 10%. If you pay this debt one year late, then a 10% late interest penalty of \$10 should be added, making your debt payment \$110. This is no different from your friend receiving the \$100 today and investing it himself at 10% interest so that it accumulates to \$110 in one year.
- **Early Payments.** If a debt is paid early, there should be some financial incentive (otherwise, why bother?). Therefore, an interest benefit, one reflecting a current rate of interest on the early payment, should be deducted from the original payment. Assume you owe your friend \$110 one year from now and that a fair current rate of simple interest is 10%. If you pay this debt today, then a 10% early interest benefit of \$10 should be deducted, making your debt payment today \$100. If your friend then invests this money at 10% simple interest, one year from now he will have the \$110, which is what you were supposed to pay.

Notice in these examples that a simple interest rate of 10% means \$100 today is the same thing as having \$110 one year from now. This illustrates the concept that two payments are **equivalent payments** if, once a fair rate of interest is factored in, they have the same value on the same day. Thus, in general you are finding two amounts at different points in time that have the same value, as illustrated in the figure below.





How It Works

The steps required to calculate an equivalent payment are no different from those for single payments. If an early payment is being made, then you know the future value, so you solve for the present value (which removes the interest). If a late payment is being made, then you know the present value, so you solve for the future value (which adds the interest penalty).

Ultimately, you need to pick a **focal date** which is the date to which all values in the financial situation are moved in order to determine equivalency. In simple interest, it is important to note that:

- 1) Simple interest amounts can only be added together when all money is on the same date – the focal date.
- 2) Simple interest amounts can only be moved once when attempting to calculate equivalent values. You cannot take a simple interest amount and move it to one date, add/subtract interest, then move it to another date. If you do so, then you have in fact compounded interest.

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Being financially smart means paying attention to when you make your debt payments. If you receive no financial benefit for making an early payment, then why make it? The prudent choice is to keep the money yourself, invest it at the best interest rate possible, and pay the debt off when it comes due. Whatever interest is earned is yours to keep and you still fulfill your debt obligations in a timely manner!

Example 8.2D: Making a Late Payment

Erin owes Charlotte \$1,500 today. Unfortunately, Erin had some unexpected expenses and is unable to make her debt payment. After discussing the issue, they agree that Erin can make the payment nine months late and that a fair simple interest rate on the late payment is 5%. Use 9 months from now as your focal date and calculate how much Erin needs to pay. What is the amount of her late penalty?

Plan	A late payment is a future value amount (S). The late penalty is equal to the interest (I).	
Understand	What You Already Know Step 1: The payment today, the agreed-upon interest rate, and the late payment timing are known: $P = \\$1,500$ $r = 5\%$ annually $t = 9$ months (the focal date)	How You Will Get There Step 2: While the rate is annual, the time is in months. Convert the time to an annual number. Step 3: Apply Formula 8.2. Step 4: Apply Formula 8.3 to calculate the late interest penalty amount.
Perform	Step 2: Nine months out of 12 months in a year is $t = \frac{9}{12}$. Step 3: $S = \\$1,500 \times (1 + 5\% \times \frac{9}{12}) = \\$1,500 \times (1 + 0.05 \times 0.75) = \\$1,556.25$ Step 4: $I = \\$1,556.25 - \\$1,500.00 = \\$56.25$	
Present	Erin's late payment is for \$1,556.25, which includes a \$56.25 interest penalty for making the payment nine months late.	

Example 8.2E: Making an Early Payment

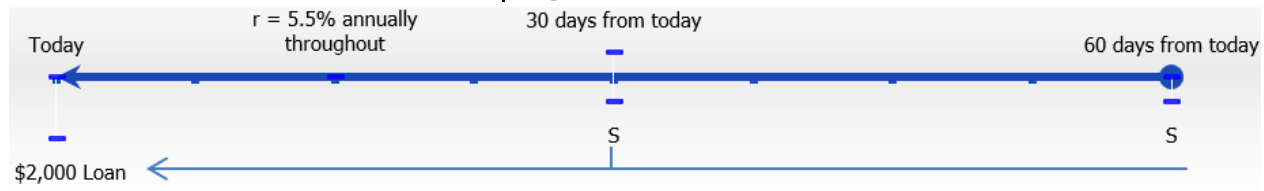
Rupert owes Aminata two debt payments: \$600 four months from now and \$475 eleven months from now. Rupert came into some money today and would like to pay off both of the debts immediately. Aminata has agreed that a fair interest rate is 7%. Using today as a focal date, what amount should Rupert pay? What is the total amount of his early payment benefit?

Plan	An early payment is a present value amount (P). Both payments will be moved to today (the focal date) and summed. The early payment benefit will be the total amount of interest removed (I).	
Understand	What You Already Know Step 1: The two payments, interest rate, and payment due dates are known: $r = 7\%$ annually First Payment: $S = \\$600$ $t = 4$ months from now Second Payment: $S = \\$475$ $t = 11$ months from now Replacement payment is being made today (the focal date).	How You Will Get There Do steps 2 and 3 for each payment: Step 2: While the rate is annual, the time is in months. Convert the time to an annual number. Step 3: Apply Formula 8.2, rearranging it to solve for P. Once you calculate both P, they can be summed to arrive at the total payment. Step 4: For each payment, apply Formula 8.3 to calculate the associated early interest benefit. Total the two values of I to get the early interest benefit overall.
Perform	Payment #1: Step 2: Four months out of 12 months in a year is $t = \frac{4}{12}$ Step 3: $\\$600 = P \left(1 + 7\% \times \frac{4}{12}\right)$ $P = \frac{\$600}{(1 + 0.07 \times 0.3)} = \$586.32$ Payment #2: Step 2: Eleven months out of 12 months in a year is $t = \frac{11}{12}$ Step 3: $\\$475 = P \left(1 + 7\% \times \frac{11}{12}\right)$ $P = \frac{\$475}{(1 + 0.07 \times 0.91\bar{6})} = \$446.36$	Totals: P today = $\\$586.32 + \\$446.36 = \\$1,032.68$ Step 4: Payment #1: $I = S - P = \\$600 - \\$586.32 = \\$13.68$ Payment #2: $I = S - P = \\$475 - \\$446.36 = \\$28.64$ $I = \\$13.68 + \\$28.64 = \\$42.32$
Present	To clear both debts today, Rupert pays \$1,032.68, which reflects a \$42.32 interest benefit reduction for the early payment.	

Example 8.2F: Making Loan Payments

A \$2,000 loan at 5.5% is repaid by two equal payments made 30 days and 60 days after the start of the loan. Determine the amount of each payment.

Plan	A focal date of today will be used for the loan. Both payments (S), when brought back to the start of the loan with all interest removed, will need to re-pay the money borrowed (P).

Understand	What You Already Know Step 1: The loan amount, interest rate, and payment dates are known: $P = \\$2,000$ (today – the focal date) $r = 5.5\%$ annually $t_1 = 30$ days from now $t_2 = 60$ days from now	How You Will Get There Do steps 2 and 3 for each payment: Step 2: While the rate is annual, the time is in days. Convert the time to an annual number. Step 3: Apply Formula 8.2, rearranging it to solve for P. Once you calculate both P, they can be equated to the original loan amount and solved. Step 4: Not needed. 
Perform	Payment #1: Step 2: 30 days out of 365 days in a year is $t = \frac{30}{365}$ Step 3: $S = P \left(1 + 5.5\% \times \frac{30}{365}\right)$ $S = 1.004520P$ therefore $P = 0.995499S$ Payment #2: Step 2: 60 days out of 365 days in a year is $t = \frac{60}{365}$ Step 3: $S = P \left(1 + 5.5\% \times \frac{60}{365}\right)$ $S = 1.009041P$ therefore $P = 0.991039S$	Totals: $P \text{ today} = 1^{\text{st}} \text{ payment today} + 2^{\text{nd}} \text{ payment today}$ $\\$2,000 = 0.995499S + 0.991039S$ $\\$2,000 = 1.986539S$ $S = \\$1,006.78$
Present	The two equal payments that will re-pay the loan are each \$1,006.78, occurring 30 days and 60 days respectively after the start of the loan.	



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8.3: Application: Savings Accounts and Short-Term GICs

(Safe and Secure)

When the stock market offers potentially huge returns, why would anyone put money into an investment that earns just a few percent annually? The answer of course is that sometimes stocks decrease in value. For example, after reaching a historic high in June 2008, the Toronto Stock Exchange plunged 40% over the course of the next six months. So if you are not willing to take the risk in the stock market, there are many options for investing your money that are safe and secure, particularly savings accounts and short-term guaranteed investment certificates (GICs). Some of the benefits and drawbacks associated with safe and secure investments are listed in the table below.

Benefits	Drawbacks
- Investments are virtually 100% safe, meaning that the chances of losing your investment are almost zero. - They always have a positive rate of interest. - It is easy to access the money when the investor needs it. - Any interest earned is better than zero interest. - It is suitable for small amounts.	- Interest rates are extremely low and offer little growth for your money. - Many come with stiff fees and interest rate penalties if certain conditions are not met. - Minimum balances are a common requirement, otherwise the interest rate is zero. - Balances are only insured up to \$100,000.

In this section, you are exposed to both savings accounts and short-term GICs. You will learn a little about the characteristics of each along with how to perform simple interest calculations on these financial tools.

Savings Accounts

A **savings account** is a deposit account that bears interest and has no stated maturity date. These accounts are found at most financial institutions, such as commercial banks (Royal Bank of Canada, TD Canada Trust, etc.), trusts (Royal Trust, Laurentian Trust, etc.), and credit unions (FirstOntario, Steinbach, Assiniboine, Servus, etc.). Owners of such accounts make deposits to and withdrawals from these accounts at any time, usually accessing the account at an automatic teller machine (ATM), at a bank teller, or through online banking.

A wide variety of types of savings accounts are available. This textbook focuses on the most common features of most savings accounts, including how interest is calculated, when interest is deposited, insurance against loss, and the interest rate amounts available.

1. **How Interest Is Calculated.** There are two common methods for calculating simple interest:
 1. Accounts earn simple interest that is calculated based on the daily closing balance of the account. The closing balance is the amount of money in the account at the end of the day. Therefore, any balances in the account throughout a single day do not matter. For example, if you start the day with \$500 in the account and deposit \$3,000 at 9:00 a.m., then withdraw the \$3,000 at 4:00 p.m., your closing balance is \$500. That is the principal on which interest is calculated, not the \$3,500 in the account throughout the day.
 2. Accounts earn simple interest based on a minimum monthly balance in the account. For example, if in a single month you had a balance in the account of \$900 except for one day, when the balance was \$500, then only the \$500 is used in calculating the entire month's worth of interest.
2. **When Interest Is Deposited.** Interest is accumulated and deposited (paid) to the account once monthly, usually on the first day of the month. Thus, the interest earned on your account for the month of January appears as a deposit on February 1.
3. **Insurance against Loss.** Canadian savings accounts at commercial banks are insured by the national Canada Deposit Insurance Corporation (CDIC), which guarantees up to \$100,000 in savings. At credit unions, this insurance is usually provided provincially by institutions such as the Deposit Insurance Corporation of Ontario (DICO), which also guarantees up to \$100,000. This means that if your bank were to fold, you could not lose your money (so long as your deposit was within the maximum limit). Therefore, savings accounts carry almost no risk.

4. **Interest Rate Amounts.** Interest rates are higher for investments that are riskier. Savings accounts carry virtually no risk, which means the interest rates on savings accounts tend to be among the lowest you can earn. At the time of writing, interest rates on savings accounts ranged from a low of 0.05% to a high of 1.95%. Though this is not much, it is better than nothing and certainly better than losing money!

While a wide range of savings accounts are available, these accounts generally follow one of two common structures when it comes to calculating interest. These structures are flat rate savings accounts and tiered savings accounts. Each of these is discussed separately.



How It Works

Flat-Rate Savings Accounts. A flat-rate savings account has a single interest rate that applies to the entire balance. The interest rate may fluctuate in synch with short-term interest rates in the financial markets.

Follow these steps to calculate the monthly interest for a flat-rate savings account:

Step 1: Identify the interest rate, opening balance, and the monthly transactions in the savings account.

Step 2: Set up a flat-rate table as illustrated here. Create a number of rows equalling the number of monthly transactions (deposits or withdrawals) in the account plus one.

Dates	Closing Balance in Account	# of Days	Simple Interest Earned
			$I = Prt$
Total Monthly Interest Earned			

Step 3: For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.

Step 4: Calculate the number of days that the closing balance is maintained for each row.

Step 5: Apply Formula 8.1, $I = Prt$, to each row in the table. Ensure that rate and time are expressed in the same units. Do not round off the resulting interest amounts (I).

Step 6: Sum the Simple Interest Earned column and round off to two decimals.

When you are calculating interest on any type of savings account, pay careful attention to the details on how interest is calculated and any restrictions or conditions on the balance that is eligible to earn the interest.

Example 8.3A: Savings at the Royal Bank

The RBC High Interest Savings Account pays 0.75% simple interest on the daily closing balance in the account and the interest is paid on the first day of the following month. On March 1, the opening balance in the account was \$2,400. On March 12, a deposit of \$1,600 was made. On March 21, a withdrawal of \$2,000 was made. Calculate the total simple interest earned for the month of March.

Plan	Calculate the total interest amount for the month (I).	
Understand	What You Already Know Step 1: The following transactions, dates, and interest rate are known: $r = 0.75\%$ per year March 1 opening balance = \$2,400 March 12 deposit = \$1,600 March 21 withdrawal = \$2,000	How You Will Get There Step 2: Set up a flat-rate table. Step 3: Determine the date ranges for each balance throughout the month and calculate the closing balances. Step 4: For each row of the table, calculate the number of days involved. Step 5: Apply Formula 8.1 to calculate simple interest on each row. Step 6: Sum the Simple Interest Earned column.

Perform	Steps 2 and 3		Step 4	Step 5
	Dates	Closing Balance in Account	# of Days	Simple Interest Earned ($I = Prt$)
	March 1 to March 12	\$2,400	$12 - 1 = 11$	$I = \$2,400(0.0075)(\frac{11}{365})$ $I = \$0.542465$
	March 12 to March 21	$\$2,400 + \$1,600 = \$4,000$	$21 - 12 = 9$	$I = \$4,000(0.0075)(\frac{9}{365})$ $I = \$0.739726$
	March 21 to April 1	$\$4,000 - \$2,000 = \$2,000$	$31 + 1 - 21 = 11$	$I = \$2,000(0.0075)(\frac{11}{365})$ $I = \$0.452054$
Present	Step 6: Total Monthly Interest Earned			$I = \$0.542465 + \$0.739726 + \$0.452054$ $I = \$1.73$
	For the month of March, the savings account earned a total simple interest of \$1.73, which was deposited to the account on April 1.			



How It Works

Tiered Savings Accounts. A tiered savings account pays higher rates of interest on higher balances in the account. This is very much like a graduated commission on gross earnings, as discussed in Section 4.1. For example, you might earn 0.25% interest on the first \$1,000 in your account and 0.35% for balances over \$1,000. Most of these tiered savings accounts use a portioning system. This means that if the account has \$2,500, the first \$1,000 earns the 0.25% interest rate and it is only the portion above the first \$1,000 (hence, \$1,500) that earns the higher interest rate.

Follow these steps to calculate the monthly interest for a tiered savings account:

Step 1: Identify the interest rate, opening balance, and the monthly transactions in the savings account.

Step 2: Set up a tiered interest rate table as illustrated below. Create a number of rows equalling the number of monthly transactions (deposits or withdrawals) in the account plus one. Adjust the number of columns to suit the number of tiered rates. Fill in the headers for each tiered rate with the balance requirements and interest rate for which the balance is eligible.

Dates	Closing Balance in Account	# of Days	Tier Rate #1 Balance Requirements and Interest Rate	Tier Rate #2 Balance Requirements and Interest Rate	Tier Rate #3 Balance Requirements and Interest Rate
			Eligible P = $I = Prt$	Eligible P = $I = Prt$	Eligible P = $I = Prt$
Total Monthly Interest Earned					

Step 3: For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.

Step 4: For each row, calculate the number of days that the closing balance is maintained.

Step 5: Assign the closing balance to the different tiers, paying attention to whether portioning is being used. In each cell with a balance, apply Formula 8.1, where $I = Prt$. Ensure that rate and time are expressed in the same units. Do not round off the resulting interest amounts (I).

Step 6: To calculate the Total Monthly Interest Earned, sum all interest earned amounts from all tier columns and round off to two decimals.

Example 8.3B: A Rate Builder Tiered Account

The Rate Builder savings account at your local credit union pays simple interest on the daily closing balance as indicated in the table below:

Balance	Interest Rate
\$0.00 to \$500.00	0% on entire balance
\$500.01 to \$2,500.00	0.5% on entire balance
\$2,500.01 to \$5,000.00	0.95% on this portion of balance only
\$5,000.01 and up	1.35% on this portion of balance only

In the month of August, the opening balance on an account was \$2,150.00. Deposits were made to the account on August 5 and August 15 in the amounts of \$3,850.00 and \$3,500.00. Withdrawals were made from the account on August 12 and August 29 in the amounts of \$5,750.00 and \$3,000.00. Calculate the simple interest earned for the month of August.

Plan	Calculate the total interest amount for the month of August (<i>I</i>).						
Understand	What You Already Know			How You Will Get There			
	Step 1: The interest rate structure is in the table above. The transactions and dates are also known: August 1 opening balance = \$2,150.00 August 5 deposit = \$3,850.00 August 12 withdrawal = \$5,750.00 August 15 deposit = \$3,500.00 August 29 withdrawal = \$3,000.00			Step 2: Set up a tiered interest rate table with four columns for the tiered rates. Step 3: Determine the date ranges for each balance throughout the month and calculate the closing balances. Step 4: Calculate the number of days involved on each row of the table. Step 5: Assign the closing balance to each tier accordingly. Apply Formula 8.1 to any cell containing a balance. Step 6: Total up all of the interest from all cells of the table.			
Perform	Steps 2 and 3		Step 4		Step 5		
	Dates	Closing Balance in Account	# of Days	0% \$0 to \$500 Entire Balance	0.5% \$500.01 to \$2,500 Entire Balance	0.95% \$2,500.01 to \$5,000 This portion only	1.35% \$5,000.01 and up This portion only
	August 1 to August 5	\$2,150.00	5 − 1 = 4		P = \$2,150.00 $I = \$2,150(0.005)(\frac{4}{365})$ $I = \$0.117808$		
	August 5 to August 12	\$2,150.00 + \$3,850.00 = \$6,000.00	12 − 5 = 7		P = \$2,500.00 $I = \$2,500(0.005)(\frac{7}{365})$ $I = \$0.239726$	P = \$2,500.00 $I = \$2,500(0.0095)(\frac{7}{365})$ $I = \$0.455479$	P = \$1,000.00 $I = \$1,000(0.0135)(\frac{7}{365})$ $I = \$0.258904$
	August 12 to August 15	\$6,000.00 − \$5,750.00 = \$250.00	15 − 12 = 3	P = \$250.00 $I = \$0.00$			
	August 15 to August 29	\$250.00 + \$3,500.00 = \$3,750.00	29 − 15 = 14		P = \$2,500.00 $I = \$2,500(0.005)(\frac{14}{365})$ $I = \$0.479452$	P = \$1,250.00 $I = \$1,250(0.0095)(\frac{14}{365})$ $I = \$0.455479$	
	August 29 to September 1	\$3,750.00 − \$3,000.00 = \$750.00	31 + 1 − 29 = 3		P = \$750.00 $I = \$750(0.005)(\frac{3}{365})$ $I = \$0.030821$		
	Step 6: Total Monthly Interest Earned			$I = \$0.117808 + \$0.239726 + \$0.455479 + \$0.258904 + \$0.00 + \$0.479452 + \$0.455479 + \$0.030821$ $I = \$2.04$			
Present	For the month of August, the tiered savings account earned a total simple interest of \$2.04, which was deposited to the account on September 1.						

Short-Term Guaranteed Investment Certificates (GICs)

A **guaranteed investment certificate (GIC)** is an investment that offers a guaranteed rate of interest over a fixed period of time. In many countries around the world, a GIC is often called a time or term deposit. GICs are found mostly at commercial banks, trust companies, and credit unions. Just like savings accounts, they have a very low risk profile and tend to

have much lower rates than are available through other investments such as the stock market or bonds. They are also unconditionally guaranteed, much like savings accounts. In this section, you will deal only with short-term GICs, defined as those that have a time frame of less than one year.

The table below summarizes three factors that determine the interest rate on a short-term GIC: principal, time, and redemption privileges.

	Higher Interest Rates	Lower Interest Rates
Principal Amount	Large	Small
Time	Longer	Shorter
Redemption Privileges	Nonredeemable	Redeemable

1. **Amount of Principal.** Typically, a larger principal is able to realize a higher interest rate than a smaller principal.
2. **Time.** The length of time that the principal is invested affects the interest rate. Short-term GICs range from 30 days to 364 days in length. A longer term usually realizes higher interest rates.
3. **Redemption Privileges.** The two types of GICs are known as redeemable and nonredeemable. A redeemable GIC can be cashed in at any point before the maturity date, meaning that you can access your money any time you want it. A nonredeemable GIC "locks in" your money for the agreed-upon term. Accessing that money before the end of the term usually incurs a stiff financial penalty, either on the interest rate or in the form of a financial fee. Nonredeemable GICs carry a higher interest rate.

To summarize, if you want to receive the most interest it is best to invest a large sum for a long time in a nonredeemable short-term GIC.



How It Works

Short-term GICs involve a lump sum of money (the principal) invested for a fixed term (the time) at a guaranteed interest rate (the rate). Most commonly the only items of concern are the amount of interest earned and the maturity value. Therefore, you need the same four steps as for single payments involving simple interest shown in Section 8.2.

Example 8.3C: GIC Choices

Your parents have \$10,000 to invest. They can either deposit the money into a 364-day nonredeemable GIC at Assiniboine Credit Union with a posted rate of 0.75%, or they could put their money into back-to-back 182-day nonredeemable GICs with a posted rate of 0.7%. At the end of the first 182 days, they will reinvest both the principal and interest into the second GIC. The interest rate remains unchanged on the second GIC. Which option should they choose?

Plan	For both options, calculate the future value (S), of the investment after 364 days. The one with the higher future value is your parents' better option.	
Understand	What You Already Know Step 1: For the first GIC investment option: $P = \$10,000$ $r = 0.75\%$ per year $t = 364$ days For the second GIC investment option: Initial $P = \$10,000$ $r = 0.7\%$ per year $t = 182$ days each	How You Will Get There Step 2: The rate is annual, the time is in days. Convert the time to an annual number. Step 3 (1st GIC option): Calculate the maturity value (S_1) of the first GIC option after its 364-day term by applying Formula 8.2. Step 3 (2nd GIC option, 1st GIC): Calculate the maturity value (S_2) after the first 182-day term by applying Formula 8.2. Step 3 (2nd GIC option, 2nd GIC): Reinvest the first maturity value as principal for another term of 182 days applying Formula 8.2 and calculate the final future value (S_3).
Perform	Step 2: Transforming both time variables, $t = \frac{364}{365}$ and $t = \frac{182}{365}$ Step 3 (1st GIC option): $S_1 = \$10,000(1 + (0.0075)(\frac{364}{365})) = \$10,074.79$ Step 3 (2nd GIC option, 1st GIC): $S_2 = \$10,000(1 + (0.007)(\frac{182}{365})) = \$10,034.90$ Step 3 (2nd GIC option, 2nd GIC): $S_3 = \$10,034.90(1 + (0.007)(\frac{182}{365})) = \$10,069.93$	

Present

The 364-day GIC results in a maturity value of \$10,074.79, while the two back-to-back 182-day GICs result in a maturity value of \$10,069.93. Clearly, the 364-day GIC is the better option as it will earn \$4.86 more in simple interest.



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8.4: Application: Promissory Notes

(A Promise Is a Promise)

Have you ever lent a friend or family member some money on a promise they would pay you back on a specific date? Then you are already experienced with promissory notes!

Just before class a fellow student and close friend announces that the two of you should go for coffee afterwards. He seems a little tense, but he says there is no trouble, so you contain your curiosity until you are seated in a quiet corner of your local Starbucks. Then he asks if he can borrow \$5,000 from you to pursue a new entrepreneurial venture, explaining that he already tried to get the money from a bank but too much red tape was involved and the interest rate was outrageous. You have the money saved up in your account. After your friend lays out all of the details, you feel satisfied that he has a good idea. Before lending your friend the money, though, it might be worthwhile to put this down in writing just in case matters turn sour, right? And by loaning your friend the money, you would be forgoing at least the interest on your savings and feel that it is fair to charge your friend the same interest rate so that you are not hurt financially. A promissory note seems a good path to take.

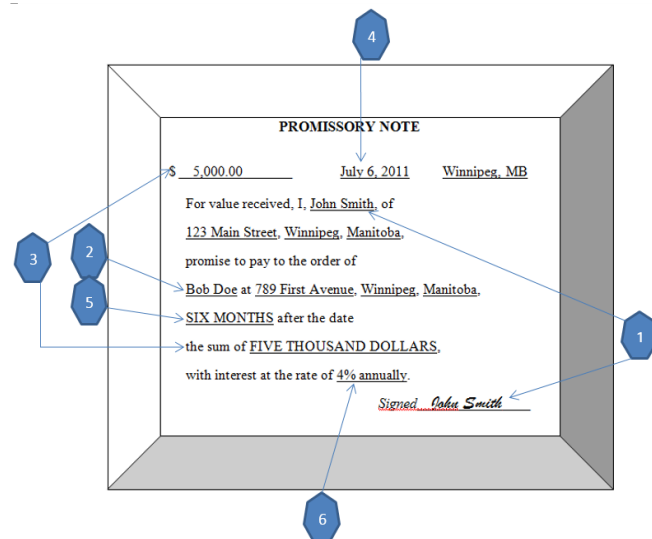
This section explains the characteristics of short-term promissory notes. You will learn how to calculate the maturity value of a promissory note along with its selling price if you want to cash it in before it matures.

What Are Promissory Notes?

Promissory notes are borrowing tools that allow both companies and people to get financing from a source other than a bank. The money is borrowed from other people or companies who are willing to provide the financing. The Bills of Exchange Act defines a **promissory note** as "an unconditional promise in writing made by one person to another person, signed by the maker, engaging to pay, on demand or at a fixed or determinable future time, a sum certain in money to, or to the order of, a specified person or to bearer."¹ Promissory notes are commonly referred to just as *notes*.

A promissory note is not the same as an IOU (I owe you). An IOU simply acknowledges a debt but contains no promise of repayment, no timelines, and no consequences. A promissory note includes these elements.

In the opening scenario, it is probably best to create a promissory note that clearly details your agreement and terms. This allows both of you to have peace of mind in the transaction, and in the event something goes wrong both of you have a legally binding written record of your transaction.



The illustration here provides an example of a promissory note. Observe the following characteristics:

1. **Borrower.** This is the individual who is borrowing the money, sometimes known as the maker, promisor, or payor. It is best to provide details such as the name and address of the borrower to clearly establish an identity.
2. **Lender.** This is the individual who is loaning the money (and should be repaid), sometimes known as the payee or issuer. As with the borrower, detailed name and address information clearly establish the identity of this person.
3. **Face Value.** This is the borrowed principal. It appears numerically and in writing to prevent it from being altered. In simple interest calculations, this is the P.
4. **Issue Date (Start Date).** This is the date on which the promissory note is issued and forms the starting date of the transaction.
5. **Term.** This is the length of time for which the face value is borrowed from the lender. In simple interest calculations, this is the basis for the time period (t).

¹ Bills of Exchange Act, R.S.C., 1985, c. B-4, Part IV Promissory Notes, §176. Available at <http://laws-lois.justice.gc.ca/eng/acts/B-4/index.html>.

6. **Interest Rate.** This is the percentage interest being charged along with the respective time frame for the interest (monthly, annually, etc.). In simple interest calculations, this is the rate (r).

Note in the illustration that the lender has decided to not have any additional consequence for failing to pay back the \$5,000 after the six months. As written, any late payment continues to be charged 4% annual interest only.

Promissory notes are issued outside the normal banking system, so it is up to the lender to determine the interest rate. This results in two types of promissory notes:

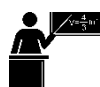
1. **Noninterest-Bearing Notes.** In these cases, the lender has decided not to charge any interest, hence making the interest rate 0%. Alternatively, the lender and borrower may decide to include an amount of interest in the face value specified on the note. For example, the lender may borrow \$5,000, but the note is written for \$5,100 with no interest. In the case of a noninterest-bearing note, of course, no calculation of interest on the promissory note is needed to determine the amount that is due, as the face value equals the maturity value.
2. **Interest-Bearing Notes.** In these cases, the lender has decided to charge interest on the face value. This requires you to calculate the simple interest on the face value. Therefore, the maturity value is the face value plus a simple interest amount. The method for this calculation is discussed next.

Maturity Value for Interest-Bearing Notes

Based on all of the elements included in the promissory note, the maturity date or end date of the promissory note must be determined before the borrower and lender can calculate simple interest. This is the date upon which the promissory note needs to be repaid. However, by Canadian law this is not the due date. The legal due date of a note is instead three days after the term specified, allowing the borrower time to repay the note without penalty in the event that the due date falls on a statutory holiday or weekend, when it might not be possible to make the payment. Interest is still charged during those three days, though. The three days' grace period can be waived by the borrower by stating "no grace" on the promissory note.

The following rules are used to determine the due date:

1. **Specific Date or Number of Days.** If the note states a specific maturity date or details the exact number of days, then the due date is three days later than the maturity date.
2. **Time Period in Months.** If the note states a number of months, then it is due on the same date in the month in which it matures plus three days. In the illustration, the note was issued on July 6, 2011. Six months later is January 6, 2012, plus three days gives a legal due date of January 9, 2012. A special situation arises, though, if the note is issued near or at the end of a month. For example, if the note is issued instead on August 31, 2011, six months later is February 31, 2012. Since this date does not exist, the rule is to use the last day of the month in which it matures. Therefore, the note has a maturity date of February 29, 2012, and that plus three days gives a legal due date of March 3, 2012.



The Formula

Once you know the legal due date, calculate the time involved in the transaction using the methods discussed in Section 8.1. You calculate the maturity value using Formula 8.2 from Section 8.2, which is reproduced below and adapted for promissory notes.

S is Maturity Value: The maturity amount is what the borrower must pay to the lender upon the maturity of the note. It includes both the face value and the interest together.

P is Face Value: The face value specified by the promissory note.

Formula 8.2 – Simple Interest for Single Payments: $S = P(1 + rt)$

r is Interest Rate: The interest rate specified by the promissory note. Ensure that the rate is expressed in the same unit as time.

t is Time: The time period used is the amount of time between the issue date and legal due date of the note, which includes the three days of grace. Ensure the time is expressed in the same unit as the rate.



How It Works

You use the same four steps to calculate the maturity value of a single payment as when calculating the maturity value of a promissory note. The only nuance requiring close attention lies in determining the time period.

Working with the previous promissory note illustration, calculate the amount that John Smith owes to Bob Doe along with the amount of interest.

Step 1: Note that this promissory note is interest-bearing because of the interest rate information provided on the note. The face value or $P = \$5,000$. The interest rate or $r = 4\%$ annually. To calculate the time, it is noted above that the legal due date of this promissory note is six months plus three days later, which is January 9, 2012. Using an appropriate date method from Section 8.1, the days between the dates equal 187 days.

Step 2: Expressing the time annually to match the annual rate, $t = 187/365$.

Step 3: Substituting into Formula 8.2, $S = \$5,000 (1 + 0.04(187/365)) = \$5,102.47$. Therefore, John needs to repay \$5,102.47 to Bob on January 9, 2012.

Step 4: The amount of interest on the note is $I = \$5,102.47 - \$5,000 = \$102.47$.

Example 8.4A: Maturity Value of an Interest-Bearing Promissory Note

Corey borrowed \$7,500 from Miller Enterprises on March 31, 2023. An interest-bearing promissory note was created, indicating that the money was to be repaid eight months later at an interest rate of 6% annually. Calculate the maturity value of the promissory note on the legal due date.

Plan	Calculate the maturity value of the promissory note (S) upon the legal due date using the stated interest rate.										
Understand	What You Already Know Step 1: The face value, interest rate, term, and issue date are known: $P = \$7,500$ $r = 6\%$ per year $t = 8$ months Issue date = March 31, 2023	How You Will Get There Step 1 (continued): Calculate the legal due date by determining the end of the term and adding three days. Transform this date into the number of days. Step 2: While the rate is annual, the time is in days. Convert the time to an annual number. Step 3: Apply Formula 8.2.									
Perform	Step 1 (continued): Eight months after the issue date of March 31, 2023, is November 31, 2023. Since November has only 30 days, use the last day of November as the maturity date. Adding three days' grace, the legal due date is December 3, 2023. Therefore, the timeframe is March 31, 2023, to December 3, 2023. By counting the days in between ($1 + 30 + 31 + 30 + 31 + 31 + 30 + 31 + 30 + 2$) or using a calculator, you find that there are 247 days between these dates. Step 2: 247 days out of 365 days in a year is $t = \frac{247}{365}$ Step 3: $S = \$7,500(1 + 0.06(\frac{247}{365})) = \$7,804.52$	Calculator Instructions For the days between dates in step 1, use DATE function: <table border="1"> <thead> <tr> <th>DT1</th><th>DT2</th><th>DBD</th><th>Mode</th></tr> </thead> <tbody> <tr> <td>03.3123</td><td>12.0323</td><td>Answer: 247</td><td>ACT</td></tr> </tbody> </table>		DT1	DT2	DBD	Mode	03.3123	12.0323	Answer: 247	ACT
DT1	DT2	DBD	Mode								
03.3123	12.0323	Answer: 247	ACT								
Present	Corey owes Miller Enterprises \$7,804.52 upon the legal due date for the promissory note.										

Selling a Promissory Note Before It Matures

Two months ago you established a promissory note to lend your friend the \$5,000 to help him start his entrepreneurial venture. Now you have unexpectedly lost your job and find yourself in need of the money you loaned out. However, the note does not require your friend to repay the debt for another four months. Is there any way you can access your money?

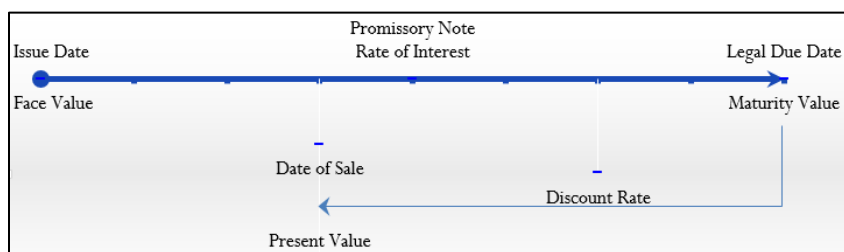
During the life of a promissory note, its holder (the lender) may be unable to wait until the end of the term to get paid back and so sell the note to another investor. It is generally up to both parties to negotiate a fair interest rate for the sale of the note. This interest rate might be based on prevailing interest rates in the market at the time of the sale or just a mutually agreed-upon rate. This negotiated interest rate is sometimes called a **discount rate**, since the process to sell the note requires the removal of interest from the maturity value of the promissory note.



How It Works

When selling a promissory note before its maturity date, always follow a three-step procedure:

Step 1: Identify the promissory note information and selling information. For the promissory note, this includes the face value of the promissory note, the agreed-upon interest rate, and the legal due date. For selling the note, the discount rate and the time of the sale must be known. If necessary, draw a timeline to illustrate the transaction as illustrated in the figure below.

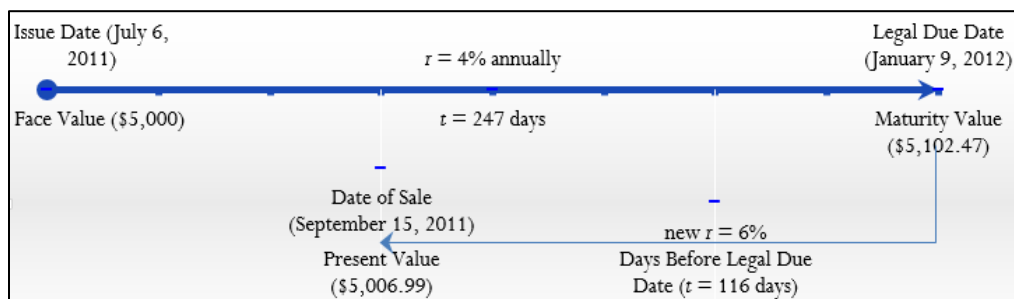


Step 2: Calculate the maturity value of the promissory note itself using the face value, promissory note rate of interest, and the time period involved with the legal due date.

Step 3: Calculate the present value of the note at the date of sale using the maturity value of the note, the new discount rate,

and the time between the date of the sale and the legal due date.

Assume in the example that you negotiated an interest rate of 6% with the buyer of the promissory note, and that the date of sale was September 15, 2011. The timeline illustrates the two steps:



Step 1: For the promissory note, the face value is \$5,000, interest rate is 4% annually, and the time is from July 6, 2011, to January 9, 2012, previously calculated as 187 days. For selling the note, the discount rate is 6%. The selling date is September

15, 2011, which works out to 116 days before the legal due date.

Step 2: Applying Formula 8.2, calculate the maturity value of the promissory note on the legal due date. From previous calculations, the promissory note is worth \$5,102.47 on January 9, 2012, when the borrower repays the note. This is the value that an investor purchasing the note receives in the future.

Step 3: Now sell the note using the discount rate and time before the legal due date. Applying Formula 8.2, rearrange and solve for $P = \$5,102.47 / (1 + (0.06 \times \frac{116}{365})) = \$5,006.99$.

The \$5,006.99 is how much the buyer of the promissory note pays based on the negotiated interest rate. This is sometimes called the **proceeds**, which is the amount of money received from a sale. The buyer now holds the promissory note and receives the full payment of \$5,102.47 on January 9, 2012, from the borrower. This earns the buyer a 6% rate of interest on the investment.



Important Notes

If the original promissory note is noninterest-bearing, remember you can skip the second step above, since the face value of the note equals the maturity value upon the legal due date.



Things To Watch Out For

When working with the sale of a promissory note to another investor, the most common mistake is to mix up the interest rates. Remember that you use the original, stated interest rate of the promissory note in calculating the maturity value on the legal due date. When calculating the selling price, you must use the new negotiated rate to discount the maturity value.

Example 8.4B: Deciding You Need the Money Sooner

On April 30, 2023, Bernie purchased some furniture from Fran's Fine Furniture on a 10-month no payment plan. A promissory note was created establishing an interest rate of 11% and it included the clause "no grace." The furniture purchase was for \$2,234.40. On August 12, 2023, Fran's Fine Furniture sold the promissory note to an investor at an agreed-upon interest rate of 13%. What were the proceeds of the sale?

Plan	Calculate the proceeds of the sale. This represents a present value calculation (P) based on the maturity value of the original promissory using the discount rate.				
Understand	What You Already Know Step 1: The following information about the promissory note and sale are known: Original Promissory Note: Issue Date = April 30, 2023 P = \$2,234.40 Term = 10 months, no grace r = 11% per year Sale of Promissory Note: Sale Date = August 12, 2023 Discount r = 13% per year		How You Will Get There Step 1 (continued): Calculate the legal due date, accounting for the no grace clause, and transform it into the number of days expressed annually to match the interest rate. Also calculate the number of days between the legal due date and the sale date, expressing it annually to match the interest rate. Step 2: To arrive at the maturity value, apply Formula 8.2. Step 3: To arrive at the present value using the new interest rate, apply Formula 8.2, rearranging for P.		
	Timeline diagram showing the dates and interest rates for the promissory note sale. A horizontal timeline starts at April 30, 2023, with a value of \$2,234.40. It goes to August 12, 2023, where a discount rate of 13% is indicated. From there, it continues to the Legal Due Date (8 months, no grace), where the maturity value FV = \$? is indicated. The interest rate for the original note is 11% annually.				
Perform	Step 1 (continued): Ten months from April 30, 2023, is February 30, 2024, which is adjusted to the last day of the month, or February 29, 2024. No grace period is added. By counting the days in between (1 + 31 + 30 + 31 + 31 + 31 + 30 + 31 + 31 + 29) or using a calculator, you find that there are 305 days between the dates. Thus $t = \frac{305}{365}$ expressed annually.				
	Step 1 (continued): Using a selling date of August 12, 2023, and the legal due date of February 29, 2024, count the days in between (19 + 30 + 31 + 30 + 31 + 31 + 29) or use a calculator to determine there are 201 days between the dates. Thus, $t = \frac{201}{365}$ expressed annually.				
	Step 2: $S = \$2,234.40 \left(1 + 0.11 \left(\frac{305}{365}\right)\right) = \$2,439.78$				
	Step 3: $P = \frac{\$2,439.78}{1 + 0.13 \left(\frac{201}{365}\right)} = \$2,276.79$				
	Calculator Instructions Use the DATE function to calculate the number of days,				
		DT1	DT2	DBD	Mode
	Original Promissory Note	04.3023	02.2924	Answer: 305	ACT
	Sale Of Promissory Note	08.1223	✓	Answer: 201	✓

Present	When Fran's Fine Furniture sold the note to the investor, the proceeds of the sale were \$2,276.79.

Example 8.4C: The Sale of a Noninterest Bearing Promissory Note

A noninterest-bearing promissory note was issued on January 31 with a face value of \$10,000 and a term of eight months. On May 28 the note was sold to an investor at an agreed-upon interest rate of 5% per year. What was the sale price of the promissory note? Assume February is in a non-leap year.

Plan	Calculate the proceeds of the sale. This represents a present value calculation (P) based on the maturity value of the original promissory using the discount rate.										
Understand	What You Already Know Step 1: The following about the promissory note and sale are known: Original Promissory Note: Issue Date =January 31 Term = 8 months P = \$10,000 r = 0% Sale of Promissory Note: Sale Date = May 28 Discount r = 5% per year		How You Will Get There Step 1 (continued): Calculate the legal due date by adding three days. Also calculate the number of days between the legal due date and the sale date, expressing it annually to match the interest rate. Step 2: The promissory note is noninterest-bearing, so the maturity value equals the face value. Step 3: To arrive at the present value using the new interest rate, apply Formula 8.2, rearranging for P.								
Perform	Step 1 (continued): Add eight months to January 31, arriving at September 31. Adjust this to the last day of the month, September 30. Adding three days' grace gives you a legal due date of October 3. Step 1 (continued): Using a selling date of May 28 and the legal due date of October 3, count the days in between (4 + 30 + 31 + 31 + 30 + 2) or use a calculator to determine that there are 128 days between the dates. Thus, $t = \frac{128}{365}$ expressed annually. Step 2: $P = S = \\$10,000$ Step 3: $P = \frac{\\$10,000}{1 + 0.05(\frac{128}{365})} = \\$9,827.68$										
	Calculator Instructions Assume a non-leap year of 2011, and use the DATE function to calculate the number of days: <table><tr><td>DT1</td><td>DT2</td><td>DBD</td><td>Mode</td></tr><tr><td>04.2811</td><td>10.0311</td><td>Answer: 128</td><td>ACT</td></tr></table>				DT1	DT2	DBD	Mode	04.2811	10.0311	Answer: 128
DT1	DT2	DBD	Mode								
04.2811	10.0311	Answer: 128	ACT								
Present	The promissory note was sold for \$9,827.68 on May 28.										



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8.5: Application: Loans

(The Bank Comes Knocking)

Almost everybody needs a little financial help from time to time. Perhaps you want to purchase a big-ticket item like a big-screen TV without having the cash in full today to pay for it up front. However, the television is drastically on sale this week and if you wait to purchase it you will lose out and potentially have to pay full price next month. Or maybe as you flip through all of your bills you notice that though you will eventually have enough income to cover them, some of your payments fall due a few days before you will actually receive your income. What will you do for those few days until the income is deposited into your account?

Businesses also find themselves in similar situations. Maybe a supplier is offering a special deal on a product line, but the business does not have the cash to stock up. Also typical of business operations is that they make purchases in advance of sales, so they need to spend the money before they can receive the revenue to pay for their expenses. How can a business get access to short-term financing?

In this section, short-term financing in the form of demand loans is explored. This will include looking at loans where payments are variable as well as loans possessing fixed payments. The section ends by looking briefly at a loan you may already have—a student loan.

Demand Loans and Characteristics

A **demand loan** is a short-term loan that generally has no specific maturity date, can be paid at any time without any interest penalty, and where the lender can demand repayment in full at any time. It allows borrowing when needed and repayment when money permits, subject to the following six characteristics:

1. **Credit Limit.** This establishes the maximum amount that can be borrowed.
2. **Variable Interest Rate.** Almost all demand loans use variable simple interest rates based on the prime rate. Only the best, most secure customers can receive prime, while others usually get "prime plus" some additional amount.
3. **Fixed Interest Payment Date.** Interest is always payable on the same date each and every month. For simplicity, the payment is usually tied to a chequing or savings account, allowing the interest payment to occur automatically.
4. **Interest Calculation Procedure.** Interest is always calculated using a simple interest procedure based on the daily closing balance in the account. This means the first day but not the last day is counted.
5. **Security.** Loans can be secured or unsecured. **Secured loans** are those loans that are guaranteed by an asset such as a building or a vehicle. In the event that the loan defaults, the asset can be seized by the lender to pay the debt. **Unsecured loans** are those loans backed up by the general goodwill and nature of the borrower. Usually a good credit history or working relationship is needed for these types of loans. A secured loan typically enables access to a higher credit limit than an unsecured loan.
6. **Repayment Structure.** The repayment of the loan is either variable or fixed.
 - a. A variable repayment structure allows the borrower to repay any amount at any time, although a minimum requirement may have to be met such as "at least 2% of the current balance each month." A **current balance** is the balance in an account plus any accrued interest. **Accrued interest** is any interest amount that has been calculated but not yet placed (charged or earned) into an account.
 - b. A fixed repayment structure requires a fixed payment amount toward the current balance on the same date each and every month.

Types of Simple Interest Financing

While many types of financial tools use simple interest, these are the four most common:

1. **Personal Line of Credit (LOC).** A demand loan for individuals is generally unsecured and is granted to those individuals who have high credit ratings and an established relationship with a financial institution. Since it is unsecured, the credit limit is usually a small amount, such as \$10,000. Repayment is variable and usually has a minimum monthly requirement based on the current balance.

2. **Home Equity Line of Credit (HELOC).** This is a special type of line of credit for individuals that is secured by residential homeownership. Typically, an amount not exceeding 80% of the equity in a home is used to establish the credit limit, thus enabling an individual access to a large amount of money. The interest rates tend to follow mortgage interest rates and are lower than personal lines of credit. Repayment is variable, usually involving only the accrued interest every month.
3. **Operating Loans.** An operating loan is the business version of a line of credit. It may or may not be secured, depending on the nature of the business and the strength of the relationship the business has with the financial institution. Repayment can be either variable or fixed.
4. **Student Loans.** A loan available to students to pursue educational opportunities. Although these are long-term in nature, the calculation of interest on a student loan uses simple interest techniques. These loans are not true demand loans since a student loan cannot be called in at any time. Repayment is fixed monthly.

Repayment Schedules

When you work with short-term loans, regardless of the repayment structure, you should always set up a repayment schedule. A **repayment schedule** is a table that details the financial transactions in an account including the balance, interest amounts, and payments. Table 8.6 presents the table structure used for setting up a repayment schedule.

Date	Balance before Transaction (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged (I)	Accrued Interest	Payment (+) or Advance (-)	Principal Amount	Balance after Transaction
Start Date								(1)
(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)

Each number in the table corresponds to an entry below that explains how to use each column or row.

1. The first row appears if the schedule has an opening balance. In these instances, list the start date in the first column and the opening balance in the last column.
2. List the date of any transaction. This could include a payment, advance, interest rate change, or an accrued interest payment.
3. Carry this number forward from (1).
4. Record the interest rate that applies to the date interval in the first column.
5. Calculate the number of days between the date on the previous row and the current row. Remember to count the first day but not the last day. Express the number annually to match the interest rate.
6. Compute the interest charges for the date interval using Formula 8.1, $I = \text{Prt}$. Use (3), (4), and (5) as your values for the formula.
7. Enter the cumulative total of any unpaid or accrued interest as of the current row's date. This amount is the sum of (6) from the current row plus any number recorded in this column from the row above.
8. Enter the amount of the transaction occurring for this row of the table. Payments should be recorded as positives (credits) as they will decrease the balance, and advances should be recorded as negatives (debits) as they will increase the balance. If this is an interest payment date, one of the following two events will happen:
 - a. Only the interest payment occurs on this date. Copy the accrued interest from (7) into this column.
 - b. An additional payment or advance is made on this date. Add the accrued interest to the advance or payment, placing the sum into this column.

Regardless of which event happens, cross out the accrued interest amount in (7) as it is considered paid, so the accrued interest balance is reduced to zero once again.
9. One of two events will happen in this column:
 - a. On an interest payment date, subtract the accrued interest from the amount in (8).
 - b. On a date other than an interest payment date, any payment or advance will have its total amount applied to the principal. Therefore, carry the amount in (8) across to this column.
10. Subtract the amount in (9) from the amount in this column on the previous row to create the new balance in the account. Copy this amount to the next row in (3).



How It Works

Although the calculations of a repayment schedule are relatively straightforward, the complexity of the repayment schedule sometimes causes grief. When a repayment schedule is required, follow these steps:

Step 1: Set up the repayment schedule as per the example table.

Step 2: Record the start date and the opening balance for the loan.

Step 3: In chronological order, make new row entries in the schedule by filling in the details provided in the question. You require a new row in the table whenever one of the following three events occurs:

- a. A payment or advance is made.
- b. An interest payment date occurs.
- c. The interest rate changes.

Step 4: Starting with the first row, work left to right across the table, filling in all information. Pay particular attention to the nuances of the "Payment or Advance" and "Principal Amount" columns as discussed previously. Once a row is complete, move to the next row until you fill in the entire table.

Step 5: Calculate any totals requested such as total interest or total principal paid.



Important Notes

For simplicity in writing the numbers into repayment schedules and performing calculations, it is this textbook's practice to round all interest calculations to two decimals throughout the table. In real-world applications, you must keep track of all decimals in the account at all times.



Things To Watch Out For

Because of the size of the repayment schedule and the large amount of information involved in the calculations, the number-one error is what most people call a "silly" error. It means that a wrong date is recorded, a wrong amount is written down, a payment is recorded as an advance or vice versa, or simply the wrong button is pressed on a calculator. Take the time to ensure you read and record the correct numbers and that you pause when performing calculations. For example, advances mean the balance should get bigger, while payments mean the balance should get smaller. Just by thinking for a second about the basic principles of debt you should be able to catch those silly errors.

Example 8.5A: Operating Loan, Fixed Repayment

On July 15, when the prime rate was set at 4%, Canadian Footwear took out an operating loan from CIBC for \$8,000 at prime plus 1.25%. The terms of the loan require a fixed payment of \$1,500 on the 15th of every month until the loan is repaid. The prime rate climbed by 0.5% on September 29. Create a repayment schedule for the loan and calculate the total interest paid.

Plan

Create the repayment schedule for the loan and sum the interest charges for the entire operating loan.

Understand	What You Already Know For a fixed repayment loan, capture three important categories of information:						How You Will Get There Step 1: Set up a repayment table. Step 2: Fill in the start date and opening balance. Step 3: Chronologically fill in all information, with one transaction per row of the table. Step 4: Work left to right and top to bottom throughout the table. Step 5: Once you have completed this step, sum the interest charges paid on the 15th of every month.		
	Opening Balance		Payments		Dates of Interest Rate Changes†				
	July 15: \$8,000		August 15: \$1,500 September 15: \$1,500 October 15: \$1,500 November 15: \$1,500 December 15: \$1,500 January 15: \$????*		July 15: 5.25% September 29: 5.75%				
*Note: You can presume there is a payment on January 15 since the first five payments total up to \$7,500, which falls short of repaying the loan. The last payment, on January 15, is still unknown as it will be based on the remaining balance in the account plus any accrued interest. ‡. Calculated as prime plus 1.25%.									
Perform	Steps 1-4:								
	Date	Balance before Transaction (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged (I = Prt)	Accrued Interest	Payment (+) or Advance (–)	Principal Amount	Balance after Transaction
	July 15								\$8,000.00
	Aug 15	\$8,000.00	5.25%	31/365 (1)	\$35.67 (2)	~~\$35.67~~	\$1,500.00	\$1,464.33 (3)	\$6,535.67 (4)
	Sept 15	\$6,535.67	5.25%	31/365	\$29.14	~~\$29.14~~	\$1,500.00	\$1,470.86	\$5,064.81
	Sept 29	\$5,064.81	5.25%	14/365	\$10.20	\$10.20	\$0.00	\$0.00	\$5,064.81
	Oct 15	\$5,064.81	5.75%	16/365	\$12.77	~~\$22.97~~ (5)	\$1,500.00	\$1,477.03	\$3,587.88
	Nov 15	\$3,587.88	5.75%	31/365	\$17.52	~~\$17.52~~	\$1,500.00	\$1,482.48	\$2,105.30
	Dec 15	\$2,105.30	5.75%	30/365	\$9.95	~~\$9.95~~	\$1,500.00	\$1,490.05	\$615.25
	Jan 15	\$615.25	5.75%	31/365	\$3.00	~~\$3.00~~ (6)	\$618.25 (7)	\$615.25	\$0.00
Select calculations and comments from the above table:									
(1) This is the number of days from the date on the previous row to the current row, or July 15 to August 15. Count the first day, but not the last.									
(2) $I = Prt = \$8,000(0.0525)(31/365) = \$35.67$									
(3) $\$1,500.00 - \$35.67 = \$1,464.33$. The accrued interest of \$35.67 is now paid and can be crossed out.									
(4) $\$8,000.00 - \$1,464.33 = \$6,535.67$									
(5) $\$10.20 + \$12.77 = \$22.97$									
(6) The last payment must clear the balance owing and the accrued interest: $\$615.25 + \$3.00 = \$618.25$									
(7) $\$618.25 - \$3.00 = \$615.25$									
Step 5: Total interest charges = $\$35.67 + \$29.14 + \$22.97 + \$17.52 + \$9.95 + \$3.00 = \$118.25$									
Present	To clear the operating loan it takes six payments: five of \$1,500 each and a final payment of \$618.25. The loan incurs total interest paid of \$118.25.								

Example 8.5B: HELOC, Variable Repayment, No Minimum Requirement

Lynne has access to a HELOC that requires only the payment of accrued interest on the first of every month. On March 1, the opening balance on her HELOC was \$15,000. She took advances of \$6,000 and \$10,000 on March 21 and May 4, respectively. She made additional payments of \$11,000 and \$15,000 on April 15 and June 17. The interest rate on her HELOC sits at prime plus 2%. On March 1, the prime rate was 3%. On April 26, it rose by 0.5%. Determine the total interest paid on her HELOC from March 1 to July 1.

Plan	This is a HELOC using simple interest. Calculate the total interest that she had to pay in March, April, May, and June and then sum the total interest paid.								
Understand	What You Already Know Capture five important categories of information:					How You Will Get There Step 1: Set up a repayment table. Step 2: Fill in the start date and opening balance. Step 3: Chronologically fill in all information, with one transaction per row of the table. Step 4: Work left to right and top to bottom throughout the table. Step 5: Once you have completed this step, sum the accrued interest payments on April 1, May 1, June 1, and July 1.			
	Opening Balance		Advances		Payments				
	March 1: \$15,000		March 21: \$6,000 May 4: \$10,000		April 15: \$11,000 June 17: \$15,000				
	Interest Payment Dates		Dates of Interest Rate Changes*						
	April 1 May 1 June 1 July 1		March 1: 5% April 26: 5.5%						
	*Calculated as prime plus 2%.								
Perform	Steps 1-4:								
	Date	Balance before Transaction (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged (I = Prt)	Accrued Interest	Payment (+) or Advance (-)	Principal Amount	Balance after Transact.
	Mar 1		5%						\$15,000
	Mar 21	\$15,000 (1)	5%	20/365 (2)	\$41.10 (3)	\$41.10 (4)	−\$6,000	−\$6,000 (5)	\$21,000 (6)
	Apr 1	\$21,000	5%	11/365	\$31.64	~~\$72.74~~ (7)	\$72.74 (8)	\$0.00 (9)	\$21,000
	Apr 15	\$21,000	5%	14/365	\$40.27	\$40.27	\$11,000	\$11,000	\$10,000
	Apr 26	\$10,000	5%	11/365	\$15.07	\$55.34	\$0.00	\$0.00	\$10,000
	May 1	\$10,000	5.5%	5/365	\$7.53	~~\$62.87~~	\$62.87	\$0.00	\$10,000
	May 4	\$10,000	5.5%	3/365	\$4.52	\$4.52	−\$10,000	−\$10,000	\$20,000
	June 1	\$20,000	5.5%	28/365	\$84.38	~~\$88.90~~	\$88.90	\$0.00	\$20,000
	June 17	\$20,000	5.5%	16/365	\$48.22	\$48.22	\$15,000	\$15,000	\$5,000
	July 1	\$5,000	5.5%	14/365	\$10.55	~~\$58.77~~	\$58.77	\$0.00	\$5,000
	Select calculations and comments from the above table: (1) This is the balance from the last column in the row above carried forward = \$15,000. (2) The date interval from the previous row to this row is March 1 to March 21: $t = 21 - 1 = 20$ days. (3) $I = Prt = \$15,000(0.05)(20/365) = \$41.10$ (4) The accrued interest from the row above plus (3). $\$0 + \$41.10 = \$41.10$ (5) Not an interest payment date, so the full payment is applied to principal = $-\$6,000$ (6) Previous balance in the column above minus the principal portion from (5). $\$15,000 - (-\$6,000) = \$21,000$ (7) The accrued interest from the row above plus interest from this row. $\$41.10 + \$31.64 = \$72.74$ (8) This is an interest payment date. Carry (7) across and cross it out, reducing the accrued interest balance to zero. (9) This is an interest payment date. Take (8) and subtract (7). $\$72.74 - \$72.74 = \$0.00$ Step 5: April 1 + May 1 + June 1 + July 1 = $\$72.74 + \$62.87 + \$88.90 + \$58.77 = \$283.28$.								
	Present	The total interest paid on the HELOC from March 1 to July 1 is \$283.28.							

Example 8.5C: Personal LoC, Variable Repayment, with Minimum Requirement

Everlyne has a personal LOC with her bank with a maximum credit limit of \$10,000. The interest rate is prime plus 3.5%, and the current prime rate is 4.5%. Regardless of any account transaction activity, her bank requires on the first of every month for her to pay "the greater of 5% of the current balance or \$100" from her chequing account. She is allowed to exceed her maximum credit limit, but if she does the entire balance is subject to 21% interest until such time as the balance is restored below the credit limit. On October 1, the opening balance on her LOC was \$2,000. She took advances of \$5,000, \$6,000, and \$1,000 on October 21, November 13, and December 1, respectively. She made payments of \$1,000, \$3,000, and \$8,500 on November 1, November 20, and December 15, respectively. The prime rate decreased by 1/4% on November 5. Calculate her total required payments (not her voluntary payments) and the portion of those payments that went toward interest for the months of October, November, and December.

Plan	Calculate a personal LOC using simple interest to determine the total required payments on November 1, December 1, and January 1 and the amount of interest included in those payments.								
Understand	What You Already Know Capture five important categories of information:					How You Will Get There Step 1: Set up a repayment table. Step 2: Fill in the start date and opening balance. Step 3: Chronologically fill in all information, with one transaction per row of the table. Step 4: Work left to right and top to bottom throughout the table. Step 5: Sum the total required payments and the interest paid.			
	Opening Balance		Advances		Payments				
	Oct 1: \$2,000		October 21: \$5,000 Nov. 13: \$6,000 December 1: \$1,000		November 1: \$1,000 November 20: \$3,000 December 15: \$8,500				
	Interest Payment Dates		Dates of Interest Rate Changes*						
	November 1 December 1 January 1		October 1: 8% November 5: 7.75% Interest rate jumps to 21% if credit limit is exceeded						
*Calculated as prime plus 4.5%									
Perform	Steps 1-4:								
	Date	Balance before Transaction (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged (I = Prt)	Accrued Interest	Payment (+) or Advance (−)	Principal Amount	Balance after Transaction
	Oct 1								\$2,000
	Oct 21	\$2,000	8%	20/365	\$8.77	\$8.77	−\$5,000	−\$5,000	\$7,000
	Nov 1	\$7,000	8%	11/365	\$16.88	~~\$25.65~~	\$1,000 + \$351.28 = \$1,351.28 (1)	\$1,325.63 (2)	\$5,674.37
	Nov 5	\$5,674.37	8%	4/365	\$4.97	\$4.97	\$0.00	\$0.00	\$5,674.37
	Nov 13	\$5,674.37	7.75%	8/365	\$9.64	\$14.61	−\$6,000	−\$6,000	\$11,674.37
	Nov 20	\$11,674.37	21% (3)	7/365	\$47.02	\$61.63	\$3,000	\$3,000	\$8,674.37
	Dec 1	\$8,674.37	7.75%	11/365	\$20.26	~~\$81.89~~	−\$1,000 + \$437.81 = −\$562.19 (4)	−\$644.08 (5)	\$9,318.45
	Dec 15	\$9,318.45	7.75%	14/365	\$27.70	\$27.70	\$8,500	\$8,500	\$818.45
	Jan 1	\$818.45	7.75%	17/365	\$2.95	~~\$30.65~~	\$100 (6)	\$69.35 (7)	\$749.10

	Select calculations and comments from the table: (1) The required payment is the greater of 5% of the current balance or \$100 regardless of her monthly account activity. The current balance is $\\$7,000 + \\$25.65 = \\$7,025.65$. Five percent of that number is \$351.28. The \$1,000 extra payment plus the required payment of \$351.28 totals \$1,351.28. (2) The $\\$1,351.28 - \\$25.65 = \\$1,325.63$ toward the principal. The accrued interest is now paid and is crossed out. (3) Note that by exceeding the credit limit of \$10,000, she changes the interest rate to the penalty rate. (4) The current balance is $\\$8,674.37 + \\$81.89 = \\$8,756.26$, of which 5% is \$437.81. Therefore, with an advance of \$1,000 and a payment of \$437.81 on the same date, the net daily result is an advance of \$562.19. (5) $-\\$562.19 - \\$81.89 = -\\$644.08$, which is added to the principal. The accrued interest is paid and is crossed out. (6) The current balance is $\\$818.45 + \\$30.65 = \\$849.10$, of which 5% is \$42.46, so pay \$100 - the greater amount. (7) $\\$100 - \\$30.65 = \\$69.35$. The accrued interest is now paid and is crossed out. Step 5: Total Required Payments = Nov 1 + Dec 1 + Jan 1 = $\\$351.28 + \\$437.81 + \\$100.00 = \\$889.09$ Total Interest Charges = Nov 1 + Dec 1 + Jan 1 = $\\$25.65 + \\$81.89 + \\$30.65 = \\$138.19$
Present	Throughout the months of October, November, and December, the total required payments on the LOC came to \$889.09. Of that amount, \$138.19 went toward the interest charges.

Student Loans

A **student loan** is a special type of loan designed to help students pay for the costs of tuition, books, and living expenses while pursuing postsecondary education. The Canada Student Loans Program (CSLP) is administered by the federal government and run by the National Student Loan Service Centre (NSLSC) under contract to Human Resources and Skills Development Canada (HRSDC).

If and when you take out a student loan, the following characteristics are in place for full-time students (defined as having a 60% workload or more):

1. While you attend an educational institution, no interest is charged on your student loan nor are any payments required.
2. If you do not attend an educational institution for a continuous period of six complete months (called your grace period), you must start to repay your student loan. The fixed monthly payments are in an amount of your choosing, so long as you will repay the student loan within a maximum of 114 months. For example, if you graduate on May 3, the six-complete-month period is from June 1 to November 30 (inclusive). The first payment is due one month later on December 31 and each of the remaining payments at the end of every month thereafter.
3. During the six months, interest accrues on the student loan at a simple rate of prime + 2.5%. At the end of the grace period, the accrued interest is either converted to principal or paid in full at the student's option.
4. At the end of the grace period, you choose the interest rate on your loan. This choice cannot be changed at a future date. Your choices are as follows:
 - a. Variable interest rate of prime + 2.5%.
 - b. Fixed interest rate of prime + 5% (where prime is determined on the day that your grace period ends).
5. Interest is calculated on a simple interest basis using the exact number of days between each payment.
6. You can make variable payments at any time without any interest penalties.
7. The loan is not a demand loan. Therefore, the government is unable to demand that the loan be paid off before its due date.

The following characteristics are different for part-time students (those with less than a 60% workload):

1. While attending school, a part-time student is charged interest (prime + 2.5%) and must make monthly interest payments, starting with the first month after taking out the student loan.
2. During the six-month grace period, the student will still be making monthly interest payments. Therefore, at the end of the grace period there is no accrued interest and no decision to be made.



How It Works

Mathematically, a part-time student loan requires a repayment schedule no different from ones already shown. With respect to full-time student loans, before you can set up the repayment schedule you require an additional step, namely, calculating the opening balance:

Pre-Step 1: Calculate the balance of the student loan at the end of the grace period by applying simple interest to the principal using Formulas 8.1 or 8.2 as necessary. When you perform this calculation, carry all decimals throughout the

required calculations, rounding the balance only at the end. Based on the student's decision at the end of the grace period, the interest accrued over the six months will either be paid off or added to the principal.

Steps 1 through 5: These steps remain unchanged from repayment schedules. Note in step 3 that there can be no advances on a student loan when it is being repaid.

Example 8.5D: Your Student Loan

Rufaro has been a full-time student at the University of Manitoba for the past four years. She has just completed her bachelor of commerce degree from the Asper School of Business and her last day of exams was April 28. Her total student loan is \$30,000. She has decided to take her six-month grace period and convert it to principal, then start making payments of \$400 per month using the variable interest rate of prime + 2.5%. The current prime rate is 4.25%. On January 10, she will make an additional payment of \$250 toward her loan. On August 27 and again on February 22, the prime rate rises by 0.5%. Construct a repayment schedule displaying only the first six months of payments. Calculate the total interest on her student loan charged for the entire year (April 30 to April 30). Assume February has 28 days.

Plan	You need to calculate a student loan using simple interest. First, determine the amount of interest charged during the grace period to calculate her balance owing, then set up the repayment schedule with the payments. Once you have completed this step, total the interest amounts from the table plus the grace period interest to arrive at the annual interest.				
Understand	What You Already Know			How You Will Get There	
	Capture five important categories of information:			Pre-Step 1: Calculate the total simple interest charged May 1 to Oct. 31. Add this amount to the opening balance as per Rufaro's choice.	
	Opening Balance	Grace Period	Payments	Dates of Interest Rate Changes	Step 1: Set up a repayment table.
	\$30,000	May 1 to October 31 inclusive—interest charged to be converted to principal on October 31.	Nov. 30: \$400 Dec. 31: \$400 Jan. 10: \$250 Jan. 31: \$400 Feb. 28: \$400 Mar. 31: \$400 Apr. 30: \$400	May 1: 6.75% Aug. 27: 7.25% Feb. 22: 7.75%	Step 2: Fill in the start date and opening balance. Step 3: Chronologically fill in all information, with one transaction per row of the table. Step 4: Work left to right and top to bottom throughout the table. Step 5: Sum the six amounts of interest paid at the end of every month from November 30 to April 30 inclusive, and add the total interest from the pre-step to arrive at the total interest charged for the year.
	*Calculated as prime plus 2.5%.				
Perform	Pre-Step 1:				
	Date Range	Balance (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged (I = Prt)
	May 1 to Aug. 27	\$30,000	6.75%	31 + 30 + 31 + 26 = 118 days or 118/365	\$654.657534
	Aug. 27 to Oct. 31 inclusive	\$30,000	7.25%	5 + 30 + 31 = 66 days or 66/365	\$393.287671
	Total Simple Interest (rounded)				\$1,047.95
	Adding to the balance, the starting balance becomes \$30,000 + \$1,047.95 = \$31,047.95.				

Steps 1-4:

Date	Balance before Transaction (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged ($I = Prt$)	Accrued Interest	Payment (+) or Advance (-)	Principal Amount	Balance after Transaction
Nov 1								\$31,047.95
Nov 30	\$31,047.95	7.25%	29/365	\$178.84	~~\$178.84~~	\$400.00	\$221.16	\$30,826.79
Dec 31	\$30,826.79	7.25%	31/365	\$189.82	~~\$189.82~~	\$400.00	\$210.18	\$30,616.61
Jan 10	\$30,616.61	7.25%	10/365	\$60.81	\$60.81	\$250.00	\$250.00	\$30,366.61
Jan 31	\$30,366.61	7.25%	21/365	\$126.67	~~\$187.48~~	\$400.00	\$212.52	\$30,154.09
Feb 22	\$30,154.09	7.25%	22/365	\$131.77	\$131.77	\$0.00	\$0.00	\$30,154.09
Feb 28	\$30,154.09	7.75%	6/365	\$38.42	~~\$170.19~~	\$400.00	\$229.81	\$29,924.28
Mar 31	\$29,924.28	7.75%	31/365	\$196.97	~~\$196.97~~	\$400.00	\$203.03	\$29,721.25
Apr 30	\$29,721.25	7.75%	30/365	\$189.32	~~\$189.32~~	\$400.00	\$210.68	\$29,510.57

Step 5: Total interest = \$1,047.95 + \$178.84 + \$189.82 + \$187.48 + \$170.19 + \$196.97 + \$189.32 = \$2,160.57

Present

Over the entire year, Rufaro will be charged \$2,160.57 of interest on her student loan.



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8.6: Application: Treasury Bills & Commercial Papers

(When Governments and Businesses Borrow)

Where does the government go to borrow money? In an article, you read that Canada's national debt ballooned from \$458 billion in 2008 to over \$654 billion in August 2012. That meant the government had to borrow this money from somewhere. You begin to wonder where . . . it can't exactly walk into the country's largest bank (RBC) and ask for a \$654 billion loan! Although RBC is Canada's largest banking institution and in 2012 it ranked 70th on the *Forbes* Global 2000 list of the world's biggest public companies, it still does not have enough money to lend to the government. If a government cannot get a loan from a financial institution, then how do federal and provincial governments borrow money in such large amounts to finance their debt?

Treasury Bills: The Basics

The answer lies in **treasury bills**, better known as T-bills, which are short-term financial instruments that both federal and provincial governments issue with maturities no longer than one year. Approximately 27% of the national debt is borrowed through T-bills.

Who purchases these T-bills? It is a two-tiered process. First, major investment companies and banks purchase the T-bills in very large denominations (such as \$1 million or \$10 million). Then these organizations break the T-bills down into bite-size chunks (such as \$1,000 denominations) and sell them to their customers and other investors like you and me, at a profit of course. In essence, it is you and I who finance the debt of Canada!

Here are some of the basics about T-bills:

1. The Government of Canada regularly places T-bills up for auction every second Tuesday. Provincial governments issue them at irregular intervals.
2. The most common terms for federal and provincial T-bills are 30 days, 60 days, 90 days, 182 days, and 364 days.
3. T-bills do not earn interest. Instead, they are sold at a discount and redeemed at full value. This follows the principle of "buy low, sell high." The percentage by which the value of the T-bill grows from sale to redemption is called the **yield** or rate of return. From a mathematical perspective, the yield is calculated in the exact same way as an interest rate is calculated, and therefore the yield is mathematically substituted as the discount rate in all simple interest formulas. Up-to-date yields on T-bills can be found at www.bankofcanada.ca/en/rates/monmrt.html.
4. The **face value of a T-bill** (also called *par value*) is the maturity value, payable at the end of the term. It includes both the principal and yield together.
5. T-bills do not have to be retained by the initial investor throughout their entire term. At any point during a T-bill's term, an investor is able to sell it to another investor through secondary financial markets. Prevailing yields on T-bills at the time of sale are used to calculate the price.

Commercial Papers - The Basics

A **commercial paper** (or paper for short) is the same as a T-bill except that it is issued by a large corporation instead of a government. It is an alternative to short-term bank borrowing for large corporations. Most of these large companies have solid credit ratings, meaning that investors bear very little risk that the face value will not be repaid upon maturity.

For a corporation, papers tend to be a cheaper source of financing than borrowing from a bank. The yields paid to investors are less than the interest rate the corporation is charged by the bank. The corporation can also make its financing even cheaper by offering the commercial paper directly to investors to avoid intermediary brokerage fees and charges. Finally, commercial papers are not subject to strict financial requirements and registration, such as stocks and bonds require; therefore, they pose easy access to funds in the short-term for a corporation.

Commercial papers carry the same properties as T-bills. The only fundamental differences lie in the term and the yield:

1. The terms are usually less than 270 days but can range from 30 days to 364 days. The most typical terms are 30 days, 60 days, and 90 days.

2. The yield on commercial papers tends to be slightly higher than on T-bills since corporations do carry a higher risk of default than governments.

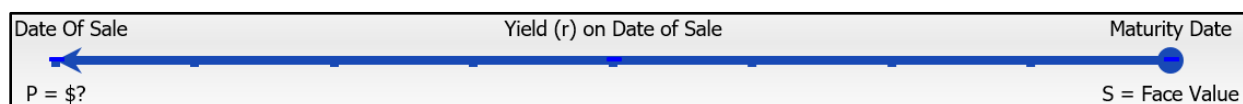


How It Works

Mathematically, T-bills and commercial papers operate in the exact same way. The future value for both of these investment instruments is always known since it is the face value. Commonly, the two calculated variables are either the present value (price) or the yield (interest rate). The yield is explored later in this section. Follow these steps to calculate the price:

Step 1: The face value, yield, and time before maturity must be known. Draw a timeline if necessary, as illustrated below, and identify the following:

1. The face value (S).
2. The yield (r) on the date of the sale, which is always expressed annually. Remember that mathematically the yield is the same as the discount rate.
3. The number of days (t) remaining between the date of the sale and the maturity date. Count the first day but not the last day. Express the number of days annually to match the annual yield.



Step 2: Apply Formula 8.2, rearranging and solving for the present value, which is the price of the T-bill or commercial paper. This price is always less than the face value.



Paths To Success

When you calculate the price of a T-bill or commercial paper, the only important pieces of information are the face value, the yield or discount rate *on the date of sale*, and how many days remain until maturity. Any prior history of the T-bill or commercial paper does not matter in any way. What the market rate was yesterday or last week or upon the date of issue does not impact today's price. Likewise, the number of days since the date of issue does not matter. What an investor paid for it three weeks ago is irrelevant. When calculating the price, pay attention only to the current market information and the future. Ignore the history!



Give It Some Thought

1. All other variables remaining stable, what happens to the price (present value) of a T-bill or commercial paper if the discount rate increases?
2. An investor purchases a T-bill when the yield is 3%. A few months later, the investor sells the T-bill when the yield has dropped to 2%. Will the investor realize a yield of 3%, more than 3%, or less than 3% on the investment?

Solutions:

1. The price is lower because more yield must be realized on the investment.

2. More than 3%. Since the T-bill yield is lower when sold, the price of the T-bill increases, resulting in the investor receiving more money on the investment. This raises the investor's return on investment.

Example 8.6A: Price of a Treasury Bill

A Government of Canada 182-day issue T-bill has a face value of \$100,000. Market yields on these T-bills are 1.5%. Calculate the price of the T-bill on its issue date.

Plan	Calculate the price that an investor will pay for the T-bill, or P.	
Understand	What You Already Know Step 1: The T-bill maturity value, term, and yield are known: $S = \$100,000$ $r = 1.5\%$ $t = 182/365$	How You Will Get There Step 2: Apply Formula 8.2, rearranging for P.
Perform	Step 2: $\$100,000 = P(1 + 0.015 \times \frac{182}{365})$ $P = \frac{\$100,000}{1 + (0.015)(\frac{182}{365})} = \$99,257.61$	
Present	An investor will pay \$99,257.61 for the T-bill. If the investor holds onto the T-bill until maturity, the investor realizes a yield of 1.5% and receives \$100,000.	

Example 8.6B: Selling a Commercial Paper during Its Term

Pfizer Inc. issued a 90-day, \$250,000 commercial paper on April 18 when the market rate of return was 3.1%. The paper was sold 49 days later when the market rate of return was 3.63%. Calculate the price of the commercial paper on its date of sale.

Plan	Calculate the price that an investor pays for the commercial paper on its date of sale, or P.	
Understand	What You Already Know Step 1: Note that the historical rate of return of 3.1% is irrelevant to the price of the commercial paper today. The number of days elapsed since the date of issue is also unimportant. The number of days before maturity is the key piece of information. The commercial paper maturity value, term, and yield are: $S = \$250,000$ $r = 3.63\%$ $t = 90 - 49 = 41$ days or $41/365$	How You Will Get There Step 2: Apply Formula 8.2, rearranging for P.
Perform	Step 2: $\$250,000 = P(1 + 0.0363 \times \frac{41}{365})$ $P = \frac{\$250,000}{1 + (0.0363)(\frac{41}{365})} = \$248,984.76$	
Present	An investor pays \$248,984.76 for the commercial paper on the date of sale. If the investor holds onto the commercial paper for 41 more days (until maturity), the investor realizes a yield of 3.63% and receives \$250,000.	

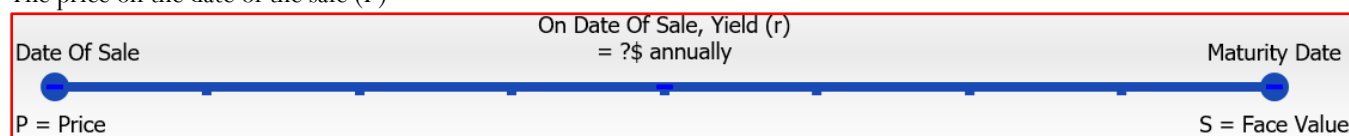


How It Works

Calculating a Rate of Return: Sometimes the unknown value when working with T-bills and commercial papers is the yield, or rate of return. In these cases, follow these steps to solve the problem:

Step 1: The face value, price, and time before maturity must be known. Draw a timeline if necessary, as illustrated below, and identify:

1. The face value (S)
2. The price on the date of the sale (P)



3. The number of days (t) remaining between the date of the sale and the maturity date. Count the first day but not the last day. Express the number of days annually so that the calculated yield will be annual.

Step 2: Apply Formula 8.3, $I = S - P$, to calculate the interest earned during the investment.

Step 3: Apply Formula 8.1, $I = Prt$, rearranging for r to solve for the interest rate (or yield or rate of return).

**TOP
SECRET**

Paths To Success

When you solve for yield, you will almost always find that the purchase price has been rounded to two decimals. This means the calculation of the yield is based on an imprecise number. The results are likely to show small decimals such as 3.4005%. In this example, the most likely yield is 3.40% with the 0.0005% appearing because of rounding. Yields on T-bills and commercial papers typically do not have more than two decimal places, which means that decimals in the third position or more are most likely caused by a rounding error and should be rounded off to two decimals accordingly.

Example 8.6C: Figuring Out Rates of Return for Multiple Investors

Marlie paid \$489,027.04 on the date of issue for a \$500,000 face value T-bill with a 364-day term. Marlie received \$496,302.21 when he sold it to Josephine 217 days after the date of issue. Josephine held the T-bill until maturity.

Determine the following:

- a. Marlie's actual rate of return
- b. Josephine's actual rate of return
- c. If Marlie held onto the T-bill for the entire 364 days instead of selling it to Josephine, what would his rate of return have been?
- d. Comment on the answers to (a) and (c).

Plan Calculate three yields or rates of return (r) involving Marlie and the sale to Josephine, Josephine herself, and Marlie without the sale to Josephine. Afterwards, comment on the rate of return for Marlie with and without the sale.

What You Already Know

Step 1: The present values, maturity value, and terms are known:

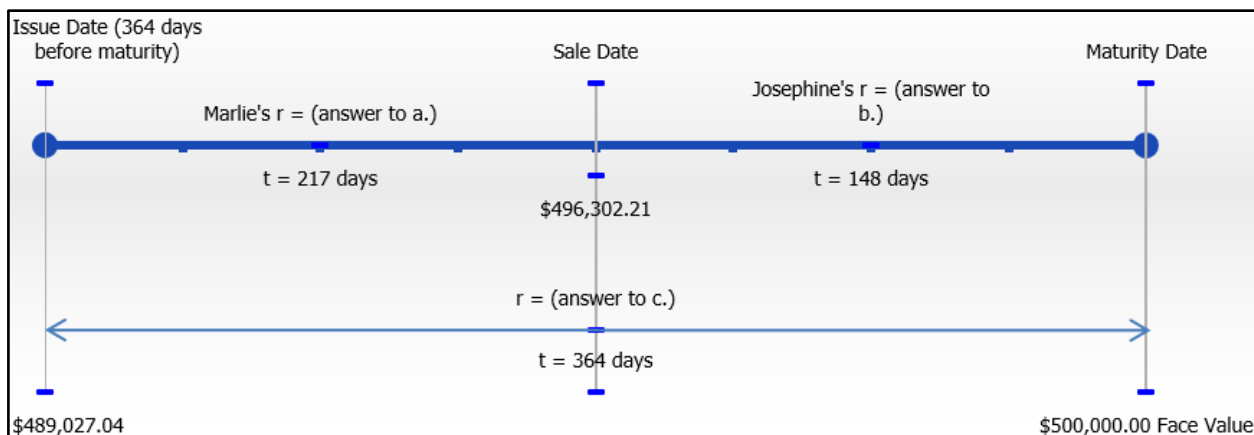
a. Marlie with sale	b. Josephine	c. Marlie without sale
$P = \$489,027.04$	$P = \$496,302.21$	$P = \$489,027.04$
$S = \$496,302.21$	$S = \$500,000.00$	$S = \$500,000.00$
$t = 217/365$	$364 - 217 = 147$ days remaining $t = 147/365$	$t = 364/365$

How You Will Get There

Step 2: For each situation, calculate the I by applying Formula 8.3.

Step 3: For each situation, apply Formula 8.1, rearranging for r .

Step 4: Compare the answers for (a) and (c) and comment.



	a. Marlie with sale to Josephine	b. Josephine by herself	c. Marlie without sale to Josephine
Step 2: Calculate I	$I = \$496,302.21 - \$489,027.04 = \$7,275.17$	$I = \$500,000.00 - \$496,302.21 = \$3,697.79$	$I = \$500,000.00 - \$489,027.04 = \$10,972.96$
Step 3: Calculate r	$r = \frac{\$7,275.17}{(\$489,027.04)\left(\frac{217}{365}\right)} = 2.50\%$	$r = \frac{\$3,697.79}{(\$496,302.21)\left(\frac{147}{365}\right)} = 1.85\%$	$r = \frac{\$10,972.96}{(\$489,027.04)\left(\frac{364}{365}\right)} = 2.25\%$
Present	When Marlie sold the T-bill after holding it for 217 days, he realized a 2.50% rate of return. Josephine then held the T-bill for another 148 days to maturity, realizing a 1.85% rate of return. If Marlie hadn't sold the note to Josephine and instead held it for the entire 364 days, he would have realized a 2.25% rate of return. Step 4: The yield on the date of issue was 2.25%. Marlie realized a higher rate of return because the interest rates in the market decreased during the 217 days he held it (to 1.85%, which is what Josephine is able to obtain by holding it until maturity). This raises the selling price of the T-bill. If his investment of \$489,027.04 grows by 2.25% for 217 days, he has \$6,541.57 in interest. The additional \$733.60 of interest (totalling \$7,275.17) is due to the lower yield in the market, increasing his rate of return to 2.50% instead of 2.25%.		



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Chapter 8 Summary

Key Concepts Summary

Section 8.1: Principal, Rate, Time (How Does Interest Work?)

- Calculating the amount of simple interest either earned or charged in a simple interest environment
- Calculating the time period when specific dates or numbers of days are involved
- Calculating the simple interest amount when the interest rate is variable throughout the transaction

Section 8.2: Moving Money Involving Simple Interest (Move and Nobody Gets Hurt)

- Putting the principal and interest together into a single calculation known as maturity value
- Altering a financial agreement and establishing equivalent payments

Section 8.3: Application: Savings Accounts and Short-Term GICs (Safe and Secure)

- How to calculate simple interest for flat-rate and tiered savings accounts
- How to calculate simple interest on a short-term GIC

Section 8.4: Application: Promissory Notes (A Promise Is a Promise)

- The characteristics of a promissory note
- Calculating the maturity value of a promissory note
- Selling a promissory note before its maturity date

Section 8.5: Application: Loans (The Bank Comes Knocking)

- Demand loans, their characteristics, and the common forms they take
- Establishing a repayment schedule for demand loans
- The characteristics of a student loan and how it is repaid

Section 8.6 Application: Treasury Bills and Commercial Papers (When Governments and Businesses Borrow)

- The characteristics of treasury bills
- The characteristics of commercial papers
- Calculating the price of T-Bills and commercial papers
- Calculating the yield of T-Bills and commercial papers

The Language of Business Mathematics

accrued interest Any interest amount that has been calculated but not yet placed (charged or earned) into an account.

commercial paper A short-term financial instrument with maturity no longer than one year that is issued by large corporations.

compound interest A system for calculating interest that primarily applies to long-term financial transactions with a time frame of one year or more; interest is periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.

current balance The balance in an account plus any accrued interest.

demand loan A short-term loan that generally has no specific maturity date, may be paid at any time without any interest penalty, and where the lender may demand repayment at any time.

discount rate An interest rate used to remove interest from a future value.

equivalent payments Two payments that have the same value on the same day factoring in a fair interest rate.

face value of a T-bill The maturity value of a T-bill, which is payable at the end of the term. It includes both the principal and interest together.

fixed interest rate An interest rate that is unchanged for the duration of the transaction.

focal date A single date that is chosen to locate all values in a financial scenario so that equivalent amounts can be determined.

future value The amount of principal with interest at a future point of time for a financial transaction. If this future point is the same as the end date of the financial transaction, it is also called the *maturity value*.

guaranteed investment certificate (GIC) An investment that offers a guaranteed rate of interest over a fixed period of time.

interest amount The dollar amount of interest that is paid or earned.

interest rate The rate of interest that is charged or earned during a specified time period.

legal due date of a note Three days after the term specified in an interest-bearing promissory note is the date when a promissory note becomes legally due. This grace period allows the borrower to repay the note without penalty in the event that the due date falls on a statutory holiday or weekend.

maturity date The date upon which a transaction, such as a promissory note, comes to an end and needs to be repaid.

maturity value The amount of money at the end of a transaction, which includes both the interest and the principal together.

present value The amount of money at the beginning of a time period in a transaction. If this is in fact the amount at the start of the financial transaction, it is also called the *principal*. Or it can simply be the amount at some time earlier before the future value was known. In any case, the amount excludes the interest.

prime rate An interest rate set by the Bank of Canada that usually forms the lowest lending rate for the most secure loans.

principal The original amount of money that is borrowed or invested in a financial transaction.

proceeds The amount of money received from a sale.

promissory note An unconditional promise in writing made by one person to another person to pay a sum of money on demand or at a fixed or determinable future time.

repayment schedule A table that details the financial transactions in an account, including the balance, interest amounts, and payments.

savings account A deposit account that bears interest and has no stated maturity date.

secured loan Those loans that are guaranteed by an asset such as a building or a vehicle that can be seized to pay the debt in case of default.

simple interest A system for calculating interest that primarily applies to short-term financial transactions with a time frame of less than one year.

student loan A special type of loan designed to help students pay for the costs of tuition, books, and living expenses while pursuing postsecondary education.

time period The length of the financial transaction for which interest is charged or earned. It may also be called the term.

treasury bills Short-term financial instruments with maturities no longer than one year that are issued by both federal and provincial governments.

unsecured loan Those loans backed up by the general goodwill and nature of the borrower.

variable interest rate An interest rate that is open to fluctuations over the duration of a transaction.

yield The percentage increase between the sale price and redemption price on an investment such as a T-bill or commercial paper.

The Formulas You Need to Know

Symbols Used

S = maturity value or future value in dollars

I = interest amount in dollars

P = principal or present value in dollars

r = interest rate (in decimal format)

t = time or term

Formulas Introduced

Formula 8.1 Simple Interest: $I = Prt$ (Section 8.1)

Formula 8.2 Simple Interest for Single Payments: $S = P(1 + rt)$ (Section 8.2)

Formula 8.3 Interest Amount for Single Payments: $I = S - P$ (Section 8.2)

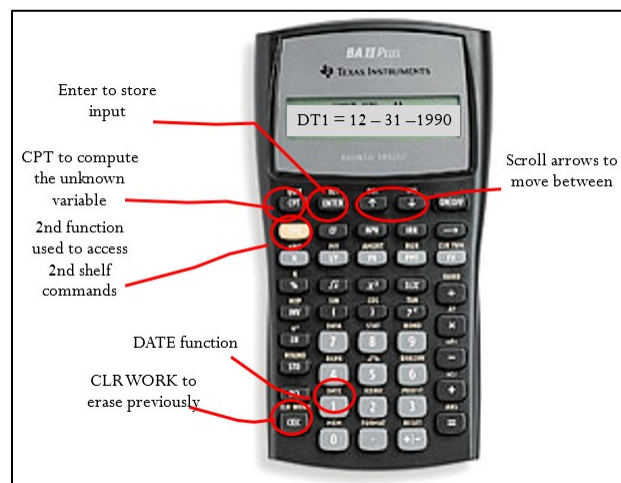
Technology

Calculator

The following calculator functions were introduced in this chapter:

Date Function

- 2nd Profit to access.
- Enter two of the three variables (DT1, DT2, DBD) by pressing Enter after each input and using $\uparrow$ and $\downarrow$ to scroll through the display. The variables are:
 - DT1 = The starting date of the transaction
 - DT2 = The ending date of the transaction
 - DBD = The days between the dates, counting the first day but not the last, which is the time period of the transaction.
 - $ACT / 360$ = A setting for determining how the calculator determines the DBD. In Canada, you should maintain this setting on ACT, which is the actual number of days. In other countries, such as the United States, they treat each year as having 360 days (the 360 setting) and each month as having 30 days. If you need to toggle this setting, press 2nd SET.
- Enter all dates in the format of MM.DDYY, where MM is the numerical month, DD is the day, and YY is the last two digits of the year. DD and YY must always be entered with both digits.
- Press CPT on the unknown (when it is on the screen display) to compute the answer.



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Chapter 9: Compound Interest: Working with Single Payments

(A Dollar Today Is Not Worth a Dollar Tomorrow)

Do you dream of owning a home? What about a car or a home theatre system? Big purchases such as these require long-term financial planning. Compound interest means that you will pay substantially more money for your purchases or earn more money on investments than you would with simple interest.

Compound interest is used for most transactions lasting one year or more. In simple interest, interest is converted to principal at the end of the transaction. Therefore, all interest is based solely on the original principal amount of the transaction. **Compound interest**, by contrast, involves interest being periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest. How significant is the interest when it compounds? Let's examine your big purchases from above:

- A home theatre system with the big-screen TV, Blu-Ray player, stereo surround sound, and more can retail for \$5,000. Did you know that if you put that amount on the retail store's 18% interest credit card and pay it off monthly for three years, you will pay over \$1,500 of compound interest?
- The average Canadian new car price is \$25,683.¹ Unless you have cash, you will join the ranks of other Canadians in taking six years to pay it off through \$450 monthly payments, resulting in almost \$7,000 of compound interest charges!
- The average Canadian home price is \$365,000,² though of course it varies widely across our country. If it takes 25 years to pay off your mortgage at 6% interest, your \$365,000 house becomes a \$700,000 acquisition thanks to compound interest. That is almost twice the original price!

And it is not any different for businesses. Whether they are local companies or multinational conglomerates they must invest and borrow at compound interest rates in their attempt to achieve long-term financial strategies. Some examples of these business activities include the following:

- Borrowing \$480,000 to open a new Tim Hortons' single restaurant franchise operation.³
- Spending \$1,000,000 on a fleet of rigs and semi-trailers for product distribution.
- Constructing new production plants or warehouses costing \$10,000,000 or more.

The money for these types of transactions does not appear out of thin air. It must be borrowed or taken from savings, and either approach involves compound interest.

Throughout the rest of this textbook, you will study compound interest as it relates to three distinct but interconnected concepts.

1. Calculating interest on a single amount (called a lump-sum amount or single payment). Chapter 9 introduces the basic mathematics, which are applied in Chapter 10.
2. Calculating interest on a series of regular, equal payments, called annuities. Chapter 11 introduces the basic mathematics, which are applied in Chapter 12.
3. Specialized applications including amortization, mortgages, bonds, sinking funds, and internal rates of return are covered in Chapters 13 through 15.

Now turn the page to get started with the basics.

¹ Statistics Canada, *New Motor Vehicle Sales, June 2011* (Catalogue no. 63-007-X, p. 15), accessed August 18, 2013, http://publications.gc.ca/collections/collection_2011/statcan/63-007-X/63-007-x2011006-eng.pdf.

² The Canadian Real Estate Association, "National Average Price Map," accessed September 27, 2013, <http://crea.ca/content/national-average-price-map>.

³ Tim Hortons, "Frequently Asked Questions," accessed May 13, 2010, http://www.timhortons.com/ca/en/join/franchise_ca_faq.html.

Outline of Chapter Topics

- 9.1: Compound Interest Fundamentals (What Can a Dollar Buy?)
- 9.2: Determining the Future Value (I Want to Pay Later)
- 9.3: Determining the Present Value (I Want to Pay Earlier)
- 9.4: Equivalent Payments (Let's Change the Deal)
- 9.5: Determining the Interest Rate (Is a Lot of Interest Accumulating or Just a Little?)
- 9.6: Equivalent and Effective Interest Rates (How Do I Compare Different Rates?)
- 9.7: Determining the Number of Compounds (How Far Away Is That?)

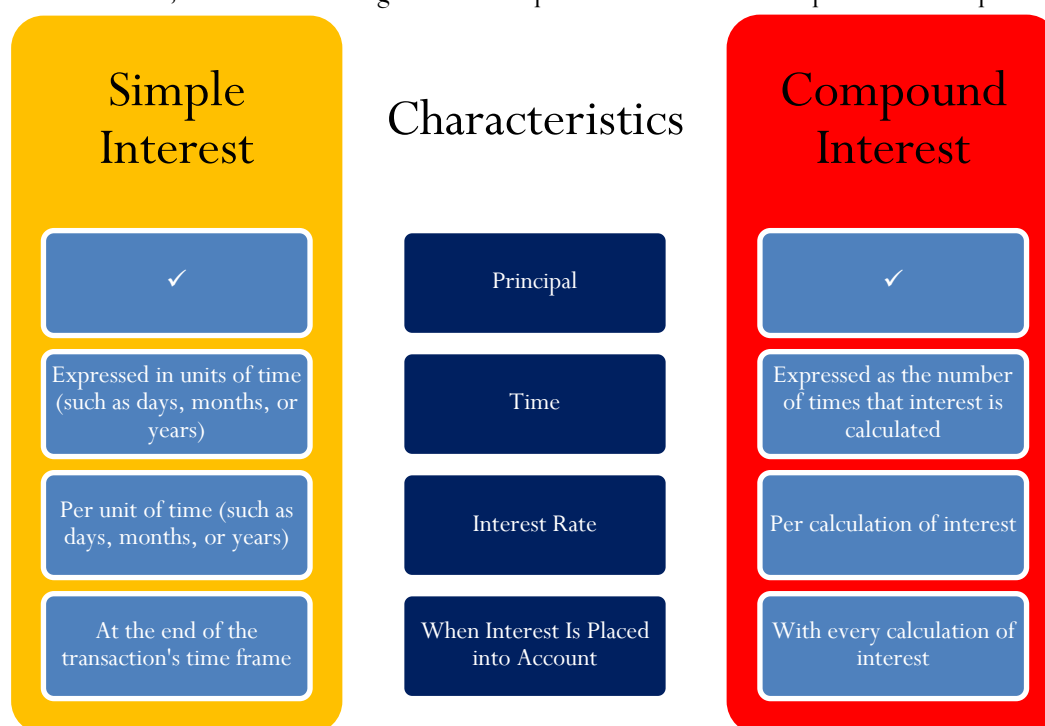
9.1: Compound Interest Fundamentals

(What Can a Dollar Buy?)

Five years ago you started a long-term GIC. The bank statement shows that you originally placed \$15,000 into the account earning 5.95% in annually compounded interest. You now have \$20,026.09. In the first year of the GIC, you earned \$892.50 in interest. Subsequent years earned annual interest amounts of \$945.60, \$1,001.87, \$1,061.48, and finally \$1,124.64. Of course, you like that the interest amount increased each year even though you did not make any more deposits to the GIC, but why did this happen?

The Concept of Compounding

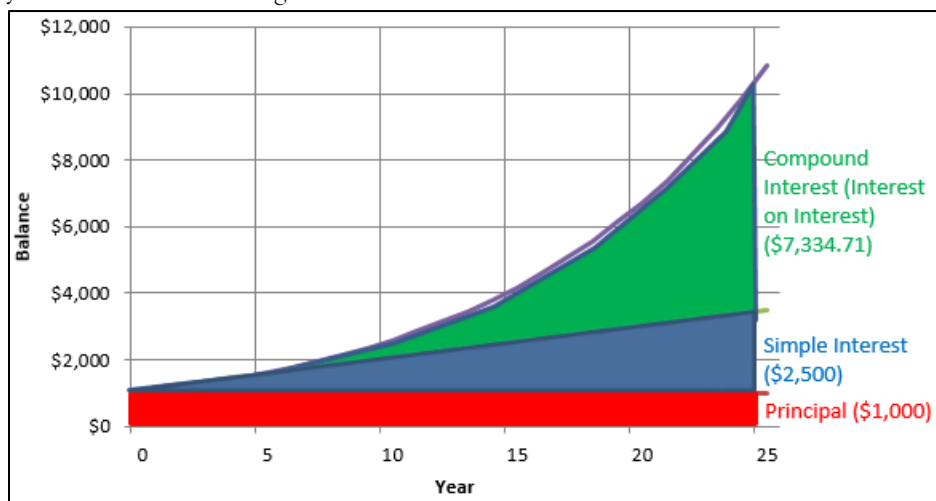
Simple interest, compound interest—what is the big difference? As you can see in the figure below, simple and compound interest calculations share the same fundamentals of time, interest rate, and placing interest into the account. But the accumulation is not the same, and over time the growth of compound interest will far outpace that of simple interest.



In both forms of interest, the principal is the starting amount that accumulates interest for a length of time at a specified interest rate. But you calculate simple interest in direct proportion to the starting amount, the interest rate, and the time period, whereas the calculation for compound interest is, well, not so simple!

The critical difference is the placement of interest into the account. Under simple interest, you convert the interest to principal at the end of the transaction's time frame. For example, in a six-month simple interest GIC the balance in its account at any point before the maturity date is the original principal and nothing more. Only upon maturity does the interest appear. In contrast, a five-year compound interest GIC such as the one discussed in the section opener receives an interest deposit annually. After one year the principal increases from \$15,000 to \$15,892.50. This higher principal in the second year explains why the interest earned in the second year then increases as well.

Invest \$1,000 for 25 years at both 10% simple interest and 10% interest compounded annually. As demonstrated in the figure and table below, \$1,000 invested at 10% per year of simple interest has a \$3,500 balance after 25 years. This consists of the original \$1,000 principal plus \$2,500 in total interest (\$100 for each year the money was invested). However, \$1,000 invested at 10% compounded annually has a balance of \$10,834.71, which is \$7,334.71 more! This difference is the result of interest being converted to principal and thus earning even more interest for you. A close examination of the first two years reveals the following:



- In the first year, both investments have a \$1,100 balance. With simple interest, the \$100 of interest remains as accrued interest and is not placed into the account while the principal remains at \$1,000. With compound interest, the \$100 of interest is converted to principal, resulting in a \$1,100 principal for the second year.
- In the second year, the simple interest account still has a \$1,000 principal, which earns another \$100 of interest ($\$1,000 \times 10\% = \$100$). The account now

has \$1,000 of principal plus \$200 in accrued interest. The compound interest account earns 10% on the new principal of \$1,100, or \$110 of interest ($\$1,100 \times 10\% = \$110$). This interest is placed into the account at the end of year two, making the principal \$1,210 for the third year . . . and so on.

End of Year	At 10% Simple Interest			At 10% Compound Interest		
	Principal	Accrued Interest	Balance	Principal	Accrued Interest	Balance
Start						\$1,000.00
1	\$1,000.00	\$100.00	\$1,100.00	\$1,000.00	\$100.00	\$1,100.00
2	\$1,000.00	\$200.00	\$1,200.00	\$1,100.00	\$110.00	\$1,210.00
3	\$1,000.00	\$300.00	\$1,300.00	\$1,210.00	\$121.00	\$1,331.00
4	\$1,000.00	\$400.00	\$1,400.00	\$1,331.00	\$133.10	\$1,464.10
5	\$1,000.00	\$500.00	\$1,500.00	\$1,464.10	\$146.41	\$1,610.51
						
10	\$1,000.00	\$1,000.00	\$2,000.00	\$2,357.95	\$235.79	\$2,593.74
						
15	\$1,000.00	\$1,500.00	\$2,500.00	\$3,797.50	\$379.75	\$4,177.25
						
20	\$1,000.00	\$2,000.00	\$3,000.00	\$6,115.91	\$611.59	\$6,727.50
						
25	\$1,000.00	\$2,500.00	\$3,500.00	\$9,849.73	\$984.98	\$10,834.71

Ultimately, as the principal in the compound interest account increases, the amount of interest earned also increases. Observe that the difference in the balance between simple interest and compound interest in the first five years is slight, but the gap widens over time.

Compound Interest Rates

If an equal amount of principal invested for 25 years earns an interest rate of 9% compounded annually versus 3% compounded annually, would the interest accumulated by the 9% investment be three times as large?

In a *simple interest* environment, this would be true since the principal never changes: in each of the 25 years the amounts of simple interest would be constant, and the amount for the 9% investment would be three times larger than for the 3% investment. However, in *compound interest* each subsequent year's interest is compounded on an increasingly larger principal. This chart shows \$1,000 invested at five different annually

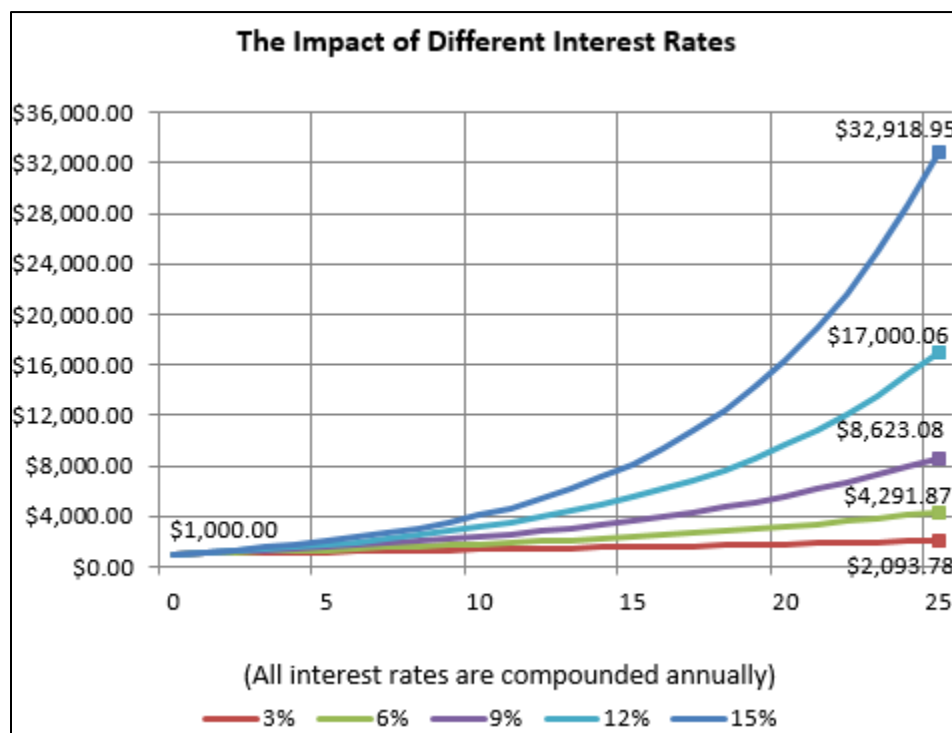
compounded rates. Observe the following after 25 years:

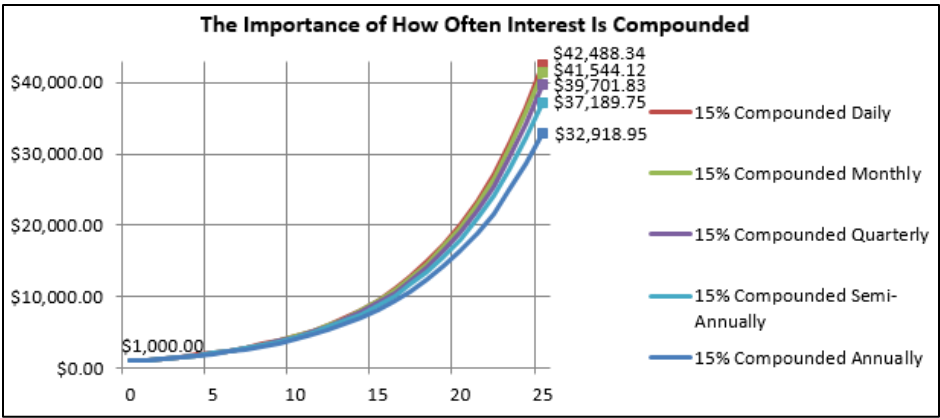
- The investment at 9% compounded annually has a balance of \$8,623.08 compared to \$2,093.78 for the 3% compounded annually. Therefore, the interest earned is \$7,623.08 versus \$1,093.78, reflecting a ratio approximating 7 : 1.
- Comparing the 15% interest rate versus the 6% interest rate, many people imagine the final balances would be in the same ratio as the ratio of the rates, 15% : 6%, which simplifies to $2\frac{1}{2} : 1$. However, in the chart, the final balances are \$32,918.95 versus \$4,291.87, reflecting a ratio approximating $7\frac{2}{3} : 1$!

These examples illustrate that in compound interest scenarios, the ratio of interest earned is not directly proportionate to the numerical ratio of the interest rates. Higher interest rates result in higher interest amounts, therefore yielding more principal upon which future interest is earned. Over a long time period, this growth in the balance is exponential.

How Often Interest Compounds

The examples so far have involved compound interest that has been compounded annually—the accrued interest is being converted to principal at the end of every year. But this is not the only option. Interest can be converted to principal at any frequency, including daily, weekly, monthly, quarterly (every three months), or semi-annually (every six months). Under any of these options the principal increases more frequently, which in turn results in more interest being earned.





The chart to the left illustrates the impact of different compounding frequencies. In the case of monthly versus annual compounding, the monthly compound increases the account balance every month whereas the annual compound increases the balance once every 12 months. To understand why the monthly compound earns more interest, note that after each month the principal increases and therefore

earns more interest. Each of the 12 increases is small on its own, but the cumulative effect is large: by having the interest deposited every month instead of every year, over the course of 25 years an additional $\$41,544.12 - \$32,918.95 = \$8,625.17$ in interest is earned!

TOP SECRET **Paths To Success**

The table below summarizes the desirable characteristics when either investing or borrowing.

Investing	Characteristic	Borrowing
Compound	Type of Interest	Simple
You want your interest to earn you more interest.		You don't want interest being converted to principal.
	Interest Rate	
The higher the rate, the more you earn.		The lower the rate, the less you pay.
	Compounding	
The more often it compounds, the more principal that can earn interest. A daily compound would be best!		The less often it compounds, the less principal that can earn interest. An annual compound would be best!
	Time (or Term)	
As the principal continually grows, you earn more interest. Longer time frames are desirable.		You don't want the principal to grow and earn more interest. Short time frames are desirable.

Calculating the Periodic Interest Rate

The first step in learning about investing or borrowing under compound interest is to understand the interest rate used in converting interest to principal. You commonly need to convert the posted interest rate to find the exact rate of interest earned or charged in any given time period.



The Formula

Formula 9.1 involves four key concepts, which are explained next, to do with conversion of the interest rate.

Formula 9.1 - Periodic Interest Rate: $i = \frac{IY}{CY}$

CY is Compounds Per Year (Compounding Frequency): The determination of CY involves two of the key concepts:

Key Concept #1: The Compounding Period. The amount of time that elapses between the dates of successive conversions of interest to principal is known as the **compounding period**. For example, a quarterly compounded interest rate converts interest to principal every three months. Therefore, the compounding period is three months.

Key Concept #2: The Compounding Frequency. The number of compounding periods in a complete year is known as the **compounding frequency**. For example, a quarterly compounded interest rate compounds every three months, or $CY = 4$ times in a single year.

IY is Nominal Interest Rate Per Year: Key Concept #3: The Nominal Interest Rate. A compound interest rate consists of two elements: a nominal number for the annual interest rate, known as the **nominal interest rate**, and words that state the compounding frequency. For example, a 12% compounded quarterly interest rate is interpreted to mean that you accumulate 12% nominal interest per year but the interest is converted to principal every compounding period, or every three months. The word *nominal* is used because if compounding occurs more than once per year, the true amount of interest that you earn per year is greater than the nominal interest rate. Section 9.6 will help you calculate just how much more.

i is Periodic Interest Rate: Key Concept #4: The Periodic Interest Rate. The percentage of interest earned or charged at the end of each compounding period is called the **periodic interest rate**. You calculate it by taking the nominal interest rate and dividing by the compounding frequency. Continuing with the example of 12% compounded quarterly, it has a periodic interest rate of $12\% \div 4 = 3\%$. This means that at the end of every three months, you calculate 3% interest and convert it to principal.



How It Works

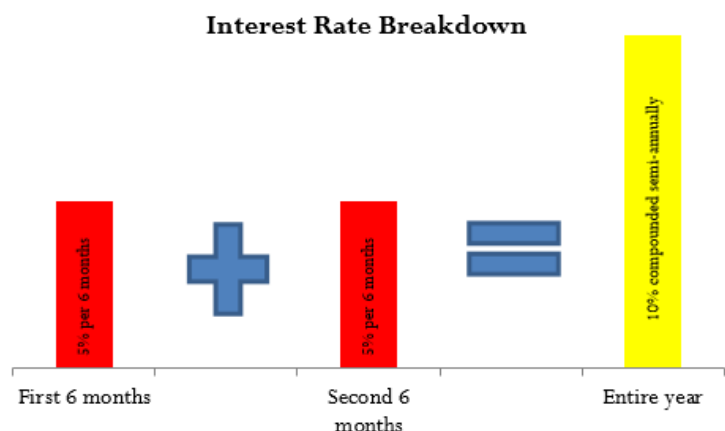
To solve any question involving the periodic interest rate, follow these steps:

Step 1: Identify two of the three key variables—the nominal interest rate (IY), compounding frequency (CY), or periodic interest rate (i).

Step 2: Substitute into Formula 9.1, rearranging if needed, and solve for the unknown variable. If you calculate the compound frequency (CY), you must convert the number back into the compounding words associated with the frequency. For example, $CY = 2$ means twice per year and is stated as “semi-annually.”

An example involving the calculation of the periodic interest rate for “10% compounded semi-annually” illustrates these steps.

Step 1: The wording “semi-annually” means the compounding period is every six months. One year



contains two such compounding periods, making the compounding frequency twice per year, or $CY = 2$. The nominal annual interest rate is 10%, or $IY = 10\%$.

Step 2: Applying Formula 9.1, calculate the periodic interest rate as $i = 10\% \div 2 = 5\%$. The figure illustrates that every six months 5% interest is converted to principal. You could look at the nominal interest rate for the year, 10%, as the total of the periodic interest rates for each of the two periods. But remember, unlike with simple interest, the actual interest amount here will increase from one period to the next.



Important Notes

When you express a compound interest rate, you must always state the words for the compounding frequency along with the nominal annual number. For example, you must say “10% compounded semi-annually” and not just “10%.” In the absence of an explicit compounding frequency, the number is interpreted by default to mean “compounded annually” except if an industry standard dictates otherwise. For example, in the case of mortgages in Canada, the default is semi-annual compounding, so when you hear of a 10% mortgage, you should assume it to be 10% compounded semi-annually. But in most industries “10%” with no words generally means “10% compounded annually.”



Things To Watch Out For

It is common to confuse the compounding period and the compounding frequency. The table below shows the relationship between compounding periods and frequencies. Remember that to calculate the periodic interest rate you need the compounding frequency, not the compounding period.

Common Compounds	Compounding Period	Compounding Frequency (CY)
Annually	Every year	1
Semi-annually	Every 6 months	2
Quarterly	Every 3 months	4
Monthly	Every 1 month	12
Weekly	Every 1 week	52*
Daily	Every day	365*

*Note that although there are not exactly 52 weeks in a year and there are 366 days in a leap year, it is common practice to use these values for compounding frequency.



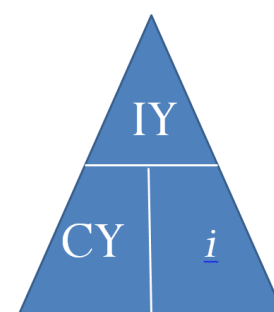
Paths To Success

For ease of manipulation, you can remember the rearrangement of Formula 9.1 using the triangle technique once again. Recall that as long as you know any two pieces of the puzzle, you can solve for the third.



Give It Some Thought

1. If you are investing money, what interest characteristics should you seek out?
2. If you are borrowing money, what interest characteristics should you seek out?
3. Of 10% compounded semi-annually or 10% compounded monthly, which earns more interest?
4. If you invested a principal at the same compounded interest rate for 20 years, would the amount of interest earned in the second 10 years be less than twice, exactly twice, or more than twice the interest earned during the first 10 years?
5. John has an interest rate of 4% compounded annually while Dwayne is earning 8% compounded annually. If both people placed the same principal into their investments for 25 years, would Dwayne have twice, less than twice, or more than twice as much money as John?
6. Of 10% simple interest or 10% interest compounded annually, which earns more interest over time?
7. Of the periodic interest rate or the nominal interest rate, which is always at least as large as the other?



- Solutions:*
1. High interest rate, high compounding frequency, long time frames, compound interest.
 2. Low interest rate, low compounding frequency, short time frames, simple interest.
 3. 10% monthly, as demonstrated in Figure 9.4.
 4. More than twice, as demonstrated in Figure 9.3.
 5. More than twice, as demonstrated in Figure 9.3.
 6. 10% compounded annually, as demonstrated in Figure 9.2.
 7. The nominal interest rate.

Example 9.1A: The Periodic Interest Rate

Calculate the periodic interest rate for the following nominal interest rates:

- a. 9% compounded monthly b. 6% compounded quarterly

Plan	For each question, calculate the periodic interest rate (i).	
Understand	What You Already Know Step 1: For each question, use the following nominal interest rate and compounding frequency: a. $IY = 9\%$ $CY = \text{monthly} = 12 \text{ times per year}$ b. $IY = 6\%$ $CY = \text{quarterly} = 4 \text{ times per year}$	How You Will Get There Step 2: For each question apply Formula 9.1.
Perform	a. $i = \frac{9\%}{12} = 0.75\% \text{ per month}$ b. $i = \frac{6\%}{4} = 1.5\% \text{ per quarter}$	
Present	a. Nine percent compounded monthly is equal to a periodic interest rate of 0.75% per month. This means that interest is converted to principal 12 times throughout the year at the rate of 0.75% each time. b. Six percent compounded quarterly is equal to a periodic interest rate of 1.5% per quarter. This means that interest is converted to principal 4 times (every three months) throughout the year at the rate of 1.5% each time.	

Example 9.1B: The Nominal Interest Rate

Calculate the nominal interest rate for the following periodic interest rates:

- a. $0.58\bar{3}\%$ per month b. 0.05% per day

Plan	For each question, calculate the nominal interest rate (IY).	
Understand	What You Already Know Step 1: For each question, use the following periodic interest rate and compounding frequency: a. $i = 0.58\bar{3}\%$ $CY = \text{monthly} = 12 \text{ times per year}$ b. $i = 0.05\%$ $CY = \text{daily} = 365 \text{ times per year}$	How You Will Get There Step 2: For each question, apply Formula 9.1 and rearrange for IY.
Perform	a. $0.58\bar{3}\% = \frac{IY}{12}$ $IY = 0.58\bar{3}\% \times 12 = 7\% \text{ compounded monthly}$ b. $0.05\% = \frac{IY}{365}$ $IY = 0.05\% \times 365 = 18.25\% \text{ compounded daily}$	
Present	a. A periodic interest rate of $0.58\bar{3}\%$ per month is equal to a nominal interest rate of 7% compounded monthly. b. A periodic interest rate of 0.05% per day is equal to a nominal interest rate of 18.25% compounded daily.	

Example 9.1C: Compounds per Year

Calculate the compounding frequency for the following nominal and periodic interest rates:

- a. nominal interest rate = 6%, periodic interest rate = 3%
 b. nominal interest rate = 9%, periodic interest rate = 2.25%

Plan	For each question, calculate the compounding frequency (CY) and convert the calculated number to words.	
Understand	What You Already Know Step 1: For each question, use the following nominal and periodic interest rates: a. $NY = 6\%$ $i = 3\%$ b. $NY = 9\%$ $i = 2.25\%$	How You Will Get There Step 2: For each question, apply Formula 9.1 and rearrange for CY.
Perform	a. $3\% = \frac{6\%}{CY}$ $CY = \frac{6\%}{3\%} = 2$ compounds per year = semi-annually b. $2.25\% = \frac{9\%}{CY}$ $CY = \frac{9\%}{2.25\%} = 4$ compounds per year = quarterly	
Present	a. For the nominal interest rate of 6% to be equal to a periodic interest rate of 3%, the compounding frequency must be twice per year, which means a compounding period of every six months, or semi-annually. b. For the nominal interest rate of 9% to be equal to a periodic interest rate of 2.25%, the compounding frequency must be four times per year, which means a compounded period of every three months, or quarterly.	

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9.2: Determining the Future Value (I Want To Pay Later)

Your company has an employee assistance plan through which employees can borrow funds at 12% compounded semi-annually, much like a loan from a bank, then repay the money at their convenience. An employee who borrowed \$4,000 two years ago has been unable to repay the loan until today. As the human resources manager in charge of the assistance plan, you must tell him the exact amount he needs to pay to return his balance to zero. How do you do this?

Now that you know how to calculate the periodic interest rate, you can compute compound interest. This chapter focuses on single amounts, also called lump-sum amounts. This knowledge will form the basis for you to work with compound interest on a series of payments, which will be covered in Chapters 11 through 15. Investing the time now to understand these fundamentals will reap dividends as you proceed in your studies of business mathematics.

Future Value Calculations with No Variable Changes

The simplest future value scenario for compound interest is for all of the variables to remain unchanged throughout the entire transaction. To understand the derivation of the formula, continue with the opening scenario. If the money was borrowed two years ago, then the employee will owe two years of compound interest in addition to the original principal of \$4,000. That means $PV = \$4,000$. The compounding frequency is semi-annually, or twice per year, which makes the periodic interest rate $i = \frac{12\%}{2} = 6\%$. Therefore, after the first six months, your employee has 6% interest converted to principal. This a future value, or FV, calculated as follows:

$$\begin{aligned}\text{Principal after one compounding period (six months)} &= \text{Principal plus interest} \\ \text{FV} &= PV + i(PV) \\ &= \$4,000 + 0.06(\$4,000) \\ &= \$4,000 + \$240 = \$4,240\end{aligned}$$

Now proceed to the next six months. The future value after two compounding periods (one year) is calculated in the same way. Note that the equation $FV = PV + i(PV)$ can be factored and rewritten as $FV = PV(1 + i)$.

$$FV \text{ (after two compounding periods)} = PV(1 + i) = \$4,240(1 + 0.06) = \$4,240(1.06) = \$4,494.40$$

Since the $PV = \$4,240$ is a result of the previous calculation where $PV(1 + i) = \$4,240$, the following algebraic substitution is possible:

$$FV = PV(1 + i)(1 + i) = \$4,000(1.06)(1.06) = \$4,240(1.06) = \$4,494.40$$

Applying exponent rules from Section 2.4 and simplifying it algebraically, you get:

$$FV = PV(1 + i)(1 + i) = PV(1 + i)^2$$

Do you notice a pattern? With one compounding period, the formula has only one $(1 + i)$. With two compounding periods involved, it has two factors of $(1 + i)$. Each successive compounding period multiplies a further $(1 + i)$ onto the equation. This makes the exponent on the $(1 + i)$ exactly equal to the number of times that interest is converted to principal during the transaction.



The Formula

First, you need to know how many times interest is converted to principal throughout the transaction. You can then calculate the future value. Use Formula 9.2 to determine the number of compound periods involved in the transaction

N is Number of Compound Periods: This is the measure of time. Calculating the maturity value requires knowing how many times interest is converted to principal throughout the transaction. Whereas simple interest expresses time in years, note that compound interest requires time to be expressed in periods.

Formula 9.2 – Number of Compound Periods For Single Payments: $N = CY \times \text{Years}$

CY is Compounds per year (Compound Frequency): Recall that the compounding frequency represents how many compound periods fall within a single year. The words that accompany the nominal interest rate determine this number.

Years is Number of Years in Transaction (The Term): Multiply the compounding frequency by the term of the transaction, expressed as a number of years. Express partial years as mixed fractions. For example, 1 year and 9 months is $1\frac{9}{12}$ years.

Once you know N, substitute it into Formula 9.3, which finds the amount of principal and interest together at the end of the transaction, or the maturity value.

FV is Future Value or Maturity Value: This is the principal and compound interest together at the future point in time.

PV is Present Value or Principal: This is the starting amount upon which compound interest is calculated. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some earlier point in time. In any case, the amount excludes the future interest.

Formula 9.3 – Compound Interest For Single Payments: $FV = PV(1 + i)^N$

i is Periodic Interest Rate: From Formula 9.1, the principal accrues interest at this rate every compounding period. For accuracy, you should never round this number.

N is Number of Compound Periods: From Formula 9.2, this is the total number of compound periods involved in the term of the transaction.

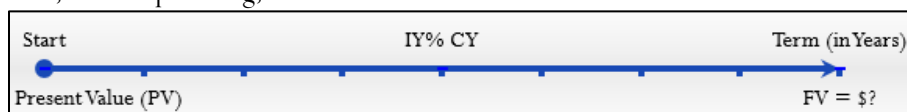
The $(1+i)^N$ performs the actual compounding of the money. The $(1 + i)$ determines the percent increase in the principal while the exponent (N) compounds the increase an appropriate number of times.



How It Works

Follow these steps to calculate the future value of a single payment:

Step 1: Read and understand the problem. If necessary, draw a timeline similar to the one here identifying the present value, the nominal interest rate, the compounding, and the term.



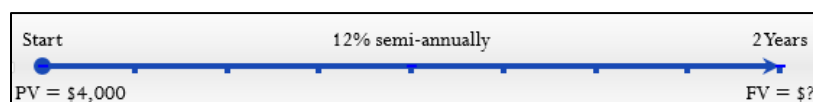
Step 2: Calculate the periodic interest rate (i) from Formula 9.1.

Step 3: Calculate the total number of compound periods (N) from Formula 9.2.

Step 4: Solve Formula 9.3.

Revisit the employee who had \$4,000 outstanding for two years with interest at 12% compounded semi-annually.

Step 1: Calculate the amount of the loan after two years (FV). Observe that $PV = \$4,000$, $IY = 12\%$, $CY = 2$ (every six months or twice per year), and $Years = 2$.



Step 2: According to Formula 9.1, $i = \frac{12\%}{2} = 6\%$. Thus, interest at a rate of 6% is converted to principal at the end of each compounding period of six months.

Step 3: Applying Formula 9.2, $N = CY \times Years = 2 \times 2 = 4$. Interest is converted to principal four times over the course of the two-year term occurring at the 6, 12, 18, and 24 month marks.

Step 4: Calculate the maturity value using Formula 9.3:

$$FV = \$4,000(1 + 0.06)^4 = \$5,049.91$$

To pay off the loan the employee owes \$5,049.91.



Important Notes

Calculating the Interest Amount: In any situation of lump-sum compound interest, you can isolate the interest amount using an adapted Formula 8.3:

$$I = S - P \text{ becomes } I = FV - PV$$

Using your employee's \$4,000 loan with a future repayment of \$5,049.91, the interest paid is calculated as

$$I = \$5,049.91 - \$4,000.00 = \$1,049.91$$

This formula applies only to compound interest situations involving lump-sum amounts. If regular payments are involved, this is called an annuity, for which a modified version of the interest formula will be introduced in Chapter 11.

The BAII Plus Calculator: Your BAII Plus calculator is a business calculator pre-programmed with compound interest formulas. These functions are called the “time value of money” buttons. The compound interest buttons are found in two areas of the calculator, as shown in the photo.

1. The five buttons located on the third row of the calculator are five of the seven variables required for time value of money calculations. This row's buttons are different in colour from the rest of the buttons on the keypad. The table on the next page relates each button to formula symbols used in this text along with what each button represents and any data entry requirements.

Frequency Functions
Time Value of Money Buttons



Calculator Symbol	Formula Symbol	Characteristic	Data Entry Requirements
N	N	The number of compounding periods (from Formula 9.2)	An integer or decimal number; no negatives
I/Y	IY	The nominal interest rate per year	Percent format without the % sign (i.e., 7% is just 7)
PV	PV	Present value or principal	An integer or decimal number
PMT	PMT	Used for annuity payment amounts (covered in Chapter 11) and is not applicable to lump-sum amounts; it needs to be set to zero for lump-sum calculations	An integer or decimal number
FV	FV	Future value or maturity value	An integer or decimal number

To enter any information into any one of these buttons, key in the data first and then press the appropriate button. For example, if you want to enter $N = 34$, then key in 34 followed by pressing N.

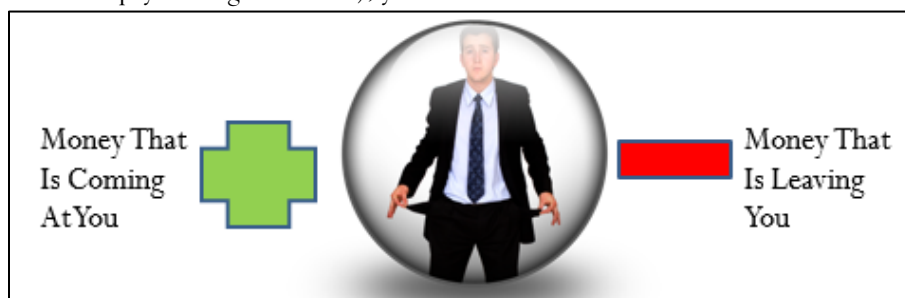
- Since frequency and interest rates are related, as shown in Formula 9.1, the frequency function is logically placed above the I/Y button and it is labelled P/Y. This function addresses compound interest frequencies, such as the compounding frequency. Access the function by pressing 2nd P/Y to find the entry fields shown in the table below, which you can scroll through using the $\downarrow$ and $\uparrow$ arrow buttons. To exit the P/Y window, press 2nd Quit.

Calculator Symbol	Formula Symbol	Characteristic	Data Entry Requirements
P/Y	PY	Annuity payments per year (payment frequency is introduced in Chapter 11); when working with lump-sum payments and not annuities, the calculator requires this variable to be set to match the C/Y	A positive, nonzero number only
C/Y	CY	Compounds per year (compounding frequency)	A positive, nonzero number only

The table relates each window variable to the formula symbols along with what each button represents and its data entry requirements. To enter information into these variables, key in the data first and then press Enter. For example, to enter a compounding frequency of 4, press 4 while C/Y is on your screen and then press Enter. Most commonly the P/Y and C/Y are the same number, as demonstrated in later chapters. Therefore, the calculator's built-in shortcut feature automatically copies any value entered into the P/Y variable to the C/Y variable. If the P/Y and C/Y are not the same, you can scroll down and re-enter the C/Y as needed.

BAlI Plus Cash Flow Sign Convention. An investment and a loan are very different. An investment earns interest and the principal increases over time. A loan is charged interest but is usually paid off through payments, resulting in the principal decreasing over time. Notice that nowhere on the calculator is there a button to enter this critical piece of information. How does the calculator distinguish between the two? You must apply a principle called the cash flow sign convention to the PV, PMT, and FV buttons:

- If you have money leaving your possession and going somewhere else (such as being put into an investment or making a payment against a loan), you must enter the number as a **NEGATIVE** number.



- If you have money coming into your possession and you are receiving it (such as borrowing money from the bank or having an investment mature and pay out to you), you must enter the number as a **POSITIVE** number.

When doing financial calculations it is important to “be somebody” in the transaction. In any compound interest

scenario, two parties are always involved—somebody is investing and somebody is borrowing. From this standpoint, think about whether the money leaves you or comes at you. This will help you place the correct sign in front of the PV, PMT, and FV when using your calculator. Who you are will not change the numbers of the transaction, just the cash flow sign convention.

- If you borrow money from your friend and then pay it back, the initial loan is received by you and hence entered as a positive PV for you. As you pay back the loan, money leaves you and therefore the FV is negative for you.
- Taking the other side of the coin and being your friend, the loaning of money is a negative PV for him. When you repay the loan, your friend receives it and therefore results in a positive FV for him.

Notice that the PV and FV always have opposite signs. Investments must always mature, providing a payback to the investor. Loans must always be repaid.

BAlI Plus Memory: Your calculator has permanent memory. Once you enter data into any of the time value buttons it is permanently stored until

- You override it by entering another piece of data and pressing the button;

- You clear the memory of the time value buttons by pressing 2nd CLR TVM before proceeding with another question; or
- The reset button on the back of the calculator is pressed.

This permanent storage is an advantage in multistep compound interest problems. If any piece of information remains constant from step to step, you need to enter it only once.

BAlI Plus Keying in a Question: Seven variables are involved with compound interest. To solve any compound interest question, you must key in six of them. To solve for the missing variable, press CPT followed by the variable. For example, if solving for future value, press CPT FV.

Returning to the employee loan example where a \$4,000 loan was taken for two years at 12% interest compounded semi-annually, recall that $N = 4$, $I/Y = 12$, $PV = \$4,000$, and $CY = 2$. Assuming the role of the employee, you would key in the problem in the following sequence:

- 2nd CLR TVM (clear the memory; this is not required, but it reduces the chance of using stale data)
- 4 N (total number of compounding periods)
- 12 I/Y (the nominal interest rate)
- 4000 PV (a positive since the employee received the money from the company)
- 0 PMT (you will not use this button until you get to annuities in Chapter 11)
- 2nd P/Y (to open the frequency window)
- 2 Enter (the P/Y and C/Y are the same when working with lump-sum amounts; by entering the C/Y here, you simultaneously set both variables to the same number)
- 2nd Quit (to close the window)
- CPT FV

Answer: -5,049.91 (it is negative since the employee owes this money)



Things To Watch Out For

The most common error in the application of Formula 9.3 is to substitute the nominal interest rate for the periodic interest rate. Hence, for 12% compounded semi-annually you might inadvertently use a mistaken value of $i = 0.12$ instead of the appropriate $i = 0.06$. Formula 9.3 encompasses Formulas 9.1 and 9.2. If you write Formula 9.3 without requiring any substitution it appears as follows:

$$FV = PV \left(1 + \frac{IY}{CY} \right)^{(CY \times \text{Years})}$$

Although you could use this equation instead of the one presented in Formula 9.3, most students find it best to use the sequence of three formulas. This requires the systematic approach involved in steps 2 to 4 of the process:

1. Calculate i .
2. Calculate N .
3. Calculate FV.

Use the phrase “iN the Future” to remember this process.



Paths To Success

When you compute solutions on the BAlI Plus calculator, one of the most common error messages displayed is "Error 5." This error indicates that the cash flow sign convention has been used in a manner that is financially impossible. Some examples of these financial impossibilities include loans with no repayment or investments that never pay out. In these cases, the PV and FV have been incorrectly set to the same cash flow sign.



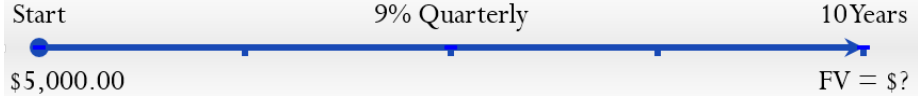
Give It Some Thought

1. Using the concept of “number of compound periods,” explain why a five-year investment earning 8% compounded monthly has a higher maturity value than an investment earning 8% compounded annually.
2. Explain why Formula 9.3, $FV = PV(1 + i)^N$, places the number of compound periods into the exponent.

- Solutions:*
1. The 8% compounded monthly investment realizes 60 compound periods of interest over the five years, while the 8% compounded annually investment realizes only five compound periods. With interest going in much more often, the principal is increasingly larger and therefore earns more interest.
 2. Each compound period increases the existing principal by the periodic rate through multiplication. Since the periodic interest rate in each compound period multiplies onto the existing principal amount, you apply exponent rules for adding exponents with the same base. The total number of increases in principal equals the number of compound periods in the transaction. See pages xxx–xxx for a full explanation.

Example 9.2A: Making an Investment

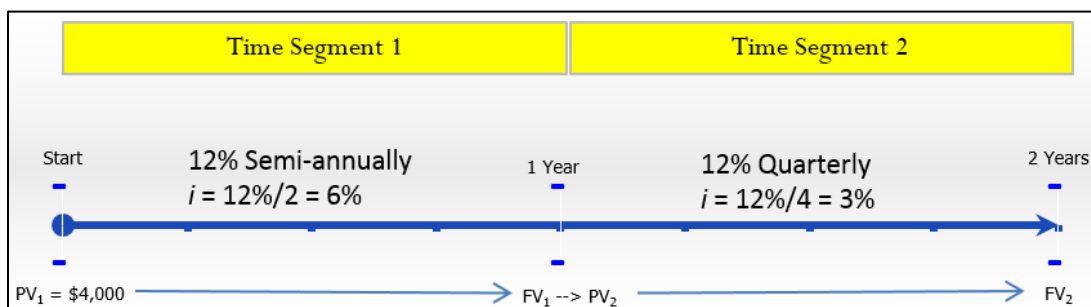
If you invested \$5,000 for 10 years at 9% compounded quarterly, how much money would you have?

Plan	Calculate the principal and the interest together, which is called the maturity value (FV).																				
Understand	What You Already Know Step 1: The principal, interest rate, and term, as illustrated in the timeline, are known.			How You Will Get There Step 2: Calculate the periodic interest by applying Formula 9.1. Step 3: Calculate the number of compound periods by applying Formula 9.2. Step 4: Calculate the maturity value by applying Formula 9.3.																	
																					
Perform	Step 2: $i = \frac{9\%}{4} = 2.25\% = 0.0225$ Step 3: $N = 4 \times 10 = 40$ Step 4: $FV = \$5,000(1 + 0.0225)^{40} = \$12,175.94$			Calculator Instructions <table border="1"><thead><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr></thead><tbody><tr><td>40</td><td>9</td><td>-5,000</td><td>0</td><td>Answer: \$12,175.94</td><td>4</td><td>4</td></tr></tbody></table>				N	I/Y	PV	PMT	FV	P/Y	C/Y	40	9	-5,000	0	Answer: \$12,175.94	4	4
N	I/Y	PV	PMT	FV	P/Y	C/Y															
40	9	-5,000	0	Answer: \$12,175.94	4	4															
Present	After 10 years, the principal grows to \$12,175.94, which includes your \$5,000 principal and \$7,175.94 of compound interest.																				

Future Value Calculations with Variable Changes

What happens if a variable such as the nominal interest rate, compounding frequency, or even the principal changes somewhere in the middle of the transaction? Formula 9.3 produces the correct final answer only when all variables remain unchanged. To illustrate this situation, assume your company modified its employee assistance plan one year after the money was borrowed, changing the interest rate in the second year from 12% compounded semi-annually to 12% compounded quarterly. Now how do you calculate the future value?

When any variable changes, you must break the timeline into separate time fragments at the point of the change. This timeline format is similar to those used in Section 8.1, involving variable simple interest rates. The timeline illustrates the employee's new scenario.



Solving for the unknown FV on the right of the timeline means that you must start at the left side of the timeline. To arrive at the solution, you need to work from left to right one time segment at a time using Formula 9.3. As noted on the timeline, at the one-year point the future value of the first time segment then becomes the present value for the second time segment since the interest is not just accrued but actually placed into the account.



How It Works

Follow these steps when variables change in calculations of future value based on lump-sum compound interest:

Step 1: Read and understand the problem. Identify the present value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the length of the time segment in years.

Step 2: For each time segment, calculate the periodic interest rate (i) using Formula 9.1.

Step 3: For each time segment, calculate the total number of compound periods (N) using Formula 9.2.

Step 4: Starting with the present value in the first time segment (starting on the left), solve Formula 9.3.

Step 5: Let the future value calculated in the previous step become the present value for the next step. If the principal changes, adjust the new present value accordingly.

Step 6: Using Formula 9.3, calculate the future value of the next time segment.

Step 7: Repeat steps 5 and 6 until you obtain the final future value from the final time segment.

In the employee's new situation, he has borrowed \$4,000 for two years with 12% compounded semi-annually in the first year and 12% compounded quarterly in the second year.

Step 1: Figure 9.15 shows a timeline. In time segment one, $PV_1 = \$4,000$, $IY = 12\%$, $CY = 2$, and the time segment is one year long. In time segment two, the only change is $CY = 4$.

Step 2: In the first time segment, the periodic interest rate is $i_1 = 12\%/2 = 6\%$. In the second time segment, the periodic interest rate is $i_2 = 12\%/4 = 3\%$.

Step 3: The first time segment is one year long; therefore, $N_1 = 2 \times 1 = 2$. The second time segment is also one year long; therefore, $N_2 = 4 \times 1 = 4$.

Step 4: Apply Formula 9.3 to the first time segment:

$$FV_1 = PV(1 + i_1)^{N_1} = \$4,000(1 + 0.06)^2 = \$4,494.40$$

Step 5: Let $FV_1 = \$4,494.40 = PV_2$.

Step 6: Apply Formula 9.3 to the second time segment:

$$FV_2 = PV_2(1 + i_2)^{N_2} = \$4,494.40(1 + 0.03)^4 = \$4,494.40 \times 1.03^4 = \$5,058.49$$

Step 7: You reach the end of the timeline. The employee needs to repay \$5,058.49 to clear the loan.

To perform the above steps on the calculator:

Segment	N	I/Y	PV	PMT	FV	C/Y	P/Y
1	2	12	4000	0	Answer: -4,494.40	2	2
2	4	✓	4494.40	✓	Answer: -5,058.49	4	4

Note: A ✓ means that the value is already entered and does not need to be re-keyed into the calculator.



Important Notes

The BAII Plus Calculator: Transforming the future value from one time segment into the present value of the next time segment does not require re-entering the computed value. Instead, apply the following technique:

1. Load the calculator with all known compound interest variables for the first time segment.
2. Compute the future value at the end of the segment.
3. With the answer still on your display, adjust the principal if needed, change the cash flow sign by pressing the $\pm$ key, and then store the unrounded number back into the present value button by pressing PV. Change the N, I/Y, and C/Y as required for the next segment.

4. Return to step 2 for each time segment until you have completed all time segments.



Things To Watch Out For

When you draw timelines, it is critical to recognize that any change in any variable requires a new time segment. This applies to changes in principal, the nominal interest rate, or the compounding frequency.

TOP SECRET

Paths To Success

Compound interest on a single payment is linked to the concept of percent change in Section 3.1. Every time interest is compounded and added to the principal, the periodic rate is the percent change in the principal. To use the percent change function on the calculator, assign Old = PV, New = FV, $\Delta\% = i$, and #PD = N. The following two-step sequence shows the percent change function applied to the working example of the employee's new loan situation.

Segment	OLD	NEW	%CH	#PD
1	4000	Answer: \$4,494.40	6	2
2	4494.40	Answer: \$5,058.49	3	4

This is an alternative way to use the calculator to compute the future value of lump-sum amounts using compound interest.



Give It Some Thought

- Which of the following investments involving the same principal results in the highest maturity value? Assume equal terms at each interest rate.
 - 8% compounded annually followed by 6% compounded semi-annually
 - 8% compounded semi-annually followed by 6% compounded semi-annually
 - 8% compounded monthly followed by 6% compounded quarterly
 - 8% compounded annually followed by 6% compounded quarterly
- How many time segments are involved in an investment where the successive interest rates are 9% compounded quarterly, 9% compounded monthly, 8% compounded monthly, and 9% compounded quarterly?

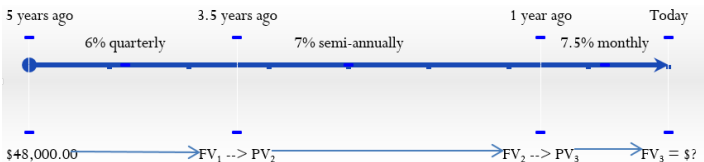
Solutions:
 1. c results in the largest maturity value. It has the highest compounding frequency in the first time segment and is tied for the highest compounding frequency in the second time segment.
 2. Four time segments are involved. Each change of interest rate requires a new time segment.

Example 9.2B: Delaying a Facility Upgrade

Five years ago Coast Appliances was supposed to upgrade one of its facilities at a quoted cost of \$48,000. The upgrade was not completed, so Coast Appliances delayed the purchase until now. The construction company that provided the quote indicates that prices rose 6% compounded quarterly for the first 1½ years, 7% compounded semi-annually for the following 2½ years, and 7.5% compounded monthly for the final year. If Coast Appliances wants to perform the upgrade today, what amount of money does it need?

Plan

Take the original quote and move it into the future with the price increases. You can view this as a single lump sum with multiple successive interest rates. The amount of money needed today is the maturity amount (FV).

Understand	What You Already Know Step 1: The timeline below shows the original quote from five years ago until today. The principal, terms, and interest rates are known: $PV_1 = \\$48,000$ First time segment: $IY = 6\%$ $CY = \text{quarterly} = 4$ Term = $1\frac{1}{2}$ years Second time segment: $IY = 7\%$ $CY = \text{semi-annually} = 2$ Term = $2\frac{1}{2}$ years Third time segment: $IY = 7.5\%$ $CY = \text{monthly} = 12$ Term = 1 year	How You Will Get There Step 2: For each time segment, calculate the periodic interest rate by applying Formula 9.1. Step 3: For each time segment, calculate the number of compound periods by applying Formula 9.2. Step 4/5: Calculate the future value of the first time segment using Formula 9.3. Let $FV_1 = PV_2$. Step 6/5: Calculate the future value of the second time segment using Formula 9.3. Let $FV_2 = PV_3$. Repeat Step 6/7: Calculate the future value of the third time segment using Formula 9.3. FV_3 is the final future value amount.																																		
																																				
Perform	Step 1st Segment 2nd Segment 3rd Segment																																			
	2.	$i = \frac{6\%}{4} = 1.5\%$	$i = \frac{7\%}{2} = 3.5\%$	$i = \frac{7.5\%}{12} = 0.625\%$																																
	3.	$N = 4 \times 1\frac{1}{2} = 6$	$N = 2 \times 2\frac{1}{2} = 5$	$N = 12 \times 1 = 12$																																
	4/5.	$FV_1 = \$48,000 \times (1 + 0.015)^6 = \$52,485.27667$																																		
	6/5.	$FV_2 = \$52,485.27667 \times (1 + 0.035)^5 = \$62,336.04435$																																		
	6/7.	$FV_3 = \$62,336.04435 \times (1 + 0.00625)^{12} = \$67,175.35$																																		
	Calculator Instructions																																			
	<table><tr><th>Segment</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>1</td><td>6</td><td>6</td><td>48000</td><td>0</td><td>Answer: -52,485.27677</td><td>4</td><td>4</td></tr><tr><td>2</td><td>5</td><td>7</td><td>52,485.27677</td><td>✓</td><td>Answer: -62,336.04435</td><td>2</td><td>2</td></tr><tr><td>3</td><td>12</td><td>7.5</td><td>62,336.04435</td><td>✓</td><td>Answer: -67,175.35</td><td>12</td><td>12</td></tr></table>	Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y	1	6	6	48000	0	Answer: -52,485.27677	4	4	2	5	7	52,485.27677	✓	Answer: -62,336.04435	2	2	3	12	7.5	62,336.04435	✓	Answer: -67,175.35	12	12			
Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y																													
1	6	6	48000	0	Answer: -52,485.27677	4	4																													
2	5	7	52,485.27677	✓	Answer: -62,336.04435	2	2																													
3	12	7.5	62,336.04435	✓	Answer: -67,175.35	12	12																													
Present	Coast Appliances requires \$67,175.35 to perform the upgrade today. This consists of \$48,000 from the original quote plus \$19,175.35 in price increases.																																			

**TOP
SECRET**

Paths To Success

When you calculate the future value of a single payment for which *only* the interest rate fluctuates, it is possible to find the maturity amount in a single multiplication:

$$FV = PV \times (1 + i_1)^{N_1} \times (1 + i_2)^{N_2} \times \dots \times (1 + i_n)^{N_n}$$

where n represents the time segment number. Note that the technique in Example 9.2B calculates each of these multiplications one step at a time, whereas this adaptation solves all time segments simultaneously. In the example above, you can calculate the same maturity value as follows:

$$FV = \$48,000 \times (1.015)^6 \times (1.035)^5 \times (1.0625)^{12} = \$67,175.35$$

Example 9.2C: Making an Additional Contribution

Two years ago Lorelei placed \$2,000 into an investment earning 6% compounded monthly. Today she makes a deposit to the investment in the amount of \$1,500. What is the maturity value of her investment three years from now?

Plan Take the original investment and move it into the future with the additional contribution. The amount of money three years from today is the maturity amount (FV).

Understand	What You Already Know Step 1: The timeline for this investment is below. First time segment: PV₁ = \$2,000, IY = 6%, CY = 12, Years = 2 Second time segment: Deposit = \$1,500, IY = 6%, CY = 12, Years = 3	How You Will Get There Step 2: For each time segment, calculate the periodic interest rate by applying Formula 9.1. Step 3: For each time segment, calculate the number of compound periods by applying Formula 9.2. Step 4: Calculate the future value of the first time segment using Formula 9.3. Step 5: Let FV₁ = PV₂. Increase PV₂ by the additional contribution amount of \$1,500. Step 6: Calculate the future value of the second time segment using Formula 9.3. Step 7: FV₂ is the final future value amount.																							
	Timeline diagram showing two segments. The first segment starts 2 years ago with PV₁ = \$2,000 and ends at 'Today' with FV₁ = PV₂. The second segment starts at 'Today' with a \$1,500 deposit and ends 3 years later with FV₂ = \$. Both segments have a 6% monthly interest rate.																								
Perform	<table><tr><th>Step</th><th>First Time Segment</th><th>Second Time Segment</th></tr><tr><td>2.</td><td>$i = \frac{6\%}{12} = 0.5\%$</td><td>$i = \frac{6\%}{12} = 0.5\%$</td></tr><tr><td>3.</td><td>N = 12 × 2 = 24</td><td>N = 12 × 3 = 36</td></tr><tr><td>4.</td><td colspan="2">FV₁ = \$2,000 × (1 + 0.005)²⁴ = \$2,254.319552</td></tr><tr><td>5.</td><td colspan="2">\$2,254.319552 + \$1,500 = \$3,754.319552 = PV₂</td></tr><tr><td>6.</td><td colspan="2">FV₂ = \$3,754.319552 × (1 + 0.005)³⁶ = \$4,492.72</td></tr></table>		Step	First Time Segment	Second Time Segment	2.	$i = \frac{6\%}{12} = 0.5\%$	$i = \frac{6\%}{12} = 0.5\%$	3.	N = 12 × 2 = 24	N = 12 × 3 = 36	4.	FV ₁ = \$2,000 × (1 + 0.005) ²⁴ = \$2,254.319552		5.	\$2,254.319552 + \$1,500 = \$3,754.319552 = PV ₂		6.	FV ₂ = \$3,754.319552 × (1 + 0.005) ³⁶ = \$4,492.72						
	Step	First Time Segment	Second Time Segment																						
	2.	$i = \frac{6\%}{12} = 0.5\%$	$i = \frac{6\%}{12} = 0.5\%$																						
	3.	N = 12 × 2 = 24	N = 12 × 3 = 36																						
	4.	FV ₁ = \$2,000 × (1 + 0.005) ²⁴ = \$2,254.319552																							
	5.	\$2,254.319552 + \$1,500 = \$3,754.319552 = PV ₂																							
6.	FV ₂ = \$3,754.319552 × (1 + 0.005) ³⁶ = \$4,492.72																								
Calculator Instructions <table><tr><th>Segment</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>1</td><td>24</td><td>6</td><td>-2000</td><td>0</td><td>Answer: 2,254.319552</td><td>12</td><td>12</td></tr><tr><td>2</td><td>36</td><td>✓</td><td>-3,754.319552</td><td>✓</td><td>Answer: 4,492.721092</td><td>✓</td><td>✓</td></tr></table>		Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y	1	24	6	-2000	0	Answer: 2,254.319552	12	12	2	36	✓	-3,754.319552	✓	Answer: 4,492.721092	✓	✓
Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y																		
1	24	6	-2000	0	Answer: 2,254.319552	12	12																		
2	36	✓	-3,754.319552	✓	Answer: 4,492.721092	✓	✓																		
Present	Three years from now Lorelei will have \$4,492.72. This represents \$3,500 of principal and \$992.72 of compound interest.																								

Example 9.2D: What Do You Still Owe?

Ruth purchased her \$5,465 diamond ring four years ago through a payment plan offered by the jeweller. This plan did not require any regular payments within the four years; however, the balance including interest must be paid in full by the end of the fourth year to avoid a large financial penalty. During the four-year period, a quarterly compounded variable interest rate was charged. Initially, the interest was 8.75% before increasing to 9.25% after two years and then 9.75% after $3\frac{1}{4}$ years. She made two payments of \$2,000 each, the first nine months after the purchase and the second after $3\frac{1}{4}$ years. What is the balance that she must pay at the end of the four years so that she doesn't incur any financial penalties?

Plan Take Ruth's initial purchase and charge it interest over the course of the four years while applying her payments to the principal at the appropriate points. Calculate the balance owing after four years; this is the future value (FV).

Understand	What You Already Know Step 1: The timeline is below. First time segment: $PV_1 = \\$5,465$, $IY = 8.75\%$, $CY = 4$, $Years = \frac{3}{4}$ Second time segment: Payment = \$2,000, $IY = 8.75\%$, $CY = 4$, $Years = 1\frac{1}{4}$ Third time segment: $IY = 9.25\%$, $CY = 4$, $Years = 1\frac{1}{4}$ Fourth time segment: Payment = \$2,000, $IY = 9.75\%$, $CY = 4$, $Years = \frac{3}{4}$	How You Will Get There Step 2: For each time segment, calculate the periodic interest rate by applying Formula 9.1. Step 3: For each time segment, calculate the number of compound periods by applying Formula 9.2. Step 4: Calculate the future value of the first time segment using Formula 9.3. Step 5: Let $FV_1 = PV_2$. Decrease PV_2 by the \$2,000 payment. Step 6: Calculate the future value of the 2nd time segment using Formula 9.3. Repeat Step 5: Let $FV_2 = PV_3$. Repeat Step 6: Calculate the future value of the third time segment using Formula 9.3. Repeat Step 5: Let $FV_3 = PV_4$. Decrease PV_4 by the \$2,000 payment. Repeat Step 6: Calculate the future value of the fourth time segment using Formula 9.3. Step 7: FV_4 is the final future value amount.																																												
	The diagram shows a horizontal timeline starting from 'Start (4 years ago)' and ending at 'Today'. Interest rates are marked at 9 months (8.75% quarterly), 2 years (8.75% quarterly), 3.25 years (9.25% quarterly), and 4 years (9.75% quarterly). Payments of \$2,000 are indicated at 1.75 years and 3.25 years. The initial present value is $PV_1 = \\$5,465.00$. The future value calculations are shown as $FV_1 = PV_2$, $FV_2 = PV_3$, $FV_3 = PV_4$, and finally $FV_4 = \\$?$. Each segment includes a box with its specific interest rate, compounding frequency (CY = 4), and term (Term = 3/4, 1 1/4, 1 1/4, 3/4).																																													
Perform	Step	1st Segment	2nd Segment	3rd Segment	4th Segment																																									
	2.	$i = \frac{8.75\%}{4} = 2.1875\%$	$i = \frac{8.75\%}{4} = 2.1875\%$	$i = \frac{9.25\%}{4} = 2.3125\%$	$i = \frac{9.75\%}{4} = 2.4375\%$																																									
	3.	$N = 4 \times \frac{3}{4} = 3$	$N = 4 \times 1\frac{1}{4} = 5$	$N = 4 \times 1\frac{1}{4} = 5$	$N = 4 \times \frac{3}{4} = 3$																																									
	4.	$FV_1 = \$5,465 \times (1 + 0.021875)^3 = \$5,465 \times 1.021875^3 = \$5,831.543094$																																												
	5.	$\$5,831.543094 - \$2,000.00 = \$3,831.543094 = PV_2$																																												
	6.	$FV_2 = \$3,831.543094 \times (1 + 0.021875)^5 = \$4,269.358126$																																												
	5-6.	$FV_3 = \$4,269.358126 \times (1 + 0.023125)^5 = \$4,786.36782$																																												
5.	$\$4,786.36782 - \$2,000.00 = \$2,786.36782 = PV_4$																																													
6.	$FV_4 = \$2,786.36782 \times (1 + 0.024375)^3 = \$2,995.13$																																													
Calculator Instructions																																														
<table><tr><th>Segment</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>1</td><td>3</td><td>8.75</td><td>5465</td><td>0</td><td>Answer: -5,831.543094</td><td>4</td><td>4</td></tr><tr><td>2</td><td>5</td><td>✓</td><td>3831.543094</td><td>✓</td><td>Answer: -4,269.358126</td><td>✓</td><td>✓</td></tr><tr><td>3</td><td>✓</td><td>9.25</td><td>4269.358126</td><td>✓</td><td>Answer: -4,786.36782</td><td>✓</td><td>✓</td></tr><tr><td>4</td><td>3</td><td>9.75</td><td>2786.36782</td><td>✓</td><td>Answer: -2,995.13</td><td>✓</td><td>✓</td></tr></table>							Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y	1	3	8.75	5465	0	Answer: -5,831.543094	4	4	2	5	✓	3831.543094	✓	Answer: -4,269.358126	✓	✓	3	✓	9.25	4269.358126	✓	Answer: -4,786.36782	✓	✓	4	3	9.75	2786.36782	✓	Answer: -2,995.13	✓	✓
Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y																																							
1	3	8.75	5465	0	Answer: -5,831.543094	4	4																																							
2	5	✓	3831.543094	✓	Answer: -4,269.358126	✓	✓																																							
3	✓	9.25	4269.358126	✓	Answer: -4,786.36782	✓	✓																																							
4	3	9.75	2786.36782	✓	Answer: -2,995.13	✓	✓																																							
Present	At the end of the four years, Ruth still owes \$2,995.13 to pay off her ring.																																													



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9.3: Determining the Present Value

(I Want to Pay Earlier)

Should you pay your bills early? If so, what amount should be paid? From a strictly financial perspective, if you are going to pay a bill earlier than its due date, the amount needs to be reduced somehow. If not, then why pay it early?

To illustrate, assume that you just received a \$3,000 bonus from your employer. Stopping at your mailbox, you pick up a large stack of envelopes that include a financial statement for your \$3,000 purchase of furniture from The Brick on its three-year, no-interest, no-payments plan. Should you use your bonus to extinguish this debt? The choices are that you can either pay \$3,000 today or \$3,000 three years from now.

- If you pay the bill today, The Brick can then take your money and invest it themselves for the next three years. At 2.75% compounded semi-annually, The Brick earns \$256.17. At the end of your agreement, The Brick then has both your \$3,000 payment and the additional interest of \$256.17! You might as well have just paid The Brick \$3,256.17 for your furniture!
- If you invest your bonus instead and pay your bill when it is due, The Brick receives its \$3,000 and you have the \$256.17 of interest left over in your bank account. Clearly, this is the financially smart choice.

For The Brick to be financially equitable in its dealings with you, it must reduce any early payment by an amount such that with interest the value of your payment accumulates to the debt amount upon the maturity of the agreement. This means that The Brick should be willing to accept a payment of \$2,763.99 today as payment in full for your \$3,000 bill. If The Brick then invests that payment at 2.75% compounded semi-annually, it grows to \$3,000 when your payment becomes due in three years.

Whether you are paying bills personally or professionally, it is important to understand present value. The amount of interest to be removed from a future value needs to reflect both how far in advance the payment occurs and an equitable interest rate. In this section, you calculate the present value of a future lump-sum amount under both conditions of constant variables and changing variables.

Present Value Calculations with No Variable Changes

As in your calculations of future value, the simplest scenario for present value is for all the variables to remain unchanged throughout the entire transaction. This still involves compound interest for a single payment or lump-sum amount and thus does not require a new formula.



The Formula

Solving for present value requires you to use Formula 9.3 once again. The only difference is that the unknown variable has changed from FV to PV.

FV is Future Value or Maturity Value: This is the future amount of money, which in the context of present value calculations is now a known variable. If the amount represents the end of the transaction, it is a maturity value.

PV is Present Value or Principal: This is the new unknown variable. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some earlier point in time than when the future value is known. In any case, the amount excludes the future interest. To calculate this variable, substitute the values for the other three variables into the formula and then algebraically rearrange to isolate PV.

Formula 9.3 – Compound Interest For Single Payments: $FV = PV(1 + i)^N$

i is Periodic Interest Rate: A result of Formula 9.1, this is the rate of interest that is used in converting the interest to principal. In a present value calculation, the nominal interest rate is sometimes called the **discount rate** since it removes interest from the future value. For accuracy, you never round this number.

N is Number of Compound Periods: The total number of compound periods involved in the transaction results from Formula 9.2. It shows how many deposits of interest you need to remove from the future value.



How It Works

Follow these steps to calculate the present value of a single payment:

Step 1: Read and understand the problem. If necessary, draw a timeline identifying the future value, the nominal interest rate, the compounding, and the term.

Step 2: Determine the compounding frequency (CY) if it is not already known, and calculate the periodic interest rate (i) by applying Formula 9.1.

Step 3: Calculate the number of compound periods (N) by applying Formula 9.2.

Step 4: Substitute into Formula 9.3, rearranging algebraically to solve for the present value.

Revisiting that furniture you bought on The Brick's three-year, no-interest and no-payments plan, if the amount owing three years from now is \$3,000 and prevailing market interest rates are 2.75% compounded semi-annually, what should The Brick be willing to accept as full payment today?

Step 1: The value of the payment today (PV) is required. The future value (FV) is \$3,000. The nominal interest rate is $iY = 2.75\%$, and the compounding frequency of semi-annually is $CY = 2$. The term is to pay it three years early.

Step 2: The periodic interest rate is $i = 2.75\% / 2 = 1.375\%$.

Step 3: The number of compounds is $N = 3 \times 2 = 6$.

Step 4: Applying Formula 9.3, $\$3,000 = PV(1 + 0.01375)^6$ or $PV = \frac{\$3,000}{1.01375^6} = \$2,763.99$.

If The Brick will accept \$2,763.99 as full payment, then pay your bill today. If not, keep your money, invest it yourself, and then pay the \$3,000 three years from now while retaining all of the interest earned.



Important Notes

You use the financial calculator in the exact same manner as described in Section 9.2. The only difference is that the unknown variable is PV instead of FV. You must still load the other six variables into the calculator and apply the cash flow sign convention carefully.

TOP SECRET

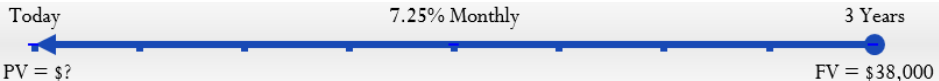
Paths To Success

Did you notice the following?

- **Future Value.** This calculation takes the present value and *multiplies* it by the interest factor. This increases the single payment by the interest earned.
- **Present Value.** This calculation takes the future value and *divides* it by the interest factor (rearranging Formula 9.3 for PV produces $\frac{FV}{(1+i)^N} = PV$). This removes the interest and decreases the single payment.

Example 9.3A: Achieving a Savings Goal

Castillo's Warehouse will need to purchase a new forklift for its warehouse operations three years from now, when its new warehouse facility becomes operational. If the price of the new forklift is \$38,000 and Castillo's can invest its money at 7.25% compounded monthly, how much money should it put aside today to achieve its goal?

Plan	You aim to calculate the amount of principal that Castillo’s must put aside today such that it can grow with interest to the desired savings goal. The principal today is the present value (PV).																				
Understand	What You Already Know Step 1: The maturity value, interest rate, and term are known: FV = \$38,000 IY = 7.25% CY = monthly = 12 times per year Term = 3 years			How You Will Get There Step 2: Calculate the periodic interest by applying Formula 9.1. Step 3: Calculate the number of compound periods by applying Formula 9.2. Step 4: Calculate the present value by substituting into Formula 9.3 and then rearranging for PV.																	
																					
Perform	Step 2: $i = \frac{7.25\%}{12} = 0.6041\bar{6}\% = 0.006041\bar{6}$ Step 3: $N = 12 \times 3 = 36$ Step 4: $\$38,000 = PV(1 + 0.006041\bar{6})^{36}$ $PV = \frac{\$38,000}{(1 + 0.006041\bar{6})^{36}} = \$30,592.06$			Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>36</td><td>7.25</td><td>Answer: \$30,592.06</td><td>0</td><td>38000</td><td>12</td><td>12</td></tr></table>				N	I/Y	PV	PMT	FV	P/Y	C/Y	36	7.25	Answer: \$30,592.06	0	38000	12	12
	N	I/Y	PV	PMT	FV	P/Y	C/Y														
36	7.25	Answer: \$30,592.06	0	38000	12	12															
Present	If Castillo’s Warehouse places \$30,592.06 into the investment, it will earn enough interest to grow to \$38,000 three years from now to purchase the forklift.																				

Present Value Calculations with Variable Changes

Addressing variable changes in present value calculations follows the same techniques as future value calculations. You must break the timeline into separate time segments, each of which involves its own calculations.

Solving for the unknown PV at the left of the timeline means you must start at the right of the timeline. You must work from right to left, one time segment at a time using Formula 9.3, rearranging for PV each time. Note that the present value for one time segment becomes the future value for the next time segment to the left.



How It Works

Follow these steps to calculate a present value involving variable changes in single payment compound interest:

Step 1: Read and understand the problem. Identify the future value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the segment's length in years.

Step 2: For each time segment, calculate the periodic interest rate (i) using Formula 9.1.

Step 3: For each time segment, calculate the total number of compound periods (N) using Formula 9.2.

Step 4: Starting with the future value in the first time segment on the right, solve Formula 9.3.

Step 5: Let the present value calculated in the previous step become the future value for the next time segment to the left. If the principal changes, adjust the new future value accordingly.

Step 6: Using Formula 9.3, calculate the present value of the next time segment.

Step 7: Repeat steps 5 and 6 until you obtain the present value from the leftmost time segment.



Important Notes

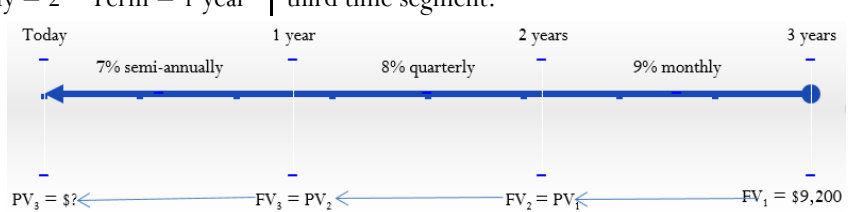
To use your calculator efficiently in working through multiple time segments, follow a procedure similar to that for future value:

1. Load the calculator with all the known compound interest variables for the first time segment on the right.
2. Compute the present value at the beginning of the segment.
3. With the answer still on your display, adjust the principal if needed, change the cash flow sign by pressing the $\pm$ key, then store the unrounded number back into the future value button by pressing FV. Change the N, I/Y, and C/Y as required for the next segment.

Return to step 2 for each time segment until you have completed all time segments.

Example 9.3B: A Variable Rate Investment

Sebastien needs to have \$9,200 saved up three years from now. The investment he is considering pays 7% compounded semi-annually, 8% compounded quarterly, and 9% compounded monthly in successive years. To achieve his goal, how much money does he need to place into the investment today?

Plan	Take the maturity amount and bring it back to today by removing the interest. Notice you are dealing with a single lump sum with multiple successive interest rates. The amount of money to be invested today is the principal (PV).
Understand	What You Already Know Step 1: The maturity amount, terms, and interest rates are on the timeline: $FV_1 = \\$9,200$ Starting from the right end of the timeline and working backwards: First time segment: IY = 9% CY = monthly = 12 Term = 1 year Second time segment: IY = 8% CY = quarterly = 4 Term = 1 year Third time segment: IY = 7% CY = semi-annually = 2 Term = 1 year
	How You Will Get There Step 2: For each time segment, calculate the periodic interest rate by applying Formula 9.1. Step 3: For each time segment, calculate the number of compound periods by applying Formula 9.2. Step 4: Calculate the present value of the first time segment using Formula 9.3 and rearrange for PV_1. Step 5: Assign $FV_2 = PV_1$. Step 6: Reapply Formula 9.3 and isolate PV_2 for the second time segment. Step 7: Assign $FV_3 = PV_2$. Reapply Formula 9.3 and isolate PV_3 for the third time segment. 

Perform	1st Segment	2nd Segment	3rd Segment																																				
	Step 2	$i = \frac{9\%}{12} = 0.75\%$	$i = \frac{8\%}{4} = 2\%$	$i = \frac{7\%}{2} = 3.5\%$																																			
	Step 3	$N = 12 \times 1 = 12$	$N = 4 \times 1 = 4$	$N = 2 \times 1 = 2$																																			
	Step 4: $\$9,200 = PV_1 \times (1 + 0.0075)^{12}$																																						
	$PV_1 = \frac{\$9,200}{1.0075^{12}} = \$8,410.991026$																																						
	Step 5: $FV_2 = \$8,410.991026$																																						
Present	Step 6: $\$8,410.991026 = PV_2 \times (1 + 0.02)^4$																																						
	$PV_2 = \frac{\$8,410.991026}{1.02^4} = \$7,770.455587$																																						
	Step 7: $FV_3 = \$7,770.455587$																																						
	$\$7,770.455587 = PV_3 \times (1 + 0.035)^2$																																						
	$PV_3 = \frac{\$7,770.455587}{1.035^2} = \$7,253.80$																																						
	Calculator Instructions																																						
<table><tr><td>Time Segment</td><td>N</td><td>I/Y</td><td>PV</td><td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>1</td><td>12</td><td>9</td><td>Answer: $-\\$8,410.991026$</td><td>0</td><td>9200</td><td>12</td><td>12</td></tr><tr><td>2</td><td>4</td><td>8</td><td>Answer: $-\\$7,770.455587$</td><td>✓</td><td>8410.991026</td><td>4</td><td>4</td></tr><tr><td>3</td><td>2</td><td>7</td><td>Answer: $-\\$7,253.803437$</td><td>✓</td><td>7770.455587</td><td>2</td><td>2</td></tr></table>								Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y	1	12	9	Answer: $-\$8,410.991026$	0	9200	12	12	2	4	8	Answer: $-\$7,770.455587$	✓	8410.991026	4	4	3	2	7	Answer: $-\$7,253.803437$	✓	7770.455587	2	2
Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y																																
1	12	9	Answer: $-\$8,410.991026$	0	9200	12	12																																
2	4	8	Answer: $-\$7,770.455587$	✓	8410.991026	4	4																																
3	2	7	Answer: $-\$7,253.803437$	✓	7770.455587	2	2																																
Sebastien needs to place \$7,253.80 into the investment today to have \$9,200 three years from now.																																							

**TOP
SECRET**

Paths To Success

When you calculate the present value of a single payment for which *only* the interest rate fluctuates, it is possible to find the principal amount in a single division:

$$PV = \frac{FV}{(1 + i_1)^{N_1} \times (1 + i_2)^{N_2} \times \dots \times (1 + i_n)^{N_n}}$$

where n represents the time segment number. Note that the technique in Example 9.3B resolves each of these divisions one step at a time, whereas this technique solves them all simultaneously. You can calculate the same principal as follows:

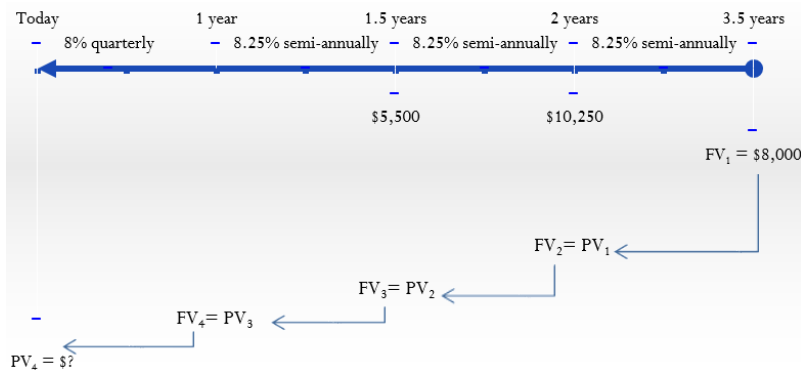
$$PV = \frac{\$9,200}{(1.0075)^{12} \times (1.02)^4 \times (1.035)^2} = \$7,253.80$$

Example 9.3C: Paying a Debt Early

Birchcreek Construction has three payments left on a debt obligation in the amounts of \$5,500, \$10,250, and \$8,000 due 1.5 years, 2 years, and 3.5 years from today. Prevailing interest rates are projected to be 8% compounded quarterly in the first year and 8.25% compounded semi-annually thereafter. If Birchcreek wants to settle the debt today, what amount should the creditor be willing to accept?

Plan

Take the three lump-sum payments in the future and remove the interest back to today to find the fair amount that Birchcreek Construction should pay. This is the present value (PV).

Understand	What You Already Know Step 1: With multiple amounts and interest rates, the timeline displays the changing variables. There are four time segments. The IY, CY, and Term are identified for each.	How You Will Get There Step 2: For each segment, calculate the periodic interest rate by applying Formula 9.1. Step 3: For each segment, calculate the number of compound periods by applying Formula 9.2. Step 4: Calculate the present value of the first time segment using Formula 9.3 and rearrange for PV_1. Step 5: Assign $FV_2 = PV_1$ and increase by the additional lump-sum payment. Step 6: Reapply Formula 9.3 and isolate PV_2 for the second time segment:. Step 7: Assign $FV_3 = PV_2$ and increase by the additional lump-sum payment. Reapply Formula 9.3 and isolate PV_3 for the third time segment. Repeat Step 7: Assign $FV_4 = PV_3$. Reapply Formula 9.3 and isolate PV_4 for the final time segment.																																								
																																										
Perform	<table><tr><th></th><th>Time Segment 1</th><th>Time Segment 2</th><th>Time Segment 3</th><th>Time Segment 4</th></tr><tr><td>Step 2</td><td>$i = \frac{8.25\%}{2} = 4.125\%$</td><td>$i = \frac{8.25\%}{2} = 4.125\%$</td><td>$i = \frac{8.25\%}{2} = 4.125\%$</td><td>$i = \frac{8\%}{4} = 2\%$</td></tr><tr><td>Step 3</td><td>$N = 2 \times 1.5 = 3$</td><td>$N = 2 \times \frac{1}{2} = 1$</td><td>$N = 2 \times \frac{1}{2} = 1$</td><td>$N = 4 \times 1 = 4$</td></tr></table>			Time Segment 1	Time Segment 2	Time Segment 3	Time Segment 4	Step 2	$i = \frac{8.25\%}{2} = 4.125\%$	$i = \frac{8.25\%}{2} = 4.125\%$	$i = \frac{8.25\%}{2} = 4.125\%$	$i = \frac{8\%}{4} = 2\%$	Step 3	$N = 2 \times 1.5 = 3$	$N = 2 \times \frac{1}{2} = 1$	$N = 2 \times \frac{1}{2} = 1$	$N = 4 \times 1 = 4$																									
		Time Segment 1	Time Segment 2	Time Segment 3	Time Segment 4																																					
Step 2	$i = \frac{8.25\%}{2} = 4.125\%$	$i = \frac{8.25\%}{2} = 4.125\%$	$i = \frac{8.25\%}{2} = 4.125\%$	$i = \frac{8\%}{4} = 2\%$																																						
Step 3	$N = 2 \times 1.5 = 3$	$N = 2 \times \frac{1}{2} = 1$	$N = 2 \times \frac{1}{2} = 1$	$N = 4 \times 1 = 4$																																						
Step 4: $\\$8,000 = PV_1 \times (1 + 0.04125)^3$ $PV_1 = \frac{\$8,000}{1.04125^3} = \$7,086.388265$ Step 5: $FV_2 = \\$7,086.388265 + \\$10,250.00 = \\$17,336.38826$ Step 6: $\\$17,336.38826 = PV_2 \times (1 + 0.04125)^1$ $PV_2 = \frac{\$17,336.38826}{1.04125} = \$16,649.59257$ Step 7: $FV_3 = \\$16,649.59257 + \\$5,500.00 = \\$22,149.59257$ $\$22,149.59257 = PV_3 \times (1 + 0.04125)^1$ $PV_3 = \frac{\$22,149.59257}{1.04125} = \$21,272.11772$ Repeat Step 7: $FV_4 = \\$21,272.11772$ $\$21,272.11772 = PV_4 \times (1 + 0.02)^4$ $PV_4 = \frac{\$21,272.11772}{1.02^4} = \$19,652.15$																																										
Present	Calculator Instructions <table><tr><th>Time Segment</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>1</td><td>3</td><td>8.25</td><td>Answer: $-\\$7,086.388265$</td><td>0</td><td>8000</td><td>2</td><td>2</td></tr><tr><td>2</td><td>1</td><td>✓</td><td>Answer: $-\\$16,649.59257$</td><td>✓</td><td>17336.38826</td><td>✓</td><td>✓</td></tr><tr><td>3</td><td>✓</td><td>✓</td><td>Answer: $-\\$21,272.11772$</td><td>✓</td><td>22149.59257</td><td>✓</td><td>✓</td></tr><tr><td>4</td><td>4</td><td>8</td><td>Answer: $-\\$19,652.14865$</td><td>✓</td><td>21272.11772</td><td>4</td><td>4</td></tr></table>		Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y	1	3	8.25	Answer: $-\$7,086.388265$	0	8000	2	2	2	1	✓	Answer: $-\$16,649.59257$	✓	17336.38826	✓	✓	3	✓	✓	Answer: $-\$21,272.11772$	✓	22149.59257	✓	✓	4	4	8	Answer: $-\$19,652.14865$	✓	21272.11772	4	4
	Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y																																		
1	3	8.25	Answer: $-\$7,086.388265$	0	8000	2	2																																			
2	1	✓	Answer: $-\$16,649.59257$	✓	17336.38826	✓	✓																																			
3	✓	✓	Answer: $-\$21,272.11772$	✓	22149.59257	✓	✓																																			
4	4	8	Answer: $-\$19,652.14865$	✓	21272.11772	4	4																																			
Using prevailing market rates, the fair amount that Birchcreek Construction owes today is \$19,652.15.																																										



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9.4: Equivalent Payments

(Let's Change the Deal)

Unforeseen events and circumstances can force you to rearrange your financial commitments. When this happens, the new deal has to be fair to all parties concerned.

Suppose you and your co-workers rely on your company's annual Christmas bonuses. However, the CEO just announced that because of tough economic conditions no one will get a bonus this year. You already earmarked that money to pay a \$5,000 debt due next week. You want to be financially responsible, but you cannot possibly make the payment. Before picking up the phone to call your creditor, you need to determine what course of action you should pursue.

You need to make alternative arrangements that leave your creditor in the same financial position as the original agreement did. You saw in the previous section that if you were going to make an early payment, the payment should be reduced by an agreed-upon discount rate. In this case, though, you are going to make a late payment, so you must grant the creditor interest. Thus, if you propose paying the debt six months late and your creditor agrees to 9% compounded monthly as a fair rate, then you owe \$5,229.26 (applying Formula 9.3).

This concept applies to all aspects of your personal and professional life. Except for gifts, personal debts to friends or family members should bear some interest. Everyone should be financially fair to each other. A business must be willing to work with its clients in the event they need to alter an agreement. A company that is inflexible tends to find itself writing off bad debt or pursuing unpleasant and sometimes expensive legal action.

This section explores the concept of **equivalent payment streams**. This involves equating two or more alternative financial streams to ensure that neither party is penalized by any choice. You then apply the concept of present value to loans and loan payments.

Fundamental Concepts

The Fundamental Concept of Time Value of Money

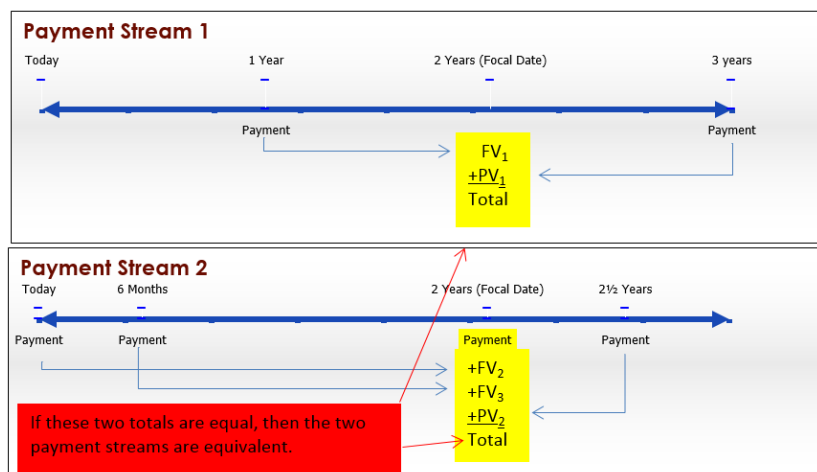
All numbers in an equation must be expressed in the same units, such as kilometres or metres. In the case of money, you treat the time at which you are considering the value as the "unit" that needs to be the same. Because of interest, a dollar today is not the same as a dollar a year ago or a dollar a year from now, so you cannot mix dollars from different times within the same equation; you must bring all monies forward or back to a common point in time, called the **focal date**.

The **fundamental concept of time value of money** states that all monies must be brought to the same focal date before any mathematical operations, decisions, or equivalencies can be determined, including the following:

1. Simple mathematics such as addition or subtraction.
2. Deciding whether to adopt alternative financial agreements.
3. Determining if payment streams are equal.

The Fundamental Concept of Equivalency

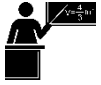
The **fundamental concept of equivalency** states that two or more payment streams are equal to each other if they have the same economic value on the same focal date. As illustrated in the figure, the two alternative financial streams are equivalent if the total of Payment Stream 1 is equal to the total of Payment Stream 2 on the same focal date. Note that the monies involved in each payment stream can be summed on the focal date because of the fundamental concept of time value of money.



Equivalent Payments

You make alternative payment streams equivalent to each other by applying a prevailing interest rate that allows for the following:

- Any late payments to be charged interest through future value calculations (Section 9.2)
- Any early payments to have interest deducted through present value calculations (Section 9.3)



The Formula

The good news is that you need no new formulas. Depending on the structure of the payment streams needing to be equated, apply Formula 9.3 as is or rearrange the formula for PV. As illustrated in the previous figure, Payment Stream 1 involves calculating one future value and one present value to move the money to the focal date. Payment Stream 2 involves calculating two future values and one present value. Once all money is moved to the same focal date, you work with the equality between the values of the two payment streams, solving for any unknown variable or variables.



How It Works

Follow these steps to solve an equivalent payment question:

Step 1: Draw as many timelines as needed to illustrate each of the original and proposed agreements. Clearly indicate dates, payment amounts, and the interest rate(s).

Step 2: Choose a focal date to which all money will be moved.

Step 3: Calculate all needed periodic interest rates using Formula 9.1.

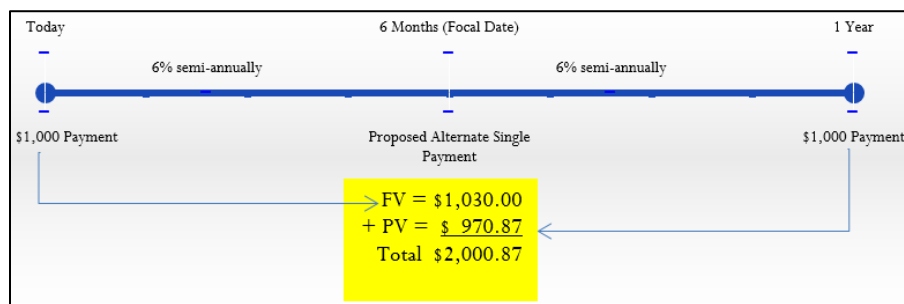
Step 4: Calculate N for each payment using Formula 9.2.

Step 5: Perform the appropriate time value calculation using Formula 9.3.

Step 6: Equate the values of the original and proposed agreements on the focal date and solve for any unknowns.

Assume you owe \$1,000 today and \$1,000 one year from now. You find yourself unable to make that payment today, so you indicate to your creditor that you want to make both payments six months from now instead. Prevailing interest rates are at 6% compounded semi-annually. What single payment six months from now (the proposed payment stream) is equivalent to the two payments (the original payment stream)?

Step 1: The timeline illustrates the scenario.



Step 2: Apply the fundamental concept of time value of money, moving all of the money to the same date. Since the proposed payment is for six months from now, you choose a focal date of six months.

Step 3: Note that semi-annual compounding means $CY = 2$. From Formula 9.1, $i = 6\%/2 = 3\%$.

Step 4: Formula 9.2 produces $N = 2 \times \frac{1}{2} = 1$ compound for both payments (each moving a $\frac{1}{2}$ year).

Step 5: Moving today's payment of \$1,000 six months into the future, you have $FV = \$1,000(1 + 0.03)^1 = \$1,030.00$.

Moving the future payment of \$1,000 six months earlier, you have $\$1,000 = PV(1 + 0.03)^1$ or $PV = \$970.87$.

Step 6: Now that money has been moved to the same date you can sum the two totals to determine the equivalent payment, which is $\$1,030.00 + \$970.87 = \$2,000.87$. Note that this is financially fair to both parties. For making your \$1,000 payment

six months late, the creditor is charging you \$30 of interest. Also, for making your second \$1,000 payment six months early, you are receiving a benefit of \$29.13. This leaves both parties compensated equitably: Neither party is financially better or worse off because of the change in the deal.

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Paths To Success

When you make two (or more) payment streams equal to each other, two tips will make the procedure easier:

1. **Proper Timelines.** Timelines help you see what to do. If you draw two or more timelines, align them vertically, ensuring that all corresponding dates are in the same columns. This allows you to see which payments need to be future valued and which need to be present valued to express them in terms of the chosen focal date.
2. **Locate the Focal Date at an Unknown.** In one sense, it does not matter what focal date you choose because two values that are equal when moved to one date in common will still be equal when both are moved together to another date. But you should simplify your calculations by selecting a focal date corresponding to the date of an unknown variable. Then when you calculate the root of the equation, it will already be in the right amount on the right date. You will not require further calculations to move the money to its correct date. For example, in the scenario above you had to determine the amount of a payment that was to be made six months from now. By setting the focal date at six months—the date of the unknown you were trying to find—you avoided any extra conversions.

Example 9.4A: Replacing a Payment Stream with a Single Payment

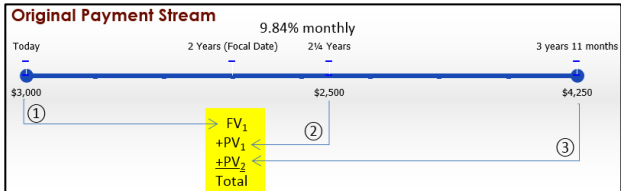
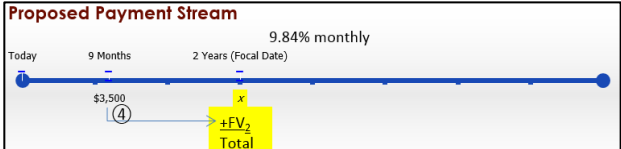
Johnson's Garden Centre has recently been unprofitable and concludes that it cannot make two debt payments of \$4,500 due today and another \$6,300 due in three months. After discussions between Johnson's Garden Centre and its creditor, the two parties agree that both payments could be made nine months from today, with interest at 8.5% compounded monthly. What total payment does Johnson's Garden Centre need nine months from now to clear its debt?

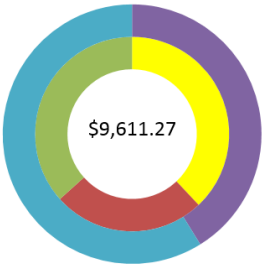
Plan	Calculate and sum the future value of each of the missed payments at the nine-month point with interest, or FV for both.	
Understand	What You Already Know Step 1: The amount of the missed payments, interest rate, and term are illustrated on the timeline. PV_1 (today) = \$4,500 PV_2 (3 months from now) = \$6,300 $IY = 8.5\%$ $CY = 12$ Step 2: Due date for all = 9 months from today. This is your focal date.	How You Will Get There Step 3: Calculate the periodic interest rate by applying Formula 9.1. Step 4: For each payment, calculate the number of compound periods (N) by applying Formula 9.2. Step 5: For each payment, calculate the maturity value (FV_1 and FV_2) by applying Formula 9.3. Step 6: Calculate the total payment by summing the two payments, FV_1 and FV_2, together.
Perform	Step 3: $i = \frac{8.5\%}{12} = 0.708\bar{3}\% = 0.00708\bar{3}$ Step 4: The first payment moves nine months into the future, or 9/12 of a year. $N = 12 \times 9/12 = 9$. The second payment moves six months into the future, or 6/12 of a year. $N = 12 \times 6/12 = 6$. Step 5: First payment: $FV_1 = \\$4,500 (1 + 0.00708\bar{3})^9 = \\$4,795.138902$ Second payment: $FV_2 = \\$6,300 (1 + 0.00708\bar{3})^6 = \\$6,572.536425$ Step 6: $FV = \\$4,795.138902 + \\$6,572.536425 = \\$11,367.68$	

	Calculator Instructions							
	Payment	N	I/Y	PV	PMT	FV	P/Y	C/Y
	1	9	8.5	4500	0	Answer: -\$4,795.138902	12	12
Present	2	6	✓	6300	✓	Answer: -\$6,572.526425	✓	✓
	With interest, the two payments total \$11,367.68. This is the \$10,800 of the original principal plus \$567.68 in interest for making the late payments.							

Example 9.4B: Replacing a Payment Stream with Multiple Payments

You have three debts to the same creditor: \$3,000 due today, \$2,500 due in $2\frac{1}{4}$ years, and \$4,250 due in 3 years 11 months. Unable to fulfill this obligation, you arrange with your creditor to make two alternative payments: \$3,500 in nine months and a second payment due in two years. You agree upon an interest rate of 9.84% compounded monthly. What is the amount of the second payment?

Plan	Determine the amount of the second payment that is due two years from today. Apply the fundamental concept of time value of money, moving all money from the original and proposed payment streams to a focal date positioned at the unknown payment. Once all money is moved to this focal date, apply the fundamental concept of equivalence, solving for the unknown payment, or x .	
Understand	What You Already Know Step 1: With two payment streams and multiple amounts all on different dates, visualize two timelines. The payment amounts, interest rate, and due dates for both payment streams are known. IY = 9.84% CY = monthly = 12	How You Will Get There Step 2: A focal date of Year 2 has been selected. Step 3: Calculate the periodic interest rate by applying Formula 9.1. Step 4: For each payment, calculate the number of compound periods by applying Formula 9.2. Step 5: For each payment, apply Formula 9.3. Note that all payments before the two-year focal date require you to calculate future values, while all payments after the two-year focal date require you to calculate present values. Step 6: Set the value of the original payment stream equal to the proposed payment stream. Solve for the unknown payment, x, using algebra. $FV_1 + PV_1 + PV_2 = x + FV_2$
	Original Payment Stream  The timeline shows cash flows at three points: Today (1), 2 Years (Focal Date) (2), and 3 years 11 months (3). At Today, there is a cash outflow of \$3,000. At 2 Years, there is a cash outflow of \$2,500. At 3 years 11 months, there is a cash outflow of \$4,250. Arrows indicate the movement of these values to the focal date: \$3,000 moves forward to become FV₁, \$2,500 moves forward to become PV₁, and \$4,250 moves backward to become PV₂. These three values are summed at the focal date.	Proposed Payment Stream  The timeline shows cash flows at four points: Today (1), 9 Months (4), 2 Years (Focal Date), and an unknown payment x. At Today, there is a cash outflow of \$3,500. At 9 Months, there is a cash outflow of \$3,500. At 2 Years, there is a cash outflow of x. At the focal date (2 Years), the value of x moves forward to become FV₂. The values at Today and 9 Months move forward to become FV₁ and FV₂ respectively. These three values are summed at the focal date.
Perform	Step 3: $i = \frac{9.84\%}{12} = 0.82\% \text{ or } 0.0082$ Steps 4 and 5: Using the circled number references from the timelines: ① $N = 12 \times 2 = 24$; $FV_1 = \\$3,000(1 + 0.0082)^{24} = \\$3,649.571607$ ② $N = 12 \times \frac{1}{4} = 3$; $\\$2,500 = PV_1(1 + 0.0082)^3$; $PV_1 = \frac{\\$2,500}{1.0082^3} = \\$2,439.494983$ ③ $N = 12 \times \frac{11}{12} = 23$; $\\$4,250 = PV_2(1 + 0.0082)^{23}$; $PV_2 = \frac{\\$4,250}{1.0082^{23}} = \\$3,522.207915$ ④ $N = 12 \times 1\frac{1}{4} = 15$; $FV_2 = \\$3,500(1 + 0.0082)^{15} = \\$3,956.110749$ Step 6: $\\$3,649.571607 + \\$2,439.494983 + \\$3,522.207915 = x + \\$3,956.110749$ $\$9,611.274505 = x + \$3,956.110749$ $\$5,655.16 = x$	

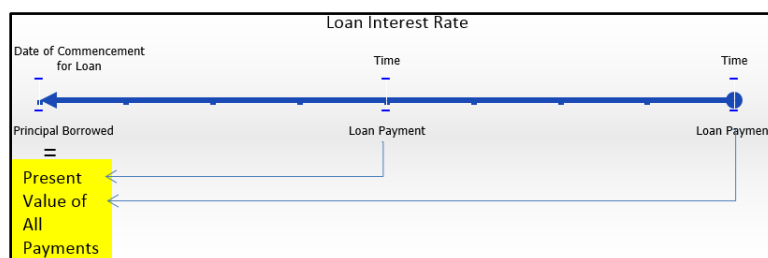
	Calculator Instructions							
	Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y
	①	24	9.84	3000	0	Answer: -\$3,649.571607	12	12
	②	3	✓	Answer: -\$2,439.494983	✓	2500	✓	✓
	③	23	✓	Answer: -\$3,522.207915	✓	4250	✓	✓
Present	④	15	✓	3500	✓	Answer: -\$3,956.110749	✓	✓
	The Equal Payment Streams at the Focal Date							
	 The second payment is \$5,655.16. This makes the original and proposed agreements equivalent to each other.							
	\$9,611.27 - Original Payment 1 ==> \$3,649.57 - Original Payment 2 ==> \$2,439.49 - Original Payment 3 ==> \$3,522.21 =====EQUALS===== - Proposed Payment 1 ==> \$3,956.11 - Proposed Payment 2 ==> \$5,655.16							

Working with Loans and Payments

The fundamental concepts of time value of money and equivalency will help you understand better how loans operate. When a loan is set up, the number of payments and the repayment amounts are established. Since the loan acquires interest throughout the transaction, this means future payments must pay back both principal and interest. Ultimately, all loan payments together need to fully reimburse the lender for the full amount of the principal.

Present Value Principle for Loans

The **present value principle for loans** states that the present value of all payments on a loan is equal to the principal that was borrowed. As illustrated in the timeline, taking all future payments and removing the interest, the payments must total to the original principal amount on the initial date of the loan. The interest rate used in this calculation is the interest rate for the loan itself. This relationship can be expressed as follows:



$$\text{Principal Borrowed} = \text{Present Value of All Payments}$$



How It Works

When you work with loans, the most common unknown variables are either the principal borrowed or the amount(s) of any unknown payment(s). You follow the same steps as for equivalent payments, discussed earlier in this section.

For example, assume a loan requires two equal payments of \$1,000 semi-annually and you want to know the original principal of the loan. The loan's interest rate is 7.1% compounded semi-annually.

Step 1: The timeline is illustrated here.

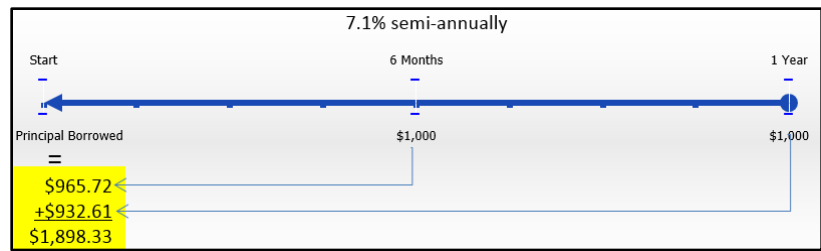
Step 2: According to the present value principle for loans, when you bring the two payments back to the starting date of the loan (the focal date), then the sum of the present values equals the initial amount borrowed.

Step 3: The IY = 7.1% and CY = 2, therefore $i = 7.1\%/2 = 3.55\%$.

Step 4: The first payment needs to come back a $\frac{1}{2}$ year, or $N = 2 \times \frac{1}{2} = 1$. The second payment needs to come back 1 year, or $N = 2 \times 1 = 2$.

Step 5: Applying Formula 9.3, for the first payment $\$1,000 = PV(1 + 0.0355)^1$ or $PV = \$965.717044$. For the second payment, $\$1,000 = PV(1 + 0.0355)^2$ or $PV = \$932.609410$.

Step 6: The amount borrowed is the sum of the two present values, or $\$965.717044 + \$932.609410 = \$1,898.33$. At 7.1% semi-annually, two payments of \$1,000 semi-annually pay off a loan of \$1,898.33.



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You can still use the time value of money buttons on your calculator to perform calculations involving unknown variables. Recall from the algebra discussion in Section 2.4 that any algebraic term consists of both a numerical and a literal coefficient. To move an unknown variable through time, enter the numerical coefficient of the variable into the calculator and compute its new value at the focal date. This new value on its focal date must then have the literal coefficient written after it before you proceed with further operations.

For example, if you want to perform a present value calculation on a future value of $2y$, enter the numerical coefficient “2” as your FV and compute the PV. Suppose the PV is 1.634. The literal coefficient has not disappeared. You just could not enter the letter on the calculator. Therefore, copying the literal coefficient to the present value yourself, you see that $PV = 1.634y$. Example 9.4C applies this concept.



Give It Some Thought

1. If four payments of \$2,000 each are needed to pay back a loan, what statement could you make about the principal of the loan?
2. The principal on a loan is \$10,000 and two payments are required. If the first payment is \$5,000, what statement could you make about the second payment?
3. Consider two loans. The first requires four \$2,500 quarterly payments. The second loan requires four \$2,500 monthly payments. Assuming equal interest rates, what statement could you make about the principals of these two loans?

Solutions:

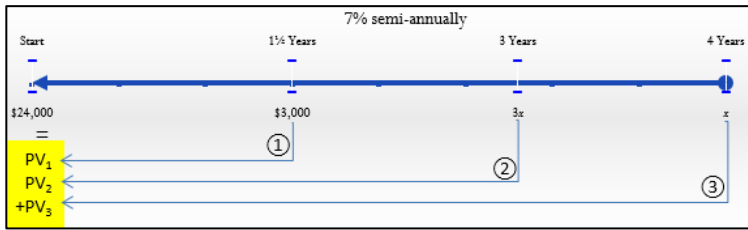
1. The principal is less than \$8,000 since all four payments must have the interest removed.
2. The second payment is more than \$5,000 since it must pay back the interest as well as the remaining principal.
3. The principal on the second loan is larger since the payments are made for only four months afterwards, so they involve relatively little interest. The first loan's payments extend for one year, so it requires the removal of larger sums of interest to arrive at the principal.

Example 9.4C: Determining Two Unknown Payments in a Loan

A \$24,000 loan at 7% compounded semi-annually requires three payments at $1\frac{1}{2}$ years, 3 years, and 4 years. The first payment is \$3,000 and the second payment is three times as large as the final payment. Calculate the values of the second and third payments on the loan.

Plan

Find the amount of the second and third payments. Since the second payment is three times the size of the third payment, solve for the third payment, or x , and then calculate the second payment afterwards.

Understand	What You Already Know Step 1: The principal, payment amounts, interest rate, and dates are known, as indicated on the timeline. IY = 7% CY = 2 Note: If the final payment is x and the second payment is three times as large, then it equals $3x$.	How You Will Get There Step 2: The focal date is the start of the loan. Step 3: Calculate the periodic interest rate by applying Formula 9.1. Step 4: For each payment, calculate the number of compound periods by applying Formula 9.2. Step 5: For each payment apply Formula 9.3. Step 6: Using the present value principle for loans, equate the payments and the principal, then solve for the unknown variable x. Principal = $PV_1 + PV_2 + PV_3$ Once you know x, calculate the size of the second payment = $3x$.																															
																																	
Perform	Step 3: $i = \frac{7\%}{2} = 3.5\% = 0.035$ Steps 4 and 5: Using the circled number references in the timeline: ① $N = 2 \times 1\frac{1}{2} = 3$; $\\$3,000 = PV_1(1 + 0.035)^3$; $PV_1 = \frac{\\$3,000}{1.035^3} = \\$2,705.828117$ ② $N = 2 \times 3 = 6$; $3x = PV_2(1 + 0.035)^6$; $PV_2 = \frac{3x}{1.035^6} = 2.440501x$ ③ $N = 2 \times 4 = 8$; $x = PV_3(1 + 0.035)^8$; $PV_3 = \frac{x}{1.035^8} = 0.759411x$ Step 6: $\\$24,000 = \\$2,705.828117 + 2.440501x + 0.759411x$ $\\$24,000 = \\$2,705.828117 + 3.199913x$ $\\$21,294.17188 = 3.199913x$ $\\$6,654.61 = x$ Solving for the $3x$ payment: $3(\\$6,654.61) = \\$19,963.83$																																
	Calculator Instructions <table><tr><th>Calculation</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>①</td><td>3</td><td>7</td><td>Answer: \$2,705.828117</td><td>0</td><td>-3000</td><td>2</td><td>2</td></tr><tr><td>②</td><td>6</td><td>✓</td><td>Answer: 2.440501</td><td>✓</td><td>-3</td><td>✓</td><td>✓</td></tr><tr><td>③</td><td>8</td><td>✓</td><td>Answer: 0.759411</td><td>✓</td><td>-1</td><td>✓</td><td>✓</td></tr></table>		Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y	①	3	7	Answer: \$2,705.828117	0	-3000	2	2	②	6	✓	Answer: 2.440501	✓	-3	✓	✓	③	8	✓	Answer: 0.759411	✓	-1	✓
Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y																										
①	3	7	Answer: \$2,705.828117	0	-3000	2	2																										
②	6	✓	Answer: 2.440501	✓	-3	✓	✓																										
③	8	✓	Answer: 0.759411	✓	-1	✓	✓																										
Present	The second payment, located three years from the start of the loan, is \$19,963.83, which is three times the size of the final payment of \$6,654.61 one year later.																																



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9.5: Determining the Interest Rate

(Is a Lot of Interest Accumulating or Just a Little?)

Whether you are borrowing or investing, it is extremely important to know what compound interest rate you are being charged or earning. Unfortunately, most consumers live for today, paying attention just to the “don’t pay for one year” clause in the deals they are offered. They do not stop to consider they might be paying too much for credit.

For example, assume you are just about to sign the purchase papers for a new 55” television. The retail sales clerk turns to you, saying, “Well, I’ve run the numbers and they show you could pay \$4,000 today or if you take advantage of our ‘don’t pay for one year’ offer, you will owe us \$4,925.76 one year from now.” You urgently want to get your new TV home, but you had better think twice about a financial decision this important. What interest rate did the sales clerk use in determining your \$4,925.76 payment? Is this interest rate fair? Could you finance the TV for less elsewhere?

This section shows how to calculate the nominal interest rate on single payments when you know both the future value and the present value.

The Nominal Interest Rate

You need to calculate the nominal interest rate under many circumstances including (but not limited to) the following:

- Determining the interest rate on a single payment loan
- Understanding what interest rate is needed to achieve a future savings goal
- Calculating the interest rate that generated a specific amount of interest
- Finding a fixed interest rate that is equivalent to a variable interest rate



The Formula

Calculating the nominal interest rate requires you to use Formula 9.3 once again. The only difference is that the unknown variable has changed from FV to IY. Note that IY is not directly part of Formula 9.3, but you are able to calculate its value after determining the periodic interest rate, or i .

FV is Future Value or Maturity Value: The ending value for the calculation.

PV is Present Value or Principal: The starting value for the calculation.

Formula 9.3 – Compound Interest For Single Payments: $FV = PV(1 + i)^N$

i is Periodic Interest Rate: The unknown variable is the periodic interest rate. Calculating the nominal interest rate requires substitution and then algebraic manipulation to solve for this variable. Once you know the periodic interest rate, recall that it is also a variable in Formula 9.1:

$$i = \frac{IY}{CY}$$

To calculate the nominal interest rate, substitute and algebraically rearrange Formula 9.1 to solve for IY.

N is Number of Compound Periods: The total number of compound periods involved in the transaction. Calculate it by applying Formula 9.2.



How It Works

Follow these steps to solve for the nominal interest rate on a single payment:

Step 1: Draw a timeline to help you visualize the question. Of utmost importance is identifying the values of PV and FV, the number of years involved, and the compounding for the interest rate.

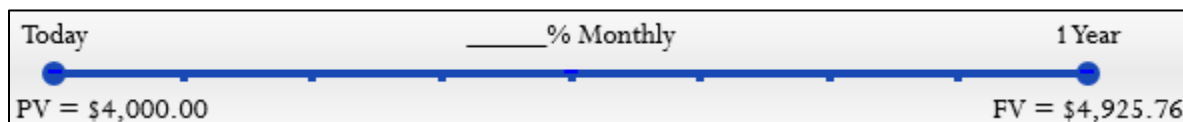
Step 2: Calculate the number of compounds, N , using Formula 9.2.

Step 3: Substitute known variables into Formula 9.3, rearrange and solve for the periodic interest rate, i .

Step 4: Substitute the periodic interest rate and the compounding frequency into Formula 9.1, rearrange, and solve for the nominal interest rate, IY . Ensure that the solution is expressed with the appropriate compounding words.

Revisiting the section opener, your \$4,000 TV cost \$4,925.76 one year later. What monthly compounded interest rate was being used?

Step 1: The timeline below illustrates the scenario and identifies the values.



The compounding is monthly, making $CY = 12$.

Step 2: The term is one year, so $N = 1 \times 12 = 12$.

Step 3: Substituting into Formula 9.3, $\$4,925.76 = \$4,000(1 + i)^{12}$. Rearranging and solving for i results in $i = 0.0175$.

Step 4: Substituting into Formula 9.1, $0.0175 = \frac{IY}{12}$. Rearranging and solving for IY calculates $IY = 0.21$, or 21%. Thus, the sales clerk used an interest rate of 21% compounded monthly.



Important Notes

Handling Decimals in Interest Rate Calculations. When you calculate interest rates, the solution rarely works out to a terminating decimal number. Since most advertised or posted interest rates commonly involve no more than a few decimals, why is this the case? Recall that when an interest dollar amount is calculated, in most circumstances this amount must be rounded off to two decimals. In single payments, a future value is always the present value plus the rounded interest amount. This results in a future value that is an imprecise number that may be up to a half penny away from its true value. When this imprecise number is used for calculating any interest rate, the result is that nonterminating decimals show up in the solutions. To express the final solution for these nonterminating decimals, you need to apply two rounding rules:

- **Rule 1: A Clear Marginal Effect.** Use this rule when it is fairly obvious how to round the interest rate. The dollar amounts used in calculating the interest rate are rounded by no more than a half penny. Therefore, the calculated interest rate should be extremely close to its true value. For example, if you calculate an IY of 7.999884%, notice this value would have a marginal difference of only 0.000116% from a rounded value of 8%. Most likely the correct rate is 8% and not 7.9999%. However, if you calculate an IY of 7.920094%, rounding to 8% would produce a difference of 0.070006%, which is quite substantial. Applying marginal rounding, the most likely correct rate is 7.92% and not 7.9201%, since the marginal impact of the rounding is only 0.000094%.
- **Rule 2: An Unclear Marginal Effect.** Use this rule when it is not fairly obvious how to round the interest rate. For example, if the calculated $IY = 7.924863\%$, there is no clear choice of how to round the rate. In these cases or when in doubt, apply the standard rule established for this book of rounding to four decimals. Hence, $IY = 7.9249\%$ in this example.

It is important to stress that the above recommendations for rounding apply to final solutions. If the calculated interest rate is to be used in further calculations, then you should carry forward the unrounded interest rate.

Your BAII Plus Calculator. Solving for the nominal interest rate requires the computation of I/Y on the BAII Plus calculator. This requires you to enter all six of the other variables, including N , PV , PMT (which is zero), FV , and both values

in the P/Y window (P/Y and C/Y) following the procedures established in Section 9.2. Ensure proper application of the cash flow sign convention to PV and FV, according to which one number must be negative while the other is positive.



Things To Watch Out For

Pay careful attention to what the situation requires you to calculate—the nominal interest rate or the periodic interest rate. The most common mistake when calculating the interest rate is to confuse these two rates. If you need the periodic interest rate, then you must rearrange and solve Formula 9.3 for i , and no further calculations are required after step 3. If you need the nominal interest rate, first calculate the periodic interest rate (i) but then substitute it into Formula 9.1 and rearrange to solve for IY , thus completing step 4 of the process.



Give It Some Thought

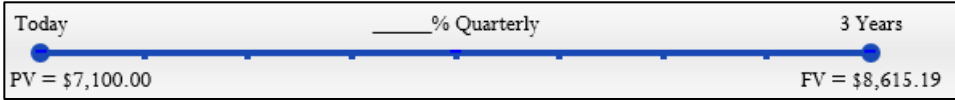
Round off the following calculated values of IY to the appropriate decimals.

- a. 4.5679998% b. 12.000138% c. 6.8499984% d. 8.0200121% e. 7.1224998%

Solutions:	a. 4.568%	b. 12%	c. 6.85%	d. 8.02%	e. 7.1225%

Example 9.5A: Straightforward Interest Rate Calculation—FV and PV Known

When Sandra borrowed \$7,100 from Sanchez, she agreed to reimburse him \$8,615.19 three years from now including interest compounded quarterly. What interest rate is being charged?

Plan	Find the nominal quarterly compounded rate of interest (IY).																				
Understand	What You Already Know Step 1: The present value, future value, term, and compounding are known, as illustrated in the timeline. CY = quarterly = 4 Term = 3 years				How You Will Get There Step 2: Calculate N using Formula 9.2. Step 3: Substitute into Formula 9.3 and rearrange for i . Step 4: Substitute into Formula 9.1 and rearrange for IY.																
																					
Perform	Step 2: $N = 4 \times 3 = 12$ Step 3: $\$8,615.19 = \$7,100(1 + i)^{12}$ $1.213407 = (1 + i)^{12}$ $1.213407^{\frac{1}{12}} = 1 + i$ $1.016249 = 1 + i$ $0.016249 = i$ Step 4: $0.016249 = \frac{IY}{4}$ $IY = 0.064999 = 0.065 \text{ or } 6.5\%$				Calculator Instructions <table border="1" data-bbox="875 1329 1463 1383"><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>12</td><td>Answer: 6.499985</td><td>-7100</td><td>0</td><td>8615.19</td><td>4</td><td>4</td></tr></table>			N	I/Y	PV	PMT	FV	P/Y	C/Y	12	Answer: 6.499985	-7100	0	8615.19	4	4
	N	I/Y	PV	PMT	FV	P/Y	C/Y														
12	Answer: 6.499985	-7100	0	8615.19	4	4															
Present	Sanchez is charging an interest rate of 6.5% compounded quarterly on the loan to Sandra.																				

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When a series of calculations involving the nominal interest rate must be performed, many people find it helpful first to rearrange Formula 9.3 algebraically for i , thus bypassing a long series of manipulations. The rearranged formula appears as follows:

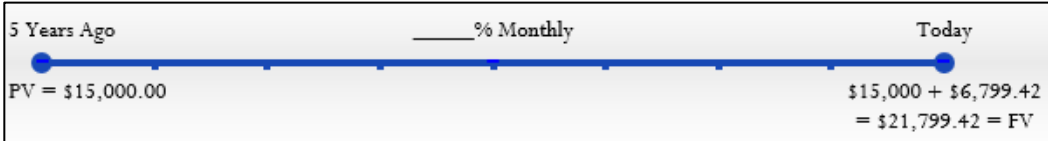
$$i = \left[\left(\frac{FV}{PV} \right)^{\frac{1}{N}} - 1 \right]$$

This rearrangement calculates the periodic interest rate. If the nominal interest rate is required, you can combine Formulas 9.3 and 9.1 together:

$$IY = \left[\left(\frac{FV}{PV} \right)^{\frac{1}{N}} - 1 \right] \times CY$$

Example 9.5B: Known Interest Amount

Five years ago, Taryn placed \$15,000 into an RRSP that earned \$6,799.42 of interest compounded monthly. What was the nominal interest rate for the investment?

Plan	Find the nominal monthly compounded rate of interest (IY).																				
Understand	What You Already Know Step 1: The present value, interest earned, term, and compounding are known, as illustrated in the timeline. Use Formula 8.3 to arrive at the FV in the figure. CY = monthly = 12 Term = 5 years				How You Will Get There Step 2: Calculate N using Formula 9.2. Step 3: Substitute into Formula 9.3 and rearrange for i . Step 4: Substitute into Formula 9.1 and rearrange for IY.																
																					
Perform	Step 2: $N = 12 \times 5 = 60$ Step 3: $\$21,799.42 = \$15,000(1 + i)^{60}$ $1.453294 = (1 + i)^{60}$ $1.453294^{\frac{1}{60}} = 1 + i$ $1.00625 = 1 + i$ $0.00625 = i$ Step 4: $0.00625 = \frac{IY}{12}$ $IY = 0.075$ or 7.5%				Calculator Instructions <table border="1" data-bbox="760 1262 1455 1333"><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>60</td><td>Answer: 7.500003</td><td>-15000</td><td>0</td><td>21799.42</td><td>12</td><td>12</td></tr></table>			N	I/Y	PV	PMT	FV	P/Y	C/Y	60	Answer: 7.500003	-15000	0	21799.42	12	12
	N	I/Y	PV	PMT	FV	P/Y	C/Y														
60	Answer: 7.500003	-15000	0	21799.42	12	12															
Present	Taryn's investment in his RRSP earned 7.5% compounded monthly over the five years.																				

Converting Variable Interest Rates to a Fixed Interest Rate

When you deal with a series of variable interest rates it is extremely difficult to determine their overall effect. This also makes it hard to choose wisely between different series. For example, assume that you could place your money into an investment earning interest rates of 2%, 2.5%, 3%, 3.5%, and 4.5% over the course of five years, or alternatively you could

invest in a plan earning 1%, 1.5%, 1.75%, 3.5%, and 7% (all rates compounded semi-annually). Which plan is better? The decision is unclear. But you can make it clear by converting the variable rates on each investment option into an equivalent fixed interest rate.



How It Works

Follow these steps to convert variable interest rates to their equivalent fixed interest rates:

Step 1: Draw a timeline for the variable interest rate. Identify key elements including any known PV or FV, interest rates, compounding, and terms.

Step 2: For each time segment, calculate i and N using Formulas 9.1 and 9.2, respectively.

Step 3: One of three situations will occur, depending on what variables are known:

1. **PV Is Known.** Calculate the future value at the end of the transaction using Formula 9.3 and solving for FV in each time segment, working left to right across the timeline.
2. **FV Is Known.** Calculate the present value at the beginning of the transaction using Formula 9.3 and rearranging to solve for PV in each time segment, working right to left across the timeline.
3. **Neither PV nor FV Is Known.** Pick an arbitrary number for PV (\$10,000 is recommended) and use Formula 9.3 in each time segment to solve for the future value at the end of the transaction, working left to right across the timeline.

Step 4: Determine the compounding required on the fixed interest rate (CY) and use Formula 9.2 to calculate a new value for N to reflect the entire term of the transaction.

Step 5: Rearrange and solve Formula 9.3 for i using the N from step 4 along with the starting PV and ending FV for the entire timeline.

Step 6: Rearrange and solve Formula 9.1 for IY.

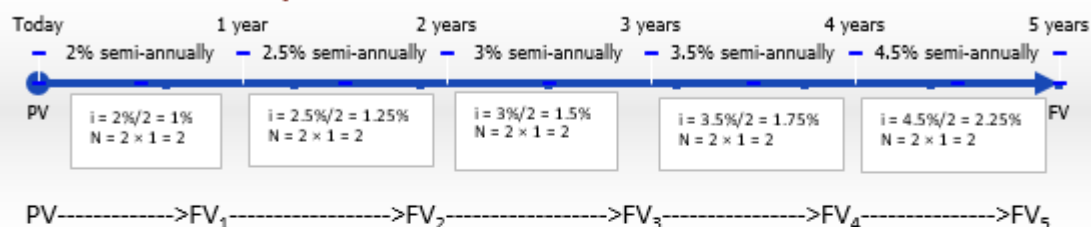
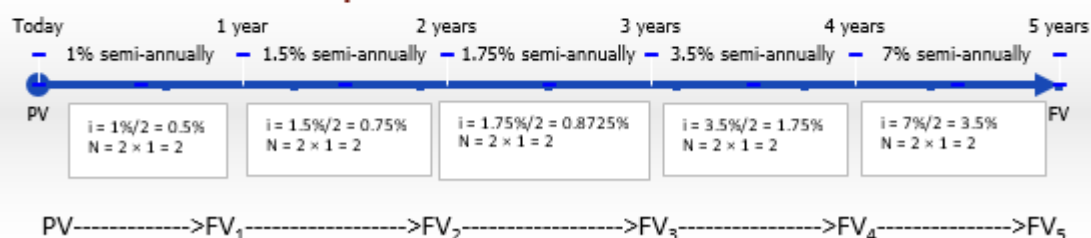
Example 9.5C: Interest Rate under Variable Rate Conditions

Continue working with the two investment options mentioned previously. The choices are to place your money into a five-year investment earning semi-annually compounded interest rates of either:

- a. 2%, 2.5%, 3%, 3.5%, and 4.5%
- b. 1%, 1.5%, 1.75%, 3.5%, and 7%

Calculate the equivalent semi-annual fixed interest rate for each plan and recommend an investment.

Plan	For each plan, solve for a fixed semi-annually compounded interest rate (IY). Since this is an investment rather than a loan, recommend the plan with the highest IY.	
Understand	What You Already Know Step 1: Draw a timeline for each investment option, as illustrated below.	How You Will Get There Step 2: For each time segment, calculate the i and N using Formulas 9.1 and 9.2. All interest rates are semi-annually compounded with $CY = 2$. Step 3: There is no value for PV or FV. Choose an arbitrary value of $PV = \\$10,000$ and solve for FV using Formula 9.3. Since only the interest rate fluctuates, solve in one calculation: $FV_5 = PV \times (1 + i_1)^{N_1} \times (1 + i_2)^{N_2} \times \dots \times (1 + i_5)^{N_5}$ Step 4: For each investment, calculate a new value of N to reflect the entire five-year term using Formula 9.2. Step 5: For each investment, substitute into Formula 9.3 and rearrange for i: Step 6: For each investment, substitute into Formula 9.1 and rearrange for IY.

First Investment Option**Second Investment Option**

Step 2: All periodic interest rate (i) and N calculations are found in Figure 9.51.

Step	First Investment	Second Investment
3.	$FV_5 = \$10,000(1 + 0.01)^2 (1 + 0.0125)^2 (1 + 0.015)^2 (1 + 0.0175)^2 (1 + 0.0225)^2 = \$11,661.65972$	$FV_5 = \$10,000(1 + 0.005)^2 (1 + 0.0075)^2 (1 + 0.00875)^2 (1 + 0.0175)^2 (1 + 0.035)^2 = \$11,570.14666$
4.	$N = 2 \times 5 = 10$	$N = 2 \times 5 = 10$
5.	$\$11,661.65972 = \$10,000(1 + i)^{10}$ $1.166165 = (1 + i)^{10}$ $1.166165^{\frac{1}{10}} = 1 + i$ $1.001549 = 1 + i$ $0.001549 = i$	$\$11,570.14666 = \$10,000(1 + i)^{10}$ $1.157014 = (1 + i)^{10}$ $1.157014^{\frac{1}{10}} = 1 + i$ $1.014691 = 1 + i$ $0.014691 = i$
6.	$0.001549 = \frac{IY}{2}$ $IY = 0.030982$ or 3.0982%	$0.014691 = \frac{IY}{2}$ $IY = 0.029382$ or 2.9382%

Calculator Instructions**Investment #1**

Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	2	2	-10000	0	Answer: 10,201	2	2
2	✓	2.5	-10201	✓	Answer: 10,457.61891	✓	✓
3	✓	3	-10457.61891	✓	Answer: 10,773.70044	✓	✓
4	✓	3.5	-10773.70044	✓	Answer: 11,154.0794	✓	✓
5	✓	4.5	-11154.0794	✓	Answer: 11661.65972	✓	✓
All	10	Answer: 0.030982	-10000	✓	11661.66	✓	✓

Investment #2

Time Segment	N	IY	PV	PMT	FV	P/Y	C/Y
1	2	1	-10000	0	Answer: 10,100.25	2	2
2	✓	1.5	-10100.25	✓	Answer: 10,252.32189	✓	✓
3	✓	1.75	-10252.32189	✓	Answer: 10,432.52247	✓	✓
4	✓	3.5	-10432.52247	✓	Answer: 10,800.85571	✓	✓
5	✓	7	-10800.85571	✓	Answer: 11,570.14666	✓	✓
All	10	Answer: 0.029382	-10000	✓	11570.15	✓	✓

Present

The variable interest rates on the first investment option are equivalent to a fixed interest rate of 3.0982% compounded semi-annually. For the second option, the rates are equivalent to 2.9382% compounded semi-annually. Therefore, recommend the first investment since its rate is higher by $3.0982\% - 2.9382\% = 0.16\%$ compounded semi-annually.



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9.6: Equivalent and Effective Interest Rates

(How Do I Compare Different Rates?)

How can you compare interest rates posted with different compounding? For example, let's say you are considering the purchase of a new home, so for the past few weeks you have been shopping around for financing. You have spoken with many banks as well as onsite mortgage brokers in the show homes. With semi-annual compounding, the lowest rate you have come across is 6.6%. In visiting another show home, you encounter a mortgage broker offering a mortgage for 6.57%. In the fine print, it indicates the rate is compounded quarterly. You remember from your business math class that the compounding is an important component of an interest rate and wonder which one you should choose—6.6% compounded semi-annually or 6.57% compounded quarterly.

When considering interest rates on loans, you clearly want the best rate. If all of your possible loans are compounded in the same manner, selecting the best interest rate is a matter of picking the lowest number. However, when interest rates are compounded differently the lowest number may in fact not be your best choice. For investments, on the other hand, you want to earn the most interest. However, the highest nominal rate may not be as good as it appears depending on the compounding.

To compare interest rates fairly and select the best, they all have to be expressed with equal compounding. This section explains the concept of an effective interest rate, and you will learn to convert interest rates from one compounding frequency to a different frequency.

Effective Interest Rates

If you place \$1,000 into an investment earning 10% compounded semi-annually, how much will you have in your account after one year? Less than \$1,100, exactly \$1,100, or more than \$1,100? If you say more than \$1,100, you are absolutely correct. In other words, the 10% nominal rate does not fully reflect the true amount of interest that the investment earns annually, which depends on how often the principal increases. This raises two questions:

1. What exact amount of interest is earned annually?
2. What annual interest rate is truly being earned?

The **effective interest rate** is the true annually compounded interest rate that is equivalent to an interest rate compounded at some other (non-annual) frequency. In other words, the amount of interest accrued at the effective interest rate once in an entire year exactly equals the amount of interest accrued at the periodic interest rate successively compounded the stated number of times in a year. To calculate the effective interest rate, you must convert the compounding on the nominal interest rate into an annual compounding.



The Formula

To see how the formula develops, take a \$1,000 investment at 10% compounded semi-annually through a full year.

Start with $PV = \$1,000$, $IY = 10\%$, and $CY = 2$ (semi-annually). Therefore, $i = 10\%/2 = 5\%$. Using Formula 9.3, after the first six-month compounding period ($N = 1$) the investment is worth

$$FV = \$1,000(1 + 0.05)^1 = \$1,050$$

With this new principal of $PV = \$1,050$, after the next six-month compounding period the investment becomes

$$FV = \$1,050(1 + 0.05)^1 = \$1,102.50$$

Therefore, after one year the investment has earned \$102.50 of interest. Notice that this answer involves compounding the periodic interest twice, according to the compounding frequency of the interest rate. What annually compounded interest rate would produce the same result? Try 10.25% compounded annually. With $PV = \$1,000$, $IY = 10.25\%$, and $CY = 1$, then $i = 10.25\%/1 = 10.25\%$. Thus, after a year

$$FV = \$1,000(1 + 0.1025)^1 = \$1,102.50$$

This alternative yields the same amount of interest, \$102.50. In other words, 10.25% compounded annually produces the same result as 10% compounded semi-annually. Hence, the effective interest rate on the investment is 10.25%, and this is what the investment truly earns annually.

Generalizing from the example, you calculate the future value and interest amount for the rate of 10% compounded semi-annually using the formulas

$$FV = PV(1 + i)(1 + i) \quad \text{and} \quad I = FV - PV$$

Notice that the periodic interest is compounded by a factor of $(1 + i)$ a number of times equalling the compounding frequency ($CY = 2$). You then rewrite the future value formula:

$$FV = PV(1 + i)^{CY}$$

Substituting this into the interest amount formula, $I = FV - PV$, results in

$$I = PV(1 + i)^{CY} - PV$$

Factor and rearrange this formula:

$$I = PV((1 + i)^{CY} - 1)$$

$$\frac{I}{PV} = (1 + i)^{CY} - 1$$

On the left-hand side, the interest amount divided by the present value results in the interest rate:

$$\text{rate} = (1 + i)^{CY} - 1$$

Formula 9.4 expresses this equation in terms of the variables for time value of money. It further adapts to any conversion between different compounding frequencies.

i_{new} is Converted Periodic Interest Rate: The new periodic interest rate expressed in a compounding frequency equal to CY_{New} . As noted below, if CY_{New} equals 1, then the new periodic interest rate is also the effective rate of interest (IY); that is, the rate compounded on an annual basis.

CY_{Old} is Original Compounding Frequency: This is how many times in a single year the original nominal interest rate is compounded.

Formula 9.4 – Interest Rate Conversion:
$$i_{\text{New}} = (1 + i_{\text{Old}})^{\frac{CY_{\text{Old}}}{CY_{\text{New}}}} - 1$$

i_{Old} is Original Periodic Interest Rate: This is the unrounded periodic rate for the original interest rate that is to be converted to its new compounding. This periodic rate results from Formula 9.1, $i = \frac{IY}{CY}$, where the original nominal interest rate is your IY and the original compounding frequency is your CY.

CY_{New} is Converted Compounding Frequency: This is how many times in a single year the newly converted interest rate compounds. If this variable is set to 1, then the result of the formula is the effective rate of interest.



How It Works

Follow these steps to calculate effective interest rates:

Step 1: Identify the known variables including the original nominal interest rate (IY) and original compounding frequency (CY_{Old}). Set the $CY_{\text{New}} = 1$.

Step 2: Apply Formula 9.1 to calculate the periodic interest rate (i_{Old}) for the original interest rate.

Step 3: Apply Formula 9.4 to convert to the effective interest rate. With a compounding frequency of 1, this makes $i_{\text{New}} = \text{IY}$ compounded annually.

Revisiting the opening scenario, comparing the interest rates of 6.6% compounded semi-annually and 6.57% compounded quarterly requires you to express both rates in the same units. Therefore, you could convert both nominal interest rates to effective rates.

	6.6% compounded semi-annually	6.57% compounded quarterly
Step 1	$IY = 6.6\%; CY_{Old} = 2; CY_{New} = 1$	$IY = 6.57\%; CY_{Old} = 4; CY_{New} = 1$
Step 2	$i_{Old} = 6.6\%/2 = 3.3\%$	$i_{Old} = 6.57\%/4 = 1.6425\%$
Step 3	$i_{New} = (1 + 0.033)^{2/1} - 1 = 6.7089\%$	$i_{New} = (1 + 0.016425)^{4/1} - 1 = 6.7336\%$

The rate of 6.6% compounded semi-annually is effectively charging 6.7089%, while the rate of 6.57% compounded quarterly is effectively charging 6.7336%. The better mortgage rate is 6.6% compounded semi-annually, as it results in annually lower interest charges.



Important Notes

The Texas Instruments BAII Plus calculator has a built-in effective interest rate converter called ICONV located on the second shelf above the number 2 key. To access it, press 2nd ICONV. You access three input variables using your $\uparrow$ or $\downarrow$ scroll buttons. Use this function to solve for any of the three variables, not just the effective rate.

Variable	Description	Algebraic Symbol
NOM	Nominal Interest Rate	IY
EFF	Effective Interest Rate	i_{New} (annually compounded)
C/Y	Compound Frequency	CY

To use this function, enter two of the three variables by keying in each piece of data and pressing ENTER to store it. When you are ready to solve for the unknown variable, scroll to bring it up on your display and press CPT. For example, use this sequence to find the effective rate equivalent to the nominal rate of 6.6% compounded semi-annually:

2nd ICONV, 6.6 Enter $\uparrow$, 2 Enter $\uparrow$, CPT

Answer: 6.7089



Paths To Success

Annually compounded interest rates can be used to quickly answer a common question: "How long does it take for my money to double?" The **Rule of 72** is a rule of thumb stating that 72% divided by the annually compounded rate of return closely approximates the number of years required for money to double. Written algebraically this is

$$\text{Approximate Years} = \frac{72\%}{IY \text{ annually}}$$

For example, money invested at 9% compounded annually takes approximately $72 \div 9\% = 8$ years to double (the actual time is 8 years and 15 days). Alternatively, money invested at 3% compounded annually takes approximately $72 \div 3\% = 24$ years to double (the actual time is 23 years and 164 days). Note how close the approximations are to the actual times.



Give It Some Thought

- If the nominal rate is 10%, what can you say about the effective rate?
- If you had effective rates of 8% and 9%, which would you select if you were
 - investing?
 - borrowing?
- If one investment takes 36 years to double while another takes 18 years to double, which has the higher effective rate?

Solutions:
 1. The effective rate is equal to or higher than the nominal rate.
 2a. 9%, since more interest is earned
 2b. 8%, since less interest is paid
 3. Eighteen years, since a higher rate takes less time to double.

Example 9.6A: Understanding Your Investment

If your investment earns 5.5% compounded monthly, what is the effective rate of interest? According to the Rule of 72, approximately how long will it take your investment to double at this effective rate?

Plan	Calculate the effective rate of interest (i_{New}). Once known, apply the Rule of 72 to approximate the doubling time.							
Understand	What You Already Know Step 1: The original nominal interest rate and compounding along with the new compounding are known: $iy = 5.5\%$ $CY_{\text{Old}} = \text{monthly} = 12$ $CY_{\text{New}} = 1$	How You Will Get There Step 2: Apply Formula 9.1 to the original interest rate. Step 3: Apply Formula 9.4, where $i_{\text{New}} = iy$ annually. Step 4: To answer approximately how long it will take for the money to double, apply the Rule of 72.						
Perform	Step 2: $i_{\text{Old}} = \frac{5.5\%}{12} = 0.458\bar{3}\%$ Step 3: $i_{\text{New}} = (1 + 0.00458\bar{3})^{\frac{12}{1}} - 1 = 0.056408 = 5.6408\%$ Step 4: Approximate Years = $\frac{72\%}{5.6408\%} = 12.76$	Calculator Instructions 2nd ICONV <table border="1"> <tr> <th>NOM</th><th>C/Y</th><th>EFF</th></tr> <tr> <td>5.5</td><td>12</td><td>Answer: 5.640786</td></tr> </table>	NOM	C/Y	EFF	5.5	12	Answer: 5.640786
NOM	C/Y	EFF						
5.5	12	Answer: 5.640786						
Present	You are effectively earning 5.6408% interest per year. At this rate of interest, it takes approximately $12\frac{3}{4}$ years, or 12 years and 9 months, for the principal to double.							

Example 9.6B: Your Car Loan

As you search for a car loan, all banks have quoted you monthly compounded rates (which are typical for car loans), with the lowest being 8.4%. At your last stop, the credit union agent says that by taking out a car loan with them, you would effectively be charged 8.65%. Should you go with the bank loan or the credit union loan?

Plan	Since it is normal for a car loan to be compounded monthly, convert the effective rate to a monthly rate (iy) so that it matches all the other quotes.							
Understand	What You Already Know Step 1: The effective rate and the compounding are as follows: $i_{\text{New}} = 8.65\%$ $CY_{\text{Old}} = \text{monthly} = 12$ $CY_{\text{New}} = 1$	How You Will Get There (Note: In this case, the i_{New} is known, so the process is reversed to arrive at the iy . Thus, steps 2 and 3 are performed in the opposite order.) Step 2: Substitute into Formula 9.4, rearrange, and solve for i_{Old} . Step 3: Substitute into Formula 9.1, rearrange, and solve for iy .						
Perform	Step 2: $0.0865 = (1 + i_{\text{Old}})^{12+1} - 1$ $1.0865 = (1 + i_{\text{Old}})^{12}$ $1.0865^{1/12} = 1 + i_{\text{Old}}$ $1.006937 = 1 + i_{\text{Old}}$ $0.006937 = i_{\text{Old}}$ Step 3: $0.006937 = \frac{iy}{12}$ $iy = 0.083249$ or 8.3249%	Calculator Instructions 2nd ICONV <table border="1"> <tr> <th>NOM</th><th>C/Y</th><th>EFF</th></tr> <tr> <td>Answer: 8.324896</td><td>12</td><td>8.65</td></tr> </table>	NOM	C/Y	EFF	Answer: 8.324896	12	8.65
NOM	C/Y	EFF						
Answer: 8.324896	12	8.65						
Present	The offer of 8.65% effectively from the credit union is equivalent to 8.3249% compounded monthly. If the lowest rate from the banks is 8.4% compounded monthly, the credit union offer is the better choice.							

Equivalent Interest Rates

At times you must convert a nominal interest rate to another nominal interest rate that is not an effective rate. For example, in the opening scenario of this section your mortgage rates were all quoted semi-annually except for one monthly rate. One way to compare these rates was to make them all effective rates. An alternative is to take the "oddball" rate and convert it to match the compounding of all the other rates. This brings up the concept of **equivalent interest rates**, which are interest rates with different compounding that produce the same effective rate and therefore are equal to each other. After one year, two equivalent rates have the same future value.



How It Works

To convert nominal interest rates you need no new formula. Instead, you make minor changes to the effective interest rate procedure and add an extra step. Follow these steps to calculate any equivalent interest rate:

Step 1: For the nominal interest rate that you are converting, identify the nominal interest rate (IY) and compounding frequency (CY_{Old}). Also identify the new compounding frequency (CY_{New}).

Step 2: Apply Formula 9.1 to calculate the original periodic interest rate (i_{Old}).

Step 3: Apply Formula 9.4 to calculate the new periodic interest rate (i_{New}).

Step 4: Apply Formula 9.1 using i_{New} and CY_{New}, rearrange, and solve for the new converted nominal interest rate (IY).

Once again revisiting the mortgage rates from the section opener, compare the 6.6% compounded semi-annually rate to the 6.57% compounded quarterly rate by converting one compounding to another. It is arbitrary which interest rate you convert. In this case, choose to convert the 6.57% compounded quarterly rate to the equivalent nominal rate compounded semi-annually.

Step 1: The original nominal interest rate IY = 6.57% and the CY_{Old} = quarterly = 4. Convert to CY_{New} = semi-annually = 2.

Step 2: Applying Formula 9.1, $i_{Old} = 6.57\% / 4 = 1.6425\%$.

Step 3: Applying Formula 9.4, $i_{New} = (1 + 0.016425)^{4 \div 2} - 1 = 0.033119$

Step 6: Applying Formula 9.1, $0.033119 = \frac{IY}{2}$ or IY = 0.06624.

Thus, 6.57% compounded quarterly is equivalent to 6.624% compounded semi-annually. Pick the mortgage rate of 6.6% compounded semi-annually since it is the lowest rate available. Of course, this is the same decision you reached earlier.



Important Notes

Converting nominal rates on the BAII Plus calculator takes two steps:

Step 1: Convert the original nominal rate and compounding to an effective rate. Input NOM and C/Y, then compute the EFF.

Step 2: Input the new C/Y and compute the NOM.

For the mortgage rate example above, use this sequence:

2nd ICONV, 6.57 Enter ↑, 4 Enter ↑, CPT ↓, 2 Enter ↓, CPT
Answer: 6.623956

A good way to memorize this technique is to remember to "go up then down the ladder." To convert a nominal rate to an effective rate, you press "↑" twice. To convert an effective rate back to a nominal rate, you press "↓" twice. Hence, you go up and down the ladder!



Things To Watch Out For

When converting interest rates, the most common source of error lies in confusing the two values of the compounding frequency, or CY. When working through the steps, clearly distinguish between the old compounding (CY_{Old}) that you want to convert from and the new compounding (CY_{New}) that you want to convert to. A little extra time spent on double-checking these values helps avoid mistakes.



Give It Some Thought

1. If you convert a monthly compounded rate into a semi-annually compounded rate, is the converted nominal rate higher or lower?
2. If you convert a quarterly rate into a monthly rate, is the converted nominal rate higher or lower?

Solutions:

1. Higher. The interest is placed into the account less frequently, so more interest needs to go in each time.
 2. Lower. The interest is placed into the account more frequently, so less interest is needed each time.

Example 9.6C: Which Investment to Choose?

You are looking at three different investments bearing interest rates of 7.75% compounded semi-annually, 7.7% compounded quarterly, and 7.76% compounded semi-annually. Which investment offers the highest interest rate?

Plan	Notice that two of the three interest rates are compounded semi-annually while only one is compounded quarterly. Although you could convert all three to effective rates (requiring three calculations), it is easier to convert the quarterly compounded rate to a semi-annually compounded rate. Then all rates are compounded semi-annually and are therefore comparable.														
Understand	What You Already Know Step 1: The original interest rate and compounding are known: $IY = 7.7\%$ $CY_{Old} = \text{quarterly} = 4$ $CY_{New} = \text{semi-annually} = 2$	How You Will Get There Step 2: Apply Formula 9.1 to the original interest rate. Step 3: Convert the original interest rate to its new periodic rate using Formula 9.4. Step 4: Substitute into Formula 9.1 and rearrange for IY .													
Perform	Step 2: $i_{Old} = \frac{7.7\%}{4} = 1.925\%$ Step 3: $i_{New} = (1 + 0.01925)^{4 \div 2} - 1 = 0.038870$ Step 4: $0.038870 = \frac{IY}{2}$ $IY = 0.077741$ or 7.7741%	Calculator Instructions 2nd ICONV <table border="1" style="width: 100%; border-collapse: collapse; text-align: center;"> <thead> <tr> <th>Steps</th><th>NOM</th><th>C/Y</th><th>EFF</th></tr> </thead> <tbody> <tr> <td>1-3</td><td>7.7</td><td>4</td><td>Answer: 7.925204</td></tr> <tr> <td>4-6</td><td>Answer: 7.774112</td><td>2</td><td>✓</td></tr> </tbody> </table>		Steps	NOM	C/Y	EFF	1-3	7.7	4	Answer: 7.925204	4-6	Answer: 7.774112	2	✓
Steps	NOM	C/Y	EFF												
1-3	7.7	4	Answer: 7.925204												
4-6	Answer: 7.774112	2	✓												
Present	The quarterly compounded rate of 7.7% is equivalent to 7.7741% compounded semi-annually. In comparison to the semi-annually compounded rates of 7.75% and 7.76%, the 7.7% quarterly rate is the highest interest rate for the investment.														



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9.7: Determining the Number of Compounds

(How Far Away Is That?)

How long will it take to reach a financial goal? At a casual get-together at your house, a close friend discusses saving for a 14-day vacation to a five star resort in the Mayan Riviera of Mexico upon graduation. The estimated cost from a travel website is \$1,998.94 including fares and taxes. He has already saved \$1,775 into a fund earning 8% compounded quarterly. Assuming the costs remain the same and he makes no further contributions, can you tell him how soon he will be basking in the sun on the beaches of Mexico?

This section shows you how to calculate the time frame for single payment compound interest transactions. You can apply this knowledge to any personal financial goal. Or in your career, if you work at a mid-size to large company, you might need to invest monies with the objective of using the funds upon maturity to pursue capital projects or even product development opportunities. So knowing the time frame for the investment to grow large enough will allow you to schedule the targeted projects.

Introductory scenarios examine situations where the number of compounding periods works out to be an integer. Then you will tackle more challenging scenarios involving time frame computations with noninteger compounding periods.

Integer Compounding Periods

Some applications of solving for the number of compounding periods include the following:

- Determining the time frame to meet a financial goal
- Calculating the time period elapsing between a present and future value
- Evaluating the performance of financial investments



The Formula

To solve for the number of compounds you need Formula 9.3 one more time. The only difference from your previous uses of this formula is that the unknown variable changes from FV to N, which requires you to substitute and rearrange the formula algebraically.

FV is Future Value or Maturity Value:
The ending value for the calculation.

PV is Present Value or Principal: The starting value for the calculation.

Formula 9.3 – Compound Interest For Single Payments: $FV = PV(1 + i)^N$

i is Periodic Interest Rate: This is the periodic rate of interest used in converting the interest to principal. It results from Formula 9.1.

N is Number of Compound Periods: The unknown variable N represents the total number of compounds required for the present value to grow to the future value. Recall from Formula 9.2 that $N = CY \times \text{Years}$. Once you calculate N, you must express it in more common terms (when was the last time you asked a bank to give you “an eight compound period loan”?). Rearranging Formula 9.2 for Years, you then calculate years as $\frac{N}{CY}$. Therefore, a quarterly compounded loan, $CY = 4$, with an $N = 8$ is a loan extending $\frac{8}{4} = 2$ years.



How It Works

Follow these steps to compute the number of compounding periods (and ultimately the time frame):

Step 1: Draw a timeline to visualize the question. Most important at this step is to identify PV, FV, and the nominal interest rate (both IY and CY).

Step 2: Solve for the periodic interest rate (i) using Formula 9.1.

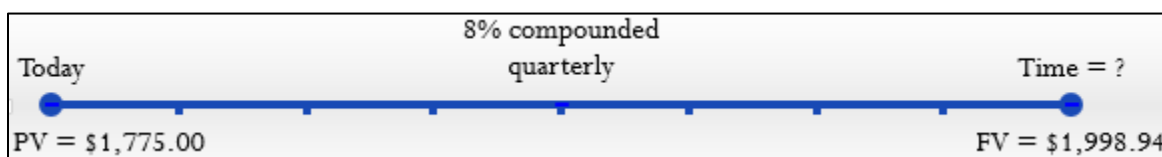
Step 3: Substitute into Formula 9.3, rearrange, and solve for N . Note that the value of N represents the number of compounding periods. For example, if the compounding is quarterly, a value of $N = 9$ is nine quarters.

Step 4: Take the value of N and convert it back to a more commonly expressed format such as years and months. When the number of compounding periods calculated in step 3 works out to an integer, handling step 4 involves applying the rearranged Formula 9.2 and solving for Years = $\frac{N}{CY}$.

1. If the Years is an integer, you are done.
2. If the Years is a noninteger, the whole number portion (the part in front of the decimal) represents the number of years. As needed, take the decimal number portion (the part after the decimal point) and multiply it by 12 to convert it to months. For example, if you have Years = 8.25 then you have 8 years plus $0.25 \times 12 = 3$ months, or 8 years and 3 months.

Revisiting the opening scenario, your friend has saved \$1,775 and needs it to become \$1,998.94 at 8% compounded quarterly. How long will it take?

Step 1: The timeline illustrates this scenario. Note that $IY = 8\%$ and $CY = \text{quarterly} = 4$.



Step 2: The periodic interest rate is $i = 8\% / 4 = 2\%$.

Step 3: Applying Formula 9.3, you have $\$1,998.94 = \$1,775(1 + 0.02)^N$ or $N = 6$ (details of the algebra can be found in subsequent examples).

Step 4: Applying the rearranged Formula 9.2, Years = $\frac{6}{4} = 1.5$. Your friend will be headed to the Mayan Riviera in 1½ years. If you prefer to express this in months, it is 1 year plus $0.5 \times 12 = 6$ months, or 1 year and 6 months.



Important Notes

You can use your financial calculator in the exact same manner as in Section 9.2 except that your unknown variable will be N instead of FV . As always, you must load the other six variables and obey the cash flow sign convention. Once you compute N , you will have to complete step 4 manually to express the solution in its more common format of years and months.



Things To Watch Out For

N rarely works out to an integer, and that causes a lot of grief. When the values of both the PV and FV are known, usually one of these numbers is rounded. Because of this rounding, the calculation of N almost always produces decimals, even if those decimals are very small. For example, in calculating $N = 6$ it is possible to come up with a value of 5.999963 or 6.000018. How can you tell if those decimals result from a rounded value (meaning that N is really an integer) or if those decimals are significant and need to be dealt with as they stand (as you will see discussed below when we consider noninteger compounding periods)?

You determine which way to go by mentally rounding your calculated N to the third decimal place. If this results in an integer, then the decimals are due to the rounding of FV and from then on you can treat the N as an integer. If a noninteger is still present, then the decimals are meaningful and you will have to address a noninteger compounding period (as discussed later in this section). For example, 5.999963 when mentally rounded to three decimals is 6.000. Therefore, you consider the N to be 6 and not 5.999963. In contrast, 5.985996 when mentally rounded to three decimals is 5.986, which is a noninteger. This means the decimals are significant, so you will not be able to round off to the closest integer.

**TOP
SECRET****Paths To Success**

In Section 9.6, the Rule of 72 allowed you to estimate the time it takes in years for money to double when you know the effective rate. When you work with any time frame that involves the doubling of money, this rule can allow you to quickly check whether your calculated value of N is reasonable.

**Give It Some Thought**

- Listed are some values of N . Determine if you should treat the N like an integer or a noninteger.
 - 9.000345
 - 5.993129
 - 6.010102
 - 11.999643
 - 10.000987
 - 3.999999
- Of \$1,000 growing to \$2,000 or \$2,000 growing to \$3,000, which will take longer at the same interest rate?

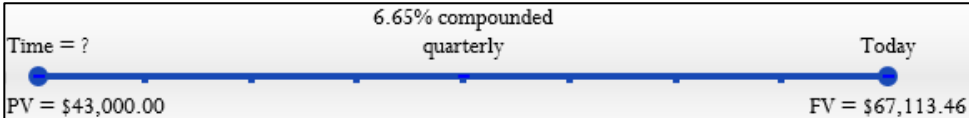
Solutions:

1. a. 9.000 integer b. 5.993 noninteger c. 6.010 noninteger d. 11.999 integer e. 10.001 noninteger f. 4.000 integer

2. \$1,000 growing to \$2,000 will take longer since it involves a doubling of money, while \$2,000 growing to \$3,000 increases the amount by less than a doubling.

Example 9.7A: Integer Compounding Period Investment

Jenning Holdings invested \$43,000 at 6.65% compounded quarterly. A report from the finance department shows the investment is currently valued at \$67,113.46. How long has the money been invested?

Plan	Determine the amount of time that the principal has been invested. This requires calculating the number of compounding periods (N).																			
Understand	What You Already Know Step 1: The principal, future value, and interest rate are known, as illustrated in the timeline. IY = 6.65% CY = 4		How You Will Get There Step 2: Apply Formula 9.1. Step 3: Substitute into Formula 9.3, rearrange, and solve for N. Recall from Section 2.6 that to solve an equation for an unknown exponent you take the logarithm of both sides. You will also need to apply the general rule for logarithm of a power $\ln(x^y) = y(\ln x)$. Step 4: Apply Formula 9.2, rearrange, and solve for Years.																	
																				
Perform	Step 2: $i = \frac{6.65\%}{4} = 1.6625\%$ Step 3: $\$67,113.46 = \$43,000(1 + 0.016625)^N$ $1.560778 = 1.016625^N$ $\ln(1.560778) = \ln(1.016625^N) = N \times \ln(1.016625)$ $0.445184 = N \times 0.016488$ $N = 26.999996$ or 27 quarterly compounds Step 4: $27 = 4 \times \text{Years}$ Years = 6.75, which is 6 years plus $0.75 \times 12 = 9$ months				Calculator Instructions															
					<table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Answer: 26.999996</td><td>6.65</td><td>-43000</td><td>0</td><td>67113.46</td><td>4</td><td>4</td></tr></table>			N	I/Y	PV	PMT	FV	P/Y	C/Y	Answer: 26.999996	6.65	-43000	0	67113.46	4
N	I/Y	PV	PMT	FV	P/Y	C/Y														
Answer: 26.999996	6.65	-43000	0	67113.46	4	4														
Present	Jenning Holdings has had the money invested for six years and nine months.																			

Noninteger Compounding Periods

When the number of compounding periods does not work out to an integer, the method of calculating N does not change. However, what changes is the method by which you convert N and express it in more common terms (step 4 of the process). Typically, the noninteger involves a number of years, months, and days.

As summarized in the table below, to convert the compounding period into the correct number of days you can make the following assumptions:

Compounding Period	# of Days in the Period
Annual	365
Semi-annual	182*
Quarter	91*
Month	30*
Week	7
Daily	1

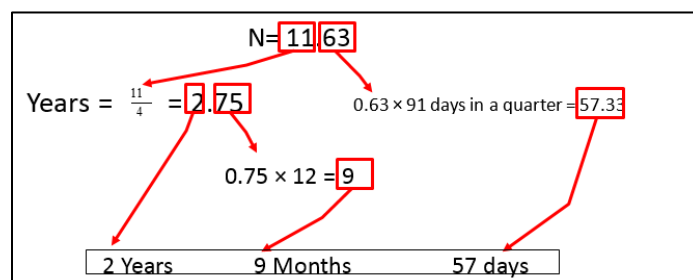


How It Works

You still use the same four steps to solve for the number of compounding periods when N works out to a noninteger as you did for integers. However, in step 4 you have to alter how you convert the N to a common expression. Here is how to convert the value:

1. Separate the integer from the decimal for your value of N .
2. With the integer portion, apply the same technique used with an integer N to calculate the number of years and months.
3. With the decimal portion, multiply by the number of days in the period to determine the number of days and round off the answer to the nearest day (treating any decimals as a result of a rounded interest amount included in the future value).

The figure to the right illustrates this process for an $N = 11.63$ with quarterly compounding, or $CY = 4$. Thus, an $N = 11.63$ quarterly compounds converts to a time frame of 2 years, 9 months, and 57 days. Note that without further information, it is impossible to simplify the 57 days into months and days since it is unclear whether the months involved have 28, 29, 30, or 31 days.



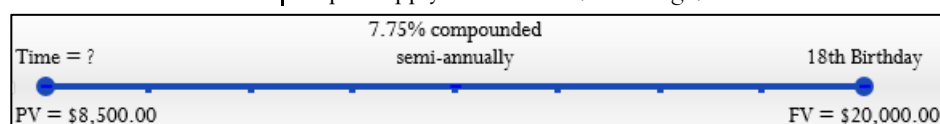
Example 9.7B: Saving for Postsecondary Education

Tabitha estimates that she will need \$20,000 for her daughter's postsecondary education when she turns 18. If Tabitha is able to save up \$8,500, how far in advance of her daughter's 18th birthday would she need to invest the money at 7.75% compounded semi-annually?

Plan Determine the amount of time in advance of the daughter's 18th birthday that the money needs to be invested. The number of compounding periods (N) must be calculated.

What You Already Know
 Step 1: The principal, future value, and interest rate are known, as illustrated in the timeline.
 $IY = 7.75\%$ $CY = 2$

How You Will Get There
 Step 2: Apply Formula 9.1.
 Step 3: Substitute into Formula 9.3, rearrange, and use the rule for logarithm of a power to solve for N .
 Step 4: Apply Formula 9.2, rearrange, and solve for Years.



Perform	Step 2: $i = \frac{7.75\%}{2} = 3.875\%$	Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Answer: 22.506828</td><td>7.75</td><td>-8500</td><td>0</td><td>20000</td><td>2</td><td>2</td></tr></table>	N	I/Y	PV	PMT	FV	P/Y	C/Y	Answer: 22.506828	7.75	-8500	0	20000	2	2
	N		I/Y	PV	PMT	FV	P/Y	C/Y								
Answer: 22.506828	7.75	-8500	0	20000	2	2										
	Step 3: $\$20,000 = \$8,500(1 + 0.03875)^N$ $2.352941 = 1.03875^N$ $\ln(2.352941) = N \times \ln(1.03875)$ $0.855666 = N \times 0.038018$ $N = 22.506828$ semi-annual compounds															
	Step 4: Take the integer: $22 = 2 \times \text{Years}$ $\text{Years} = 11$ Take the decimal: semi-annual compounding according is 182 days, so $0.506828 \times 182 \text{ days} = 92 \text{ days}$															
Present	If Tabitha invests the \$8,500 11 years and 92 days before her daughter's 18th birthday, it will grow to \$20,000.															



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Chapter 9 Summary

Key Concepts Summary

Section 9.1: Compound Interest Fundamentals (What Can a Dollar Buy?)

- How compounding works
- How to calculate the periodic interest rate

Section 9.2: Determining the Future Value (I Want to Pay Later)

- The basics of taking a single payment and moving it to a future date
- Moving single payments to the future when variables change

Section 9.3: Determining the Present Value (I Want to Pay Earlier)

- The basics of taking a single payment and moving it to an earlier date
- Moving single payments to the past when variables change

Section 9.4: Equivalent Payments (Let's Change the Deal)

- The concept of equivalent payments
- The fundamental concept of time value of money
- The fundamental concept of equivalency
- Applying single payment concepts to loans and payments

Section 9.5: Determining the Interest Rate (Is a Lot of Interest Accumulating or Just a Little?)

1. Solving for the nominal interest rate
2. How to convert a variable interest rate into its equivalent fixed interest rate

Section 9.6: Equivalent and Effective Interest Rates (How Do I Compare Different Rates?)

- The concept of effective rates
- Taking any nominal interest rate and finding its equivalent nominal interest rate

Section 9.7: Determining the Number of Compounds (How Far Away Is That?)

- Figuring out the term when N is an integer
- Figuring out the term when N is a noninteger

The Language of Business Mathematics

compound interest A system for calculating interest that primarily applies to long-term financial transactions with a time frame of one year or more; interest is periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.

compounding frequency The number of compounding periods in a complete year.

compounding period The amount of time that elapses between the dates of successive conversions of interest to principal.

discount rate An interest rate used to remove interest from a future value.

effective interest rate The true annually compounded interest rate that is equivalent to an interest rate compounded at some other (non-annual) frequency.

equivalent payment streams Equating two or more alternative financial streams such that neither party receives financial gain or harm by choosing either stream.

equivalent interest rates Interest rates with different compounding that produce the same effective rate and therefore are equal to each other.

focal date A point in time to which all monies involved in all payment streams will be moved using time value of money calculations.

fundamental concept of equivalency Two or more payment streams are equal to each other if they have the same economic value on the same focal date.

fundamental concept of time value of money All monies must be brought to the same focal date before any mathematical operations, decisions, or equivalencies can be determined.

nominal interest rate A nominal number for the annual interest rate, which is commonly followed by words that state the compounding frequency.

periodic interest rate The percentage of interest earned or charged at the end of each compounding period.

present value principle for loans The present value of all payments on a loan is equal to the principal that was borrowed.

Rule of 72 A rule of thumb stating that 72 divided by the annually compounded rate of return closely approximates the number of years required for money to double.

The Formulas You Need to Know

Symbols Used

CY or C/Y = Compounds per year, or compounding frequency

CY_{New} = the new compounding frequency an interest rate is converted to

CY_{Old} = the old compounding frequency an interest rate is converted from

FV = Future value, or maturity value

i = Periodic interest rate

i_{New} = the new periodic interest rate after a conversion

i_{Old} = the old periodic interest rate before a conversion

IY or I/Y = Nominal interest rate per year

$\ln$ = natural logarithm

N = number of compound periods

PV = Present value, or principal

Formulas Introduced

Formula 9.1 Periodic Interest Rate: $i = \frac{IY}{CY}$ (Section 9.1)

Formula 9.2 Number of Compound Periods for Single Payments: $N = CY \times \text{Years}$ (Section 9.2)

Formula 9.3 Compound Interest for Single Payments: $FV = PV(1 + i)^N$ (Section 9.2)

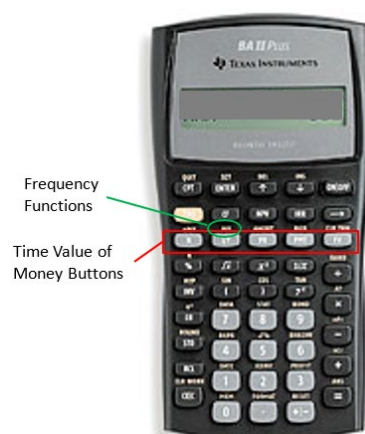
Formula 9.4 Interest Rate Conversion: $i_{New} = (1 + i_{Old})^{\frac{CY_{Old}}{CY_{New}}} - 1$ (Section 9.6)

Technology

Calculator

Time Value of Money Buttons

- The time value of money buttons are the five buttons located on the third row of your calculator.



Calculator Symbol	Formula Symbol	Characteristic	Data Entry Requirements
N	N	The number of compounding periods (from Formula 9.2)	An integer or decimal number; no negatives
I/Y	IY	The nominal interest rate per year	Percent format without the % sign (i.e., 7% is just 7)
PV	PV	Present value or principal	An integer or decimal number
PMT	PMT	Used for annuity payment amounts (covered in Chapter 11) and is not applicable to lump-sum amounts; it needs to be set to zero for lump-sum calculations	An integer or decimal number
FV	FV	Future value or maturity value	An integer or decimal number

To enter any information into any one of these buttons or variables, called loading the calculator, key in the information first and then press the appropriate button.

- The frequency function is logically placed above the I/Y button and is labelled P/Y. This function addresses compound interest frequencies, such as the compounding frequency. Access the function by pressing 2nd P/Y to find the following entry fields, through which you can scroll using your ↓ and ↑ arrow buttons.

Calculator Symbol	Formula Symbol	Characteristic	Data Entry Requirements
P/Y	PY	Annuity payments per year (payment frequency is introduced in Chapter 11); when working with lump-sum payments and not annuities, the calculator requires this variable to be set to match the C/Y	A positive, nonzero number only
C/Y	CY	Compounds per year (compounding frequency)	A positive, nonzero number only

- To enter any information into one of these fields, scroll to the field on your screen, key in the data, and press Enter.
- When you enter a value into the P/Y field, the calculator will automatically copy the value into the C/Y field for you. If in fact the C/Y is different, you can change the number manually.
- To exit the P/Y window, press 2nd Quit.

Keying in a Question

- You must load the calculator with six of the seven variables.
- To solve for the missing variable, press CPT followed by the variable.

Cash Flow Sign Convention

- The cash flow sign convention is used for the PV, PMT, and FV buttons.
- If money leaves you, you must enter it as a negative.
- If money comes at you, you must enter it as a positive.

Clearing the Memory

- Once you enter data into any of the time value buttons, it is permanently stored until one of the following happens:
 - You override it by entering another piece of data and pressing the button.
 - You clear the memory of the time value buttons by pressing 2nd CLR TVM (a step that is recommended before you proceed with a separate question).
 - You press the reset button on the back of the calculator.

Chapter 10: Compound Interest: Applications Involving Single Payments

(In the Real World, Where Do I Put This Knowledge to Work?)

Everywhere you look, financial advertisements appeal for a share of your hard-earned money. It is hard to go a day without seeing at least one of the following:

- Television ads about GICs that earn you the maximum money
- Bank billboards and signage indicating maximum interest rates
- Magazine ads for high-interest bonds
- Internet banner ads and unsolicited emails promising safe investments and high interest rates

Headlines on news websites blare that costs are rising from inflation and how your purchasing power is dropping. Revealing articles tell how underfunding of the Canada Pension Plan (CPP) and Old Age Security (OAS) will leave you without sufficient retirement income in your golden years. The CPP in 2021 offered a maximum benefit of \$1203.75 per month to a limited few, with the average CPP collector earning only \$689.17 per month. OAS pays a maximum \$618.45 per month, with the average pensioner receiving \$592.29. If you retired today your monthly income for life (to be indexed for inflation) is \$1,281.46 with a \$1,822.20 maximum. Very few people receive the maximum. To make matters worse, OAS and CPP income is taxable by the government! If you don't start investing your money into your tax-sheltered RRSP or TFSA (Tax-Free Savings Account) soon and provide your own source of retirement funds, you will be cash poor in retirement. It is no wonder that many mid-size to large companies help their loyal workers save for retirement by offering employee benefit plans from company-sponsored GIC plans to high-interest savings.

In retail, endless sources of credit and no-payment plans are available on a wide range of merchandise. These highly successful plans allow consumers to buy merchandise they probably couldn't otherwise get. The Brick offers no payments for at least 15 months, while Leon's hosts the "Don't pay a cent" event. Canadian consumers increasingly purchase merchandise through such plans. Meanwhile, behind the scenes, retailers are buying and selling these plans to finance companies to liquidate their outstanding accounts receivables.

This whole chapter uses compound interest involving single payments to deal with real-life applications. For all of them your knowledge of future values, present values, interest rates, and terms is vital. This chapter introduces a range of investment options such as long-term GICs, and strip bonds. You will see how businesses buy and sell promissory notes. You will also learn how to extend time value of money concepts to applications involving rates of inflation, purchasing power, and rates of change.

Your knowledge of compound interest concepts is about to become very practical. So buckle up your seatbelt as this chapter takes you on an exciting ride!

Outline of Chapter Topics

10.1: Application: Long-Term GICs (Keeping Your Money Safe When Investing)

10.2: Application: Long-Term Promissory Notes (IOUs)

10.3: Application: Strip Bonds (Buy Low, Sell High)

10.4: Application: Inflation, Purchasing Power, and Rates of Change (Your Grandparents Used to Go to a Movie for a Quarter)

10.1: Application: Long-Term GICs

(Keeping Your Money Safe When Investing)

Recall that Guaranteed Investment Certificates (GICs) are investments offering a guaranteed rate of interest over a predetermined time period. Whereas short-term GICs in Section 8.3 involved terms less than one year, most long-term GICs range from one to five years. Though terms longer than this are available, they are not very common.

Also recall that Section 8.3 discussed the factors that determine interest rates for short-term GICs. The same factors apply to long-term GICs: To receive the highest interest rate on a GIC, you should still invest a large principal in a nonredeemable GIC for the longest term possible.

The key difference between short- and long-term GICs lies in the compounding of interest. Long-term GICs do not wait until the end of the term for interest on them to appear and be paid out. Rather, in line with the definition of compound interest, a long-term GIC periodically converts the accrued interest into principal throughout the transaction. Although GICs come in many varieties (remember, financial institutions try to market these products attractively to investors), three structures are commonly available:

1. **Interest Payout GICs.** An **interest payout GIC** uses interest rates that by all appearances you might assume to be compounded periodically since they are listed side-by-side with compound interest rates. In practice, though (and by reading the fine print), you will find that the periodically calculated interest is never added to the principal of the GIC, and in essence the concepts of simple interest are used. Instead, the interest is paid out to the investor (perhaps into a chequing account) and does not actually compound unless the investor takes the interest payment received and invests it in another compounding investment. Interest payout GIC interest rates can take either a fixed or variable format. For example, in an online browsing of long-term GICs you may find a posted rate on a three-year GIC at 2% semi-annually. The fine print and footnotes may show that the interest is paid out on a simple interest basis at the end of each six months.
2. **Compound Interest GICs.** A **compound interest GIC** uses compound interest rates for which interest is periodically calculated and converted to the principal of the GIC for further compounding. Interest rates can be either fixed or variable.
3. **Escalator Interest GICs.** An **escalator interest GIC** uses compound interest rates that usually remain constant during each of a series of time intervals, always rising stepwise throughout the term of the investment with any accrued interest being converted to principal.

When you invest in a GIC, you are in essence lending the bank your money in return for earning an interest rate. Financial institutions commonly use the money raised from GICs to fund mortgages. As a result, the posted interest rates on GICs hover at 1% to 2% lower than the interest rates on mortgages. Thus, the bank earns the interest from its mortgagors, paying only a portion of it to its GIC investors. The rest of this section discusses each of the three types of GICs separately.

Interest Payout GIC

The interest payout GIC is mathematically the least interesting of the three types of GICs since the interest, while periodically calculated based on compound interest rates, does not get converted to principal. Instead, the interest payment is paid out to an account specified by the investor. Therefore, usually the only variable of concern on an interest payout GIC is the amount of the interest payment.

As a source of confusion, in the marketplace many financial institutions refer to interest payout GICs as simple interest GICs, because the interest never converts to principal. Examining bank websites such as CIBC (<https://www.cibc.com/en/interest-rates/gic-rates.html>), TD Canada Trust (<https://www.td.com/ca/en/personal-banking/personal-investing/products/gic/gic-rates-canada>), or HSBC (<https://www.hsbc.ca/bank-with-us/todays-rates/>) reveals a variety of long-term GICs referred to as using simple interest. Why do these GICs then appear in this chapter and not previously in Chapter 8? The answer is threefold:

1. **Periodic Interest Payments.** In interest payout GICs, interest is periodically paid out throughout the term of the GIC. This differs from the GICs discussed in Chapter 8 in that simple interest required interest to be calculated and added to the principal only at the end of the transaction's time period.
2. **Rates Used in Calculating the Interest Amount.** In interest payout GICs, the posted rates appear intermixed with the posted compound interest rates. For example, a rate of 2% semi-annually may be posted (the word *compounded* is

usually omitted to reduce confusion with compound interest GICs), which means that 1% interest is paid out every six months.

3. **Length of Time.** The terms involved with interest payout GICs are longer than one year, meeting the definition of "long-term" GICs.



The Formula

Calculating the interest payment requires a simplification of Formula 9.3 involving compound interest for single payments. As with compound interest GICs, you use the periodic interest rate calculated through Formula 9.1. However, in an interest payout GIC you never add the interest to the principal, so you do not need the $1 + i$ term in the formula. There is also no future value to calculate, just an interest amount. This changes Formula 9.3 from $FV = PV \times (1 + i)^N$ to $I = PV \times i^N$. Simplifying further, you calculate the interest one compound period at a time, where $N = 1$. This eliminates the need for the N exponent and establishes Formula 10.1.

I is Interest Payment Amount: You determine the dollar amount of any interest payment by multiplying the principal in the account by the periodic rate of interest. Note that this formula is another version of the Rate, Portion, Base formula from Section 2.3, where the base is PV , the rate is i , and the interest payment is I .

PV is Principal: The sum of money in the account upon which interest is earned. In an interest payout GIC, it is the amount of money originally deposited into the account.

Formula 10.1 – Periodic Interest Amount: $I = PV \times i$

i is Periodic Interest Rate: A result of Formula 9.1, $i = \frac{IY}{CY}$, this is the rate of interest that is used in calculating the interest payment amount. In an interest payout GIC, the periodic rate might be based on a posted rate of 2% semi-annually, interpreted as $1Y = 2\%$ and $CY = 2$



How It Works

Follow these steps to calculate the interest payment for an interest payout GIC:

Step 1: Identify the amount of principal invested. This is the PV . Also determine the nominal interest rate (IY) and interest payout frequency (CY).

Step 2: Using Formula 9.1, calculate the periodic interest rate (i).

Step 3: Apply Formula 10.1 to calculate the interest amount.

Assume \$5,000 is invested for a term of three years at 5% quarterly in an interest payout GIC. Calculate the amount of the quarterly interest payment.

Step 1: The principal invested is \$5,000, or $PV = \$5,000$. The nominal interest rate is $IY = 5\%$. The frequency is $CY = 4$ for quarterly.

Step 2: Applying Formula 9.1 results in $i = 5\% / 4 = 1.25\%$.

Step 3: Applying Formula 10.1, $I = \$5,000 \times 0.0125 = \$62.50$.

The investor receives an interest payment of \$62.50 every quarter throughout the term of the investment. A term of three years then means that there are $N = 4 \times 3 = 12$ payments totalling $\$62.50 \times 12 = \$750$ of interest. At the end of the three years, the GIC matures and the investor is paid out the principal of \$5,000.



Important Notes

If the interest rate is variable, you must apply Formula 10.1 to each of the variable interest rates in turn to calculate the interest payment amount in the corresponding time segment. For example, if a \$1,000 GIC earns 2% quarterly for the first year and 2.4% monthly for the second year, then in the first year $I = \$1,000 \times \frac{2\%}{4} = \$5$, and in the second year

$$I = \$1,000 \times \frac{2.4\%}{12} = \$2.$$



Give It Some Thought

In an interest payout GIC, is the maturity value of the investment higher than, lower than, or the same as the principal invested in the GIC?

Solutions:
The maturity value and the principal are the same since the interest is never converted to principal.

Example 10.1A: An Interest Payout GIC

Jackson placed \$10,000 into a four-year interest payout GIC earning 5.5% semi-annual interest. Calculate the amount of each interest payment and the total interest earned throughout the term.

Plan	Calculate the periodic interest payment amount (I). Then calculate the total interest paid for the term based on N .	
Understand	What You Already Know Step 1: $PV = \$10,000$ $IY = 5.5\%$ $CY = 2$	How You Will Get There Step 2: Apply Formula 9.1. Step 3: Apply Formula 10.1. Step 4: To calculate the total interest, determine N using Formula 9.2 and multiply the payment by N .
Perform	Step 2: $i = \frac{5.5\%}{2} = 2.75\%$ Step 3: $I = \$10,000 \times 0.0275 = \$275$ Step 4: $N = 2 \times 4 = 8$; total interest = $\$275 \times 8 = \$2,200$	
Present	Every six months, Jackson receives an interest payment of \$275. Over the course of four years, these interest payments total \$2,200.	

Compound Interest GIC

Throughout the term of a compound interest GIC, interest is periodically converted to principal. A starting amount, called the principal, remains in the account for the entire term and compounds interest. Therefore, you treat a compound interest GIC exactly like a future value compound interest calculation on a single payment amount.



How It Works

Compound interest GICs do not require any new formulas or techniques. Most commonly, the variables of concern are either the maturity value of the investment or the compound interest rate.

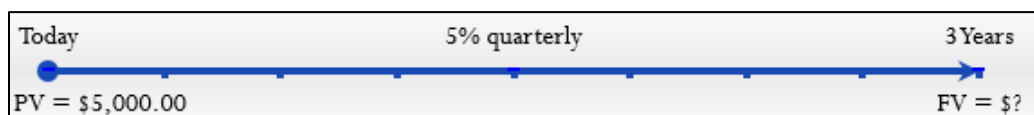
1. **Maturity Value.** If the compound interest rate is fixed, then you find the maturity value by applying Formula 9.3 once, where $FV = PV(1 + i)^N$. These 4 steps were introduced in Chapter 9.2.

If the compound interest rate is variable, then to find the maturity value you must apply Formula 9.3 once for each segment of the timeline. These 7 steps were also introduced in Chapter 9.2. Note that in step 5, no principal adjustment needs to be made since only the interest rate variable changes.

2. **Compound Interest Rate.** In the event that the unknown variable is the interest rate, recall the 6 steps were introduced in Chapter 9.5.

Assume an investment of \$5,000 is made into a three-year compound interest GIC earning 5% compounded quarterly. Solve for the maturity value.

Step 1: The timeline below illustrates this investment. This is a fixed rate compound interest GIC with a term of three years and $PV = \$5,000$. The nominal interest rate is $IY = 5\%$. The compounding frequency is $CY = 4$.



Step 2: The periodic interest rate is $i = 5\%/4 = 1.25\%$.

Step 3: The number of compound periods is $N = 4 \times 3 = 12$.

Step 4: Applying Formula 9.3, $FV = \$5,000(1 + 0.0125)^{12} = \$5,803.77$. Hence, at maturity the GIC contains \$5,803.77, consisting of \$5,000 of principal and \$803.77 of compound interest.



Give It Some Thought

Supposing that each investment is held until maturity, which earns more interest, an interest payout GIC or a compound interest GIC?

Solution:

The compound interest GIC earns more interest since the interest is converted to principal and therefore earns even more interest. The interest payout GIC does not compound.

Example 10.1B: How Much Do You Have at Maturity?

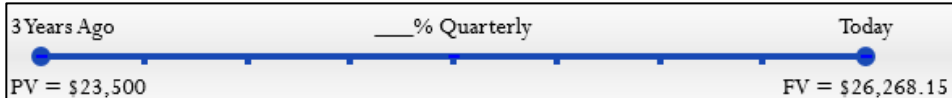
Andrej invested \$23,500 into a three-year variable compound interest GIC. The quarterly compounded interest rate was 3.8% for the first 15 months, 3.7% for the next 12 months, and 3.65% after that. What is the maturity value of Andrej's GIC?

Plan	Calculate the maturity value (FV) of Andrej's variable rate compound interest GIC.	
Understand	What You Already Know Step 1: The principal, terms, and interest rates are known, as shown in the timeline. $PV = \\$23,500$ First time segment: $IY = 3.8\%$ $CY = \text{quarterly} = 4$ Term = $1\frac{1}{4}$ years Second time segment: $IY = 3.7\%$ $CY = \text{quarterly} = 4$ Term = 1 year Third time segment: $IY = 3.65\%$ $CY = \text{quarterly} = 4$ Term = $\frac{3}{4}$ year	How You Will Get There Step 2: For each time segment, calculate the periodic interest rate by applying Formula 9.1. Step 3: For each time segment, calculate the number of compound periods by applying Formula 9.2. Step 4: Calculate the future value (FV_1) of the first time segment using Formula 9.3. Step 5: Let $FV_1 = PV_2$. Step 6: Calculate the future value (FV_2) of the second time segment using Formula 9.3. Repeat Step 5: Let $FV_2 = PV_3$. Repeat Step 6: Calculate the future value (FV_3) of the third time segment using Formula 9.3. Step 7: FV_3 is the final future value amount.

Perform	Step	First Time Segment	Second Time Segment	Third Time Segment				
	2.	$i = \frac{3.8\%}{4} = 0.95\%$	$i = \frac{3.7\%}{4} = 0.925\%$	$i = \frac{3.65\%}{4} = 0.9125\%$				
	3.	$N = 4 \times 1\frac{1}{4} = 5$	$N = 4 \times 1 = 4$	$N = 4 \times \frac{3}{4} = 3$				
	4.	$FV_1 = \$23,500 \times (1 + 0.0095)^5 = \$24,637.66119$						
	5–6.	$FV_2 = \$24,637.66119 \times (1 + 0.00925)^4 = \$25,561.98119$						
	Repeat 5–6.	$FV_3 = \$25,561.98119 \times (1 + 0.009125)^3 = \$26,268.15$						
	Calculator Instructions							
	Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
	1	5	3.8	–23500	0	Answer: –24,637.66119	4	4
	2	4	3.7	24,637.66119	✓	Answer: –25,561.98119	✓	✓
	3	3	3.65	25,561.98119	✓	Answer: –26,268.14515	✓	✓
Present	At the end of the three-year compound interest GIC, Andrej has \$26,268.15, consisting of the \$23,500 principal plus \$2,768.15 in interest.							

Example 10.1C: Equivalent Interest Rate on the GIC

Using Example 10.1B, what equivalent fixed quarterly compounded interest rate did Andrej earn on his GIC?

Plan	Calculate the fixed quarterly compounded interest rate (IY) that is equivalent to the three variable interest rates.																				
Understand	What You Already Know Step 1: From Example 10.1B, the principal, maturity value, compounding frequency, and term are known, as illustrated in the timeline. CY = 4 Term = 3 years			How You Will Get There Step 2: Calculate N using Formula 9.2. Step 3: Substitute into Formula 9.3 and rearrange for i . Step 4: Substitute into Formula 9.1 and rearrange for IY.																	
																					
Perform	Step 2: $N = 4 \times 3 = 12$ Step 3: $\$26,268.15 = \$23,500(1 + i)^{12}$ $1.117793 = (1 + i)^{12}$ $1.117793^{\frac{1}{12}} = 1 + i$ $1.009322 = 1 + i$ $0.009322 = i$ Step 4: $0.009322 = \frac{IY}{4}$ $IY = 0.037292 = 3.7292\% \text{ compounded quarterly}$			Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>12</td><td>Answer: 3.729168</td><td>-23500</td><td>0</td><td>26268.15</td><td>4</td><td>4</td></tr></table>				N	I/Y	PV	PMT	FV	P/Y	C/Y	12	Answer: 3.729168	-23500	0	26268.15	4	4
	N	I/Y	PV	PMT	FV	P/Y	C/Y														
12	Answer: 3.729168	-23500	0	26268.15	4	4															

Present

A fixed rate of 3.7292% compounded quarterly is equivalent to the three variable interest rates that Andrej realized.

Escalator Interest GIC

An escalator interest GIC is a compound interest rate GIC with four distinguishing characteristics:

1. The interest rate is variable throughout the term.
2. The nominal interest rate always increases with each change so that higher returns on longer terms will encourage the investor to keep the sum of money invested in this GIC.
3. The interest rates are known in advance and fixed for the duration of each time segment of the investment.
4. Each time segment is most commonly one year in length.

Various financial institutions call escalator interest GICs by many names to differentiate their products from others on the market. Some of the names include Rate Riser, Stepper, Multi-Rater, Stepmaker, and RateAdvantage. Regardless of the actual name used, if the GIC fits the above characteristics it is an escalator interest GIC.



How It Works

The escalator interest GIC is a special form of the compound interest GIC, so the exact same formulas and procedures used for compound interest GICs remain applicable. The most common applications with escalator GICs involve finding one of the following:

1. The maturity value of the GIC.
2. The equivalent fixed rate of interest on the GIC so that the investor can either compare it to that of other options or just better understand the interest being earned. Recall from Chapter 9 the 6 steps you need to calculate equivalent fixed rates.

Example 10.1D: Understanding Your Escalator Rate GIC

Antoine is thinking of investing \$8,000 into a five-year Escalating Rate GIC with annually compounded rates of 0.5%, 1.5%, 2%, 3.5%, and 6.5% in each subsequent year. Determine the maturity value of Antoine's investment along with the equivalent fixed annually compounded rate.

Plan

First, calculate the maturity value of Antoine's investment (FV) at the end of the five-year term. Then calculate the equivalent fixed nominal interest rate (IY).

Understand

What You Already Know

Step 1: The present value, term, and escalating nominal interest rates are known, as shown in the timeline.

How You Will Get There

Step 2: For each time segment, calculate the i and N in the timeline using Formulas 9.1 and 9.2. Note that since all rates are compounded annually ($CY = 1$), Formula 9.1 results in $i = IY$ for all time segments. As well, in Formula 9.2 the N always equals 1 since both $CY = 1$ and $\text{Years} = 1$ for every time segment.

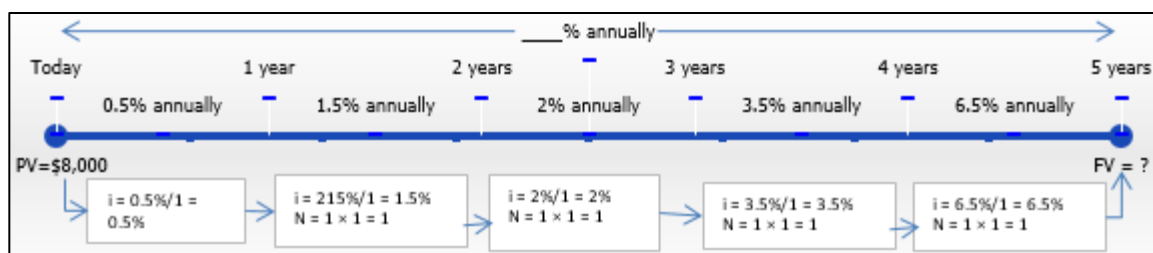
Step 3: Solve for FV using Formula 9.3. Since only the interest rate changes, expand the formula:

$$FV = PV \times (1 + i_1)^{N_1} \times (1 + i_2)^{N_2} \times \dots \times (1 + i_5)^{N_5}$$

Step 4: To reflect the entire five-year term compounded annually, calculate a new value of N using Formula 9.2.

Step 5: Substitute into Formula 9.3 and rearrange for i .

Step 6: With $CY = 1$, then $IY = i$.



Step 2: The timeline shows the successive calculated values of i and N .

Step 3: $FV_1 = \$8,000(1 + 0.005)^1 (1 + 0.015)^1 (1 + 0.02)^1 (1 + 0.035)^1 (1 + 0.065)^1 = \$9,175.13$

Step 4: $N = 1 \times 5 = 5$

Step 5: $\$9,175.13 = \$8,000(1 + i)^5$

$$1.146891 = (1 + i)^5$$

$$1.146891^{\frac{1}{5}} = 1 + i$$

$$1.027790 = 1 + i$$

$$0.027790 = i$$

Step 6: $IY = i = 0.027790$ or 2.779% compounded annually

Calculator Instructions

Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	1	0.5	-8000	0	Answer: 8,040	2	2
2	✓	1.5	-8040	✓	Answer: 8,160.60	✓	✓
3	✓	2	-8160.60	✓	Answer: 8,323.812	✓	✓
4	✓	3.5	-8323.812	✓	Answer: 8,615.14542	✓	✓
5	✓	6.5	-8,615.14542	✓	Answer: 9,175.129872	✓	✓
All	5	Answer: 2.779014	-8000	✓	9175.13	✓	✓

If Antoine invests in the Escalating Rate GIC, the maturity value after five years is \$9,175.13. He realizes 2.779% compounded annually on his investment.



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10.2: Application: Long-Term Promissory Notes (IOUs)

How do all of those "don't pay until . . ." plans work? The retail industry overflows with financing plans specifically designed to attract customers to purchase merchandise on credit. Most of these offers include terms such as "no money down" and "no payments for x months." You can find these plans at most furniture retailers such as The Brick and Leon's, electronic retailers such as Best Buy, along with many other establishments including jewellery stores and sleep centres.

But have you ever considered how this works on the business side? If every customer purchased merchandise under these no-money-required plans, how would the retailer stay in business? For example, perhaps The Brick's customers purchase furniture in January 2023 that they do not have to pay for until January 2025. During these two years, The Brick does not get paid for the products sold; however, it has paid its suppliers for the merchandise. How can a retailer stay in business while it waits for all those postponed payments?

Consumers generally do not read the fine print on their contracts with these retailers. Few consumers realize that the retailers often sell these contracts (sometimes immediately) to finance companies they have partnered with. While the consumer sees no noticeable difference, behind the scenes the retailer receives cash today in exchange for the right to collect payment in the future when the contract becomes due. That way the finance company becomes responsible for collecting on the loan to the consumer.

This section introduces the mathematics behind the sale of promissory notes between companies. Recall from Section 8.4 that a *promissory note*, more commonly called a note, is a written debt instrument that details a promise made by a buyer to pay a specified amount to a seller at a predetermined and specified time. If the debt allows for interest to accumulate, then it is called an *interest-bearing promissory note*. If there is no allowance for interest, then it is called a *noninterest-bearing promissory note*. Interest-bearing notes are covered in this section and noninterest notes are discussed in the next section. When promissory notes extend more than one year, they involve compound interest instead of simple interest.

Interest-Bearing Promissory Notes

When you participate in "don't pay until . . ." promotions, you create a promissory note in which you promise to pay for your goods within the time interval stated. These promotions commonly carry 0% interest if paid before the stated deadline, so the notes are noninterest-bearing promissory notes. However, failure to pay the note before the deadline transforms the note into an interest-bearing note for which interest is retroactive to the date of sale, usually at a very high rate of interest such as 21%.

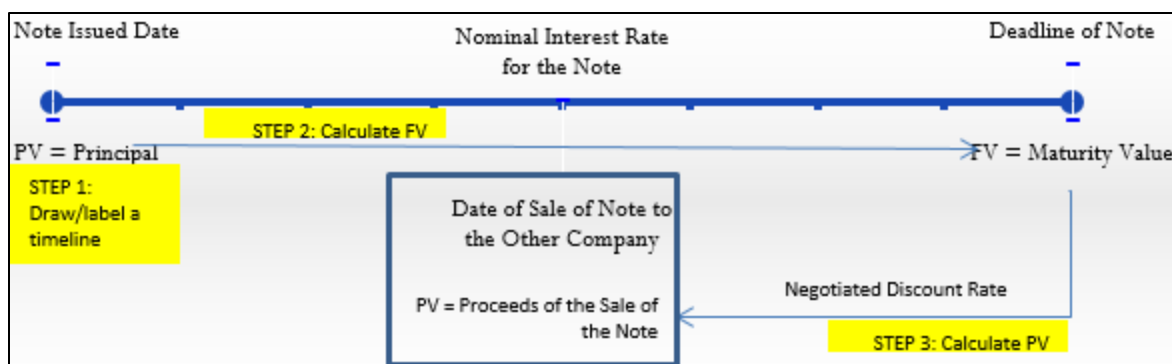
The mathematics of interest-bearing promissory notes deal primarily with the sale of long-term promissory notes between organizations. When the note is sold, the company buying it (usually a finance company) purchases the maturity value of the note and not the principal of the note. To the finance company, the transaction is an investment from which it intends to earn a profit through the difference between maturity value and purchase price. Thus, the finance company discounts the maturity value of the note using a discount rate that permits it to invest a smaller sum of money today to receive a larger sum of money in the future. The company selling the note is willing to take the smaller sum of money to cash in its accounts receivables and eliminate the risk of default on the debt.



How It Works

Recall that in simple interest the sale of short-term promissory notes involved three steps. You use the same three-step sequence for long-term compound interest promissory notes. On long-term promissory notes, a three-day grace period is not required, so the due date of the note is the same as the legal due date of the note.

Step 1: Draw a timeline, similar to the one on the next page, detailing the original promissory note and the sale of the note.



Step 2: Take the initial principal on the date of issue and determine the note's future value at the stated deadline using the stated rate of interest attached to the note. As most long-term promissory notes have a fixed rate of interest, this involves a future value calculation using Formula 9.3.

- Calculate the periodic interest rate using Formula 9.1, $i = \frac{IY}{CY}$.
- Calculate the number of compounding periods between the issue date and due date using Formula 9.2, $N = CY \times \text{Years}$.
- Solve for the future value using Formula 9.3, $FV = PV(1 + i)^N$.

Step 3: Using the date of sale, discount the maturity value of the note using a new negotiated discount rate of interest to determine the proceeds of the sale. Most commonly the negotiated discount rate is a fixed rate and involves a present value calculation using Formula 9.3.

- Calculate the new periodic interest rate using Formula 9.1, $i = \frac{IY}{CY}$.
- Calculate the number of compounding periods between the date of sale of the note and the due date using Formula 9.2, $N = CY \times \text{Years}$. Remember to use the CY for the discount rate, not the CY for the original interest rate.
- Solve for the present value using Formula 9.3, $FV = PV(1 + i)^N$, rearranging for PV.

Assume that a three-year \$5,000 promissory note with 9% compounded monthly interest is sold to a finance company 18 months before the due date at a discount rate of 16% compounded quarterly.

Step 1: The timeline to the right illustrates the situation.

Step 2a: The periodic interest rate on the note is $i = 9\% / 12 = 0.75\%$.

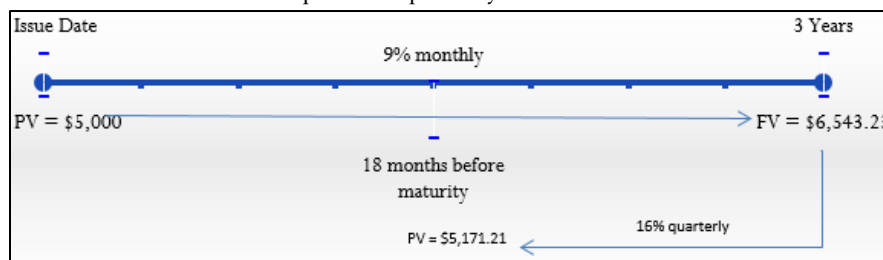
Step 2b: The term is three years with monthly compounding, resulting in $N = 12 \times 3 = 36$.

Step 2c: The maturity value of the note is $FV = \$5,000(1 + 0.0075)^{36} = \$6,543.23$.

Step 3a: Now sell the note. The periodic discount rate is $i = 16\% / 4 = 4\%$.

Step 3b: The time before the due date is $1\frac{1}{2}$ years at quarterly compounding. The number of compounding periods is $N = 4 \times 1\frac{1}{2} = 6$.

Step 3c: The proceeds of the sale of the note is $\$6,543.23 = PV(1 + 0.04)^6$, where $PV = \$5,171.21$. The finance company purchases the note (invests in the note) for \$5,171.21. Eighteen months later, when the note is paid, it receives \$6,543.23.



Important Notes

The assumption behind the three-step procedure for selling a long-term promissory note is that the process starts with the issuance of the note and ends with the proceeds of the sale. However, mathematically you may deal with any part of the transaction as an unknown. For example, perhaps the details of the original note are known, the finance company's offer on the date of sale is known, but the quarterly discounted rate used by the finance company needs to be calculated.

The best strategy in any of these scenarios is always to execute step 1 and create a timeline. Identify the known variables to visualize the process, then recognize any variable(s) remaining unknown. Keeping in mind how the selling of a promissory note works, you can adapt the three-step promissory note procedure using any of the techniques discussed in Chapter 9. Some examples of these adaptations include the following:

1. The discounted rate is unknown. Execute steps 1 and 2 normally. In step 3, solve for i (then $1Y$) instead of PV.
2. The original principal of the note is unknown. Execute step 1 normally. Work with step 3, but solve for FV instead of PV. Then work with step 2 and solve for PV instead of FV.
3. The length of time by which the date of sale precedes the maturity date is unknown. Execute steps 1 and 2 normally. In step 3, solve for N instead of PV.

As you can see, the three steps always stay intact. However, you may need to reverse steps 2 and 3 or calculate a different unknown variable.



Things To Watch Out For

In working with compound interest long-term promissory notes, the most common mistakes relate to the maturity value and the two interest rates.

1. **Maturity Value.** Remember that the company purchasing the note is purchasing the maturity value of the note, *not* its principal on the issue date. Any promissory note situation always involves the maturity value of the promissory note on its due date.
2. **Two Interest Rates.** The sale involves two interest rates: an interest rate tied to the note itself and an interest rate (the discount rate) used by the purchasing company to acquire the note. Do not confuse these two rates.

Example 10.2A: Proceeds on an Interest-Bearing Note

Jake's Fine Jewellers sold a diamond engagement ring to a customer for \$4,479.95 and established a promissory note under one of its promotions on January 1, 2023. The note requires 6% compounded semi-annually interest and is due on January 1, 2026. On October 1, 2024, Jake's Fine Jewellers needed the money and sold the note to a finance company at a discount rate of 11% compounded quarterly. What are the proceeds of the sale?

Plan	Find the proceeds of the sale for Jake's Fine Jewellers on October 1, 2024. This is the present value of the note (PV) based on the maturity value and the discount rate.
Understand	What You Already Know Step 1: The issue date, maturity date, principal, interest rate on the note, date of sale, and the discount rate are known, as illustrated in the timeline. How You Will Get There Step 2a: Working with the promissory note itself, calculate the periodic interest rate by applying Formula 9.1. Step 2b: Calculate the number of compound periods using Formula 9.2. Step 2c: Calculate the maturity value of the note using Formula 9.3. Step 3a: Working with the sale of the note, calculate the discount periodic interest rate by applying Formula 9.1. Step 3b: Calculate the number of compound periods elapsing between the sale and maturity. Use Formula 9.2. The timeline diagram illustrates the following: Start: January 1, 2023. A blue dot marks the beginning. Below it, $PV = \\$4,479.95$. End: January 1, 2026. A blue dot marks the end. Below it, $FV = ?$. Interest: A horizontal line with arrows above it is labeled "6% semi-annually". Sale Date: A vertical line marks "October 1, 2024 (date of sale)". Discount: A horizontal line with arrows below it, starting from the sale date and ending at the maturity date, is labeled "11% quarterly". Below this line, $PV = ?$ is indicated.

Perform	Step 2a: $IY = 6\%$, $CY = 2$, $i = 6\%/2 = 3\%$	Calculator Instructions <table><tr><th>Part</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Maturity</td><td>6</td><td>6</td><td>-4479.95</td><td>0</td><td>Answer: 5,349.294586</td><td>2</td><td>2</td></tr><tr><td>Sale</td><td>5</td><td>11</td><td>Answer: -4,670.753954</td><td>✓</td><td>5349.29</td><td>4</td><td>4</td></tr></table>	Part	N	I/Y	PV	PMT	FV	P/Y	C/Y	Maturity	6	6	-4479.95	0	Answer: 5,349.294586	2	2	Sale	5	11	Answer: -4,670.753954	✓	5349.29	4	4
	Part		N	I/Y	PV	PMT	FV	P/Y	C/Y																	
	Maturity		6	6	-4479.95	0	Answer: 5,349.294586	2	2																	
	Sale		5	11	Answer: -4,670.753954	✓	5349.29	4	4																	
	Step 2b: Years = January 1, 2026 – January 1, 2023 = 3 Years, $N = 2 \times 3 = 6$																									
Step 2c: $PV = \$4,479.95$, $FV = \$4,479.95(1 + 0.03)^6 = \$5,349.29$																										
Step 3a: $IY = 11\%$, $CY = 4$, $i = 11\%/4 = 2.75\%$																										
Present	Step 3b: Years = January 1, 2026 – October 1, 2024 = 1¼ Years, $N = 4 \times 1\frac{1}{4} = 5$																									
	Step 3c: $\$5,349.29 = PV(1 + 0.0275)^5$ $PV = \frac{\$5,349.29}{1.0275^5} = \$4,670.75$																									
	Jake's Fine Jewellers made the sale for \$4,479.95 on January 1, 2023. On October 1, 2024, it receives \$4,670.75 in proceeds of the sale to the finance company. The finance company holds the note until maturity and receives \$5,349.29 from the customer.																									

Example 10.2B: Finding an Unknown Discount Rate

A \$6,825 two-year promissory note bearing interest of 12% compounded monthly is sold six months before maturity to a finance company for proceeds of \$7,950.40. What semi-annually compounded discount rate was used by the finance company?

Plan	Calculate the semi-annually compounded negotiated discount rate (IY) used by the finance company when it purchased the note.	
Understand	What You Already Know Step 1: The term, principal, promissory note interest rate, date of sale, and proceeds amount are known, as shown in the timeline.	How You Will Get There Step 2a: Working with the promissory note itself, calculate the periodic interest rate by applying Formula 9.1. Step 2b: Calculate the number of compound periods using Formula 9.2. Step 2c: Calculate the maturity value of the note using Formula 9.3. Step 3a: Note the CY on the discount rate. Step 3b: Calculate the number of compound periods elapsing between the sale and before maturity. Use Formula 9.2. Step 3c: Calculate the periodic discount rate used by the finance company by applying Formula 9.3 and rearranging for i. Then substitute into Formula 9.1 and rearrange for IY.
Perform	Step 2a: $IY = 12\%$, $CY = 12$, $i = 12\%/12 = 1\%$ Step 2b: Years = 2, $N = 12 \times 2 = 24$ Step 2c: $PV = \\$6,825.00$, $FV = \\$6,825.00(1 + 0.01)^{24} = \\$8,665.94$ Step 3a: $CY = 2$ Step 3b: Years = 6 months = $\frac{1}{2}$ Year, $N = 2 \times \frac{1}{2} = 1$ Step 3c: Solve for i: $\begin{aligned} \$8,665.94 &= \$7,950.40(1 + i)^1 \\ 1.090000 &= 1 + i \\ i &= 0.090000 \end{aligned}$ Solve for IY: $0.090000 = IY \div 2$ $IY = 0.180001$ or 18.0001% compounded semi-annually (most likely 18%; the difference is due to a rounding error)	

Calculator Instructions

Part	N	I/Y	PV	PMT	FV	P/Y	C/Y
Maturity	24	12	-6825	0	Answer: 8,665.938976	12	12
Discount Rate	1	Answer: 18.0001	-7950.40	✓	8665.94	2	2

Present

The sale of the promissory note is based on a maturity value of \$8,665.94. The finance company used a discount rate of 18% compounded semi-annually to arrive at proceeds of \$7,950.40.

Noninterest-Bearing Promissory Notes

A noninterest-bearing promissory note involves either truly having 0% interest or else already including a flat fee or rate within the note's face value. Therefore, the principal amount and maturity amount of the promissory note are the same.



How It Works

A noninterest-bearing note simplifies the calculations involved with promissory notes. Instead of performing a future value calculation on the principal in step 2, your new step 2 involves equating the present value and maturity value to the same amount ($PV = FV$). You then proceed with step 3.

Assume a three-year \$5,000 noninterest-bearing promissory note is sold to a finance company 18 months before the due date at a discount rate of 16% compounded quarterly.

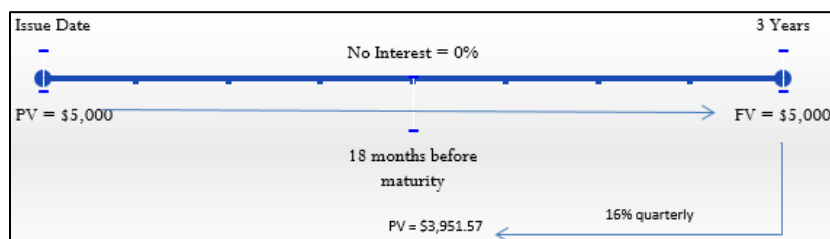
Step 1: The timeline is illustrated here.

Step 2: The maturity value of the note three years from now is the same as the principal, or $FV = \$5,000$.

Step 3a: Now sell the note. The periodic discount rate is $i = 16\%/4 = 4\%$.

Step 3b: The time before the due date is $1\frac{1}{2}$ years at quarterly compounding. The number of compounding periods is $N = 4 \times 1\frac{1}{2} = 6$.

Step 3c: The proceeds of the sale of the note is $\$5,000 = PV(1 + 0.04)^6$, or $PV = \$3,951.57$. The finance company purchases the note (invests in the note) for \$3,951.57. Eighteen months later, when the note is paid, it receives \$5,000.



Give It Some Thought

If a noninterest-bearing note is sold to another company at a discount rate, are the proceeds of the sale more than, equal to, or less than the note?

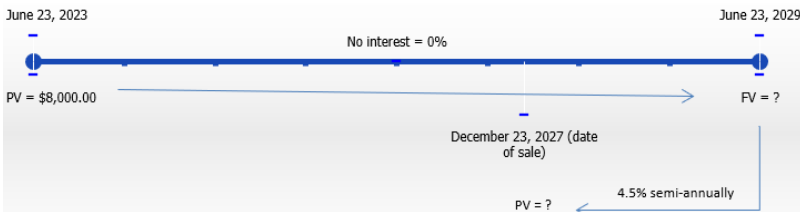
Solutions:
Less than the note. The maturity value is discounted to the date of sale.

Example 10.2C: Selling a Noninterest-Bearing Long-Term Promissory Note

A five-year, noninterest-bearing promissory note for \$8,000 was issued on June 23, 2023. The plan is to sell the note at a discounted rate of 4.5% compounded semi-annually on December 23, 2027. Calculate the expected proceeds on the note.

Plan

Calculate the proceeds on the note (PV) based on the note's maturity value and the date of the sale.

Understand	What You Already Know Step 1: The principal, noninterest on the note, issue date, due date, discount rate, and date of sale are known, as illustrated in the timeline.	How You Will Get There Step 2: Equate the FV of the note with the principal of the note. Step 3a: Working with the sale of the note, calculate the discount periodic interest rate by applying Formula 9.1. Step 3b: Calculate the number of compound periods elapsing between the sale and maturity. Use Formula 9.2. Step 3c: Calculate the proceeds of the sale by using Formula 9.3, rearranging for PV.														
																
Perform	Step 2: $FV = PV = \\$8,000$ Step 3a: $IY = 4.5\%$, $CY = 2$, $i = 4.5\%/2 = 2.25\%$ Step 3b: Years = June 23, 2029 – December 23, 2027 = $1\frac{1}{2}$ Years, $N = 2 \times 1\frac{1}{2} = 3$ Step 3c: $\\$8,000 = PV(1 + 0.0225)^3$ $PV = \frac{\$8,000}{1.0225^3} = \$7,483.42$	Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>6</td><td>4.5</td><td>Answer: -7,483.418564</td><td>0</td><td>8000</td><td>2</td><td>2</td></tr></table>	N	I/Y	PV	PMT	FV	P/Y	C/Y	6	4.5	Answer: -7,483.418564	0	8000	2	2
N	I/Y	PV	PMT	FV	P/Y	C/Y										
6	4.5	Answer: -7,483.418564	0	8000	2	2										
Present	The expected proceeds are \$7,483.42 on December 23, 2027, with a maturity value of \$8,000.00 on June 23, 2029.															



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10.3: Application: Strip Bonds

(Buy Low, Sell High)

A **strip bond** is a marketable bond that has been stripped of all interest payments and is one of the many financial tools through which you may earn nontaxable income inside your Registered Retirement Savings Plan (RRSP). A marketable bond is a long-term debt of a corporation or government and represents a means by which large sums of money are "borrowed" from investors. In a marketable bond, interest is regularly paid out to investors. A more detailed discussion on marketable bonds is found in Chapter 14.

Characteristics of Strip Bonds

Mathematically, a strip bond essentially is a long-term version of the treasury bill. Whereas T-bills are found with terms of less than one year, strip bonds have longer terms because of the large sums of money that governments or large corporations require. These large sums take a long time to pay back (think of how long it takes most Canadians to pay off a mortgage, for example). In fact, Canada holds the record of the longest strip bond, with a 99-year term!

Much like T-bills, strip bonds have the following characteristics:

1. Strip bonds do not earn interest. They are sold at a discount and redeemed at full value. This follows the line of "buy low, sell high." The compound percentage by which the value of the strip bond grows from sale to redemption is called the *yield* or rate of return. From a mathematical perspective, you calculate the yield in the exact same manner as an interest rate. Yields are normally specified in nominal form, that is, as a numerical rate followed by words stating the compounding frequency, which is semi-annual unless otherwise specified. Yields remain fixed for the entire term.
2. The face value of a strip bond is the maturity value payable at the end of the term. It includes both the principal and yield together.
3. Strip bonds are traded between investors at any point in secondary financial markets according to prevailing strip bond rates at the time of sale. Historical strip bond yields can be found at www.bankofcanada.ca/en/rates/yield_curve.html (note that the Bank of Canada refers to strip bonds as zero-coupon bonds). Current yields on strip bonds are found on any online financial source and are usually published daily in the financial sections of newspapers such as the *Globe and Mail*.
4. No minimum investment is required for strip bonds, which are available in any increment of \$1.

Purchase Price of a Strip Bond

Strip bonds are financial investment tools that have a known payout (face value) upon maturity. The most common mathematical application involving strip bonds is to calculate the price that must be paid today, or the purchase price, to acquire the strip bond.



The Formula

When you work with strip bonds, you may need up to four formulas that have been previously introduced:

1. Formula 8.3 Interest Amount for Single Payments: $I = S - P$
2. Formula 9.1 Periodic Interest Rate: $i = \frac{IY}{CY}$
3. Formula 9.2 Number of Compound Periods for Single Payments: $N = CY \times \text{Years}$
4. Formula 9.3 Compound Interest for Single Payments: $FV = PV(1 + i)^N$

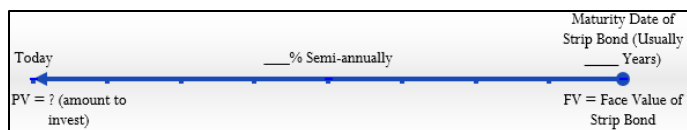


How It Works

The future value of a strip bond is always known since it is the face value. You calculate the present value or purchase price by applying the following steps:

Step 1: Determine the face value (FV) of the strip bond, the years between the date of the sale and the maturity date (Years), the yield (IY) on the date of the sale, and the compounding frequency (CY). If needed, draw a timeline similar to the one below, which illustrates a typical strip bond timeline.

Step 2: Calculate the periodic interest rate (i) and the number of compounding periods (N) using Formulas 9.1 and 9.2, respectively.



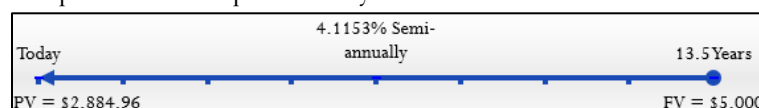
Step 3: Calculate the present value of the strip bond using Formula 9.3, rearranging for PV.

Step 4: If you are interested in the actual dollar amount

that the investor gains by holding onto the strip bond until maturity, apply Formula 8.3.

Assume that when prevailing market rates are 4.1153% you want to invest in a \$5,000 face value strip bond with 13½ years until maturity. How much money is required to purchase the strip bond today?

Step 1: The known variables are FV = \$5,000, Years = 13½, IY = 4.1153%, and CY = 2. The timeline illustrates the strip bond.



Step 2: The periodic interest rate is

$i = 4.1153\% / 2 = 2.05765\%$. The number of compounding periods remaining on the strip bond is $N = 2 \times 13\frac{1}{2} = 27$.

Step 3: The present value or purchase price of the strip bond is calculated as $\$5,000 = PV(1 + 0.0205765)^{27}$ or $PV = \$2,884.96$. Thus, you can purchase the strip bond for \$2,884.96

Step 4: If you hold onto the strip bond for the remaining 13½ years, you will receive \$5,000 upon maturity. This represents a gain of $I = \$5,000.00 - \$2,884.96 = \$2,115.04$ over the term of the investment.



Things To Watch Out For

Don't forget that the compounding on strip bonds is assumed to be semi-annual unless stated otherwise. Thus, CY = 2 in most scenarios.



Give It Some Thought

1. If the yield remains constant, what is the relationship between the purchase price of a strip bond and the time to maturity?
2. What happens to the purchase price of a strip bond if the yield increases while the time to maturity remains constant?

Solutions:

1. The longer the strip bond needs to be discounted, the lower the purchase price becomes; the shorter the time to maturity, the greater the purchase price.
2. The purchase price becomes lower since the future value is discounted using a higher yield.

Example 10.3A: Purchase Price of a Strip Bond

Johansen is considering purchasing a \$50,000 face value Government of Canada strip bond with 23½ years until maturity. The current market yield for these bonds is posted at 4.2031%. What is the price of the strip bond today, and how much money is gained if the bond is held until maturity?

Plan	Calculate the purchase price (PV) of the strip bond today along with the total interest (I).	
Understand	What You Already Know Step 1: The face value, years to maturity, and nominal interest rate are known: FV = \$50,000 Years = 23.5 IY = 4.2031% CY = default = semi-annual = 2	How You Will Get There Step 2: Apply Formulas 9.1 and 9.2. Step 3: Apply Formula 9.3 and rearrange for PV. Step 4: To calculate the total gain, apply Formula 8.3.

	Today 4.2031% Semi-annually 23.5 Years PV = ? FV = \$50,000															
Perform	Step 2: $i = \frac{4.2031\%}{2} = 2.10155\%$, $N = 2 \times 23.5 = 47$ Step 3: $PV = \\$50,000 (1 + 0.0210155)^{-47} = \\$18,812.66$ Step 4: $I = \\$50,000 - \\$18,812.66 = \\$31,187.34$	Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>47</td><td>4.2031</td><td>Answer: -18,812.66199</td><td>0</td><td>50000</td><td>2</td><td>2</td></tr></table>	N	I/Y	PV	PMT	FV	P/Y	C/Y	47	4.2031	Answer: -18,812.66199	0	50000	2	2
N	I/Y	PV	PMT	FV	P/Y	C/Y										
47	4.2031	Answer: -18,812.66199	0	50000	2	2										
Present	The strip bond has a purchase price of \$18,812.66 today. If you hold onto the strip bond until maturity, you will receive a payment of \$50,000 23½ years from today. This represents a \$31,187.34 gain.															

Nominal Yields on Strip Bonds

In two instances it is common to calculate the nominal yield on strip bonds:

- At times, various newspapers or online sites may list a purchase price per \$100 of face value but not actually publish the yield on the strip bond. A smart investment decision always requires you to know the interest rate being earned.
- Strip bonds are actively traded on a market. Many investors sell strip bonds to other investors prior to maturity. In these cases, the investor selling the bond needs to calculate the actual yield realized on the investment.

To calculate nominal yields you need the same four formulas as for the strip bond's purchase price.



How It Works

Follow these steps to calculate the nominal yield of a strip bond:

Step 1: How much was paid for the strip bond? Determine the present value or purchase price of the strip bond (PV). This may already be known, or you may have to calculate this amount using the previously introduced steps for calculating the present value.

Step 2: How much did the strip bond sell for? This is the future value (FV) of the strip bond. If it is the maturity of the strip bond, then the future value is the face value of the bond. If the investor is selling the strip bond prior to maturity, then this number is based on a present value calculation using the yield at the time of sale and time remaining until maturity.

Step 3: Determine the years (Years) between the purchase and the sale of the strip bond. Using CY = 2 (unless otherwise stated), apply Formula 9.2 and calculate the number of compounding periods between the dates (N).

Step 4: Calculate the periodic interest rate (i) using Formula 9.3, rearranging for i .

Step 5: Convert the periodic interest rate to its nominal interest rate by applying Formula 9.1 and rearranging for IY.

Step 6: If you are interested in the actual dollar amount that the investor gained while holding the strip bond, apply Formula 8.3.

Assume that when market yields were 4.254% an investor purchased a \$25,000 strip bond 20 years before maturity. The purchase price was \$10,772.52. The investor then sold the bond five years later, when current yields dropped to 3.195%. The selling price was \$15,539.94. The investor wants to know the gain and the actual yield realized on her investment.

Step 1: The PV is known at \$10,772.52. This is the purchase price paid for the strip bond.

Step 2: The FV is known at \$15,539.94. This is the selling price received for the strip bond.

Step 3: Between the purchase and sale, the Years = 5. With CY = 2, the strip bond was held for $N = 2 \times 5 = 10$ compounding periods.

Step 4: Applying Formula 9.3, $\$15,539.94 = \$10,772.52(1 + i)^{10}$ or $i = 0.037321$.

Step 5: Converting to the nominal rate, $0.037321 = IY \div 2$ or $IY = 7.4642\%$.

Step 6: The actual gain is $I = \$15,539.94 - \$10,772.52 = \$4,767.42$.

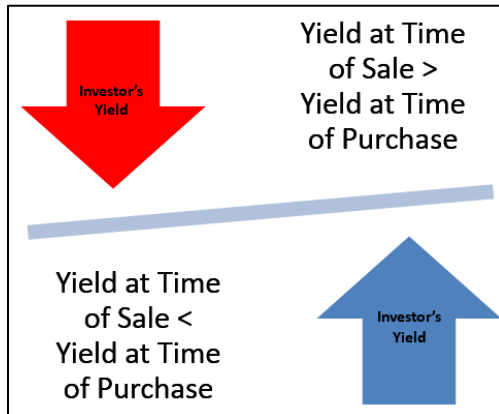
The investor gained \$4,767.42 on the investment, representing an actual yield of 7.4642% compounded semi-annually.



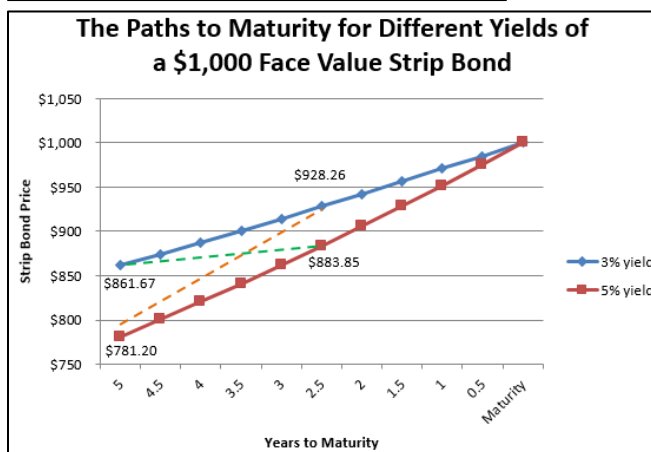
Important Notes

The relationship between the yield at which you purchased the strip bond versus the current yield in the market at the time of sale is illustrated in the figure below. Each relationship illustrated in the figure is discussed below:

Current Yield > Purchase Yield. If the current yield is higher at the time of sale than the yield at time of purchase, then the strip bond price at the time of sale is pushed downward from the original path to maturity. Consider two situations to



illustrate this point, demonstrated in the figure on the next page. If \$1,000 is present valued over five years at a 3% semi-annual yield, the price is \$861.67. At a 5% semi-annual yield, the same strip bond would have a lower price of \$781.20. In both of these cases, though, as the time of sale advances toward the maturity date over the next five years, the prices will gradually increase toward \$1,000. However, the price of the 5% yield strip bond will always be lower than that of the 3% strip bond. Thus, if you purchase a 3% strip bond and sell when yields are 5%, the original sale price is lowered from the blue line to the red line. Thus, if the sale occurs with 2.5 years remaining until maturity, the price drops from \$928.26 to \$883.85. Since the future value drops, the actual investor's yield will become lower than the original purchase yield (as reflected by the dashed green arrow).



Current Yield < Purchase Yield. If the current yield is lower at the time of sale than the yield at the time of purchase, then correspondingly the strip bond price at the time of sale is pushed upward from the original path to maturity. Using the same example from above, recall that the price on the 3% yield strip bond is always higher than the 5% yield strip bond. Thus, if you had purchased the strip bond when yields were 5%, selling the strip bond when yields are 3% means that the future value jumps upward from the red line to the blue line. Thus, if the sale occurs with 2.5 years remaining until maturity, the price increases from \$883.85 to \$928.26. Therefore, the actual investor's yield will become higher than the original purchase yield (as reflected by the dashed orange line). This concept is exemplified by the investor above. She

originally purchased when discount rates were 4.254% and sold when rates had dropped to 3.195%. This increased the sale price of her strip bond and translated into the investor actually realizing 7.4642% instead of the original yield of 4.254%!



Give It Some Thought

In each of the following situations, determine whether the investor achieves a yield higher than, lower than, or equal to the yield at the time of purchase.

1. Yield at time of purchase = 5.438%; Yield at time of sale = 6.489%
2. Yield at time of purchase = 2.888%; Yield at time of sale = 2.549%
3. Yield at time of purchase = 4.137%; Yield at time of sale = 4.137%

- Solutions:**
1. Lower than; current yield > purchase yield.
 2. Higher than; current yield < purchase yield.
 3. Equal to; both yields are the same.

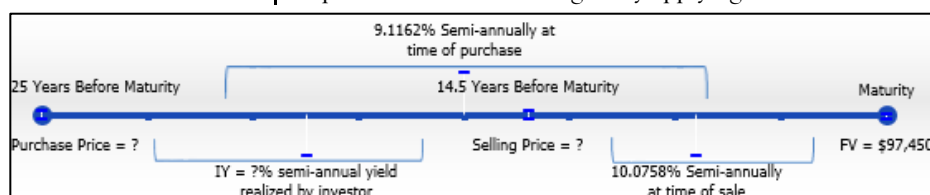
Example 10.3B: The Investor's Actual Yield

Cameron invested in a \$97,450 face value Government of British Columbia strip bond with 25 years until maturity. Market yields at that time were 9.1162%. Ten and half years later Cameron needed the money, so he sold the bond when current yields were 10.0758%. Calculate the actual yield and gain that Cameron realized on his investment.

Plan Calculate the nominal interest rate (IY) that Cameron realized while in possession of the strip bond. Also calculate the total gain (I).

What You Already Know
 The following information is known about the bond at the time of purchase:
 $FV = \$97,450$ Years = 25
 $IY = 9.1162\%$ CY = 2
 The following information is known about the bond at the time of sale:
 Years strip bond held = 10.5
 $FV = \$97,450$ CY = 2
 Years = $25 - 10.5 = 14.5$
 $IY = 10.0758\%$

How You Will Get There
 Step 1: To figure out the purchase price, apply Formulas 9.1, 9.2, and 9.3 based on the purchase date.
 Step 2: To figure out the selling price, apply Formulas 9.1, 9.2, and 9.3 based on the selling date. Note that once the PV is calculated, it becomes the FV for subsequent calculations.
 Step 3: Taking the years the strip bond was held, apply Formula 9.2 to determine the number of compounding periods.
 Step 4: Calculate the periodic interest rate by applying Formula 9.3, rearranging for i .
 Step 5: Calculate the nominal interest rate by applying Formula 9.1, rearranging for IY.
 Step 6: Calculate the total gain by applying Formula 8.3.



Perform

Step 1: $i = 9.1162\% / 2 = 4.5581\%$; $N = 2 \times 25 = 50$
 $\$97,450 = PV(1 + 0.045581)^{50}$
 $PV = \$97,450 \div 1.045581^{50} = \$10,492.95$

Step 2: $i = 10.0758\% / 2 = 5.0379\%$; $N = 2 \times 14.5 = 29$
 $\$97,450 = PV(1 + 0.050379)^{29}$
 $PV = \$97,450 \div 1.050379^{29} = \$23,428.63$

Step 3: $CY = 2$; $N = 2 \times 10.5 = 21$

Step 4: $PV = \$10,492.95$; $FV = \$23,428.63$
 $\$23,428.63 = \$10,492.95(1 + i)^{21}$

$$i = \left(\frac{\$23,428.63}{\$10,492.95} \right)^{\frac{1}{21}} = 0.038991$$

Step 5: $0.038991 = IY \div 2$ or $IY = 0.038991 \times 2 = 7.7982\%$

Step 6: $I = \$23,428.63 - \$10,492.95 = \$12,935.68$

Calculator Instructions

Step	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	50	9.1162	Answer: -10,492.95246	0	97450	2	2
2	29	10.0758	Answer: -23,428.63365	✓	✓	✓	✓
4 and 5	21	Answer: 7.798240	-10492.95	✓	23428.63	✓	✓

Present Cameron sold the strip bond when prevailing market rates were higher than the rate at the time of purchase. He has realized a lower yield of 7.7982% compounded semi-annually, which is a gain of \$12,935.68.



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10.4: Application: Inflation, Purchasing Power, and Rates of Change

(Your Grandparents Used to Go to a Movie for a Quarter)

How much does inflation cut into people's incomes and wealth? In 1955, the average Canadian worker earned an annual gross salary of about \$2,963.¹ By 2010, the average Canadian worker brought home annual gross earnings approximating \$44,366.² Does this mean that Canadians today are 15 times richer than our grandparents? From earlier discussions in Section 4.3 on inflation, real income, and purchasing power, you already can tell we are not. The actual increase is by something less than 15 times. But how much less is it?

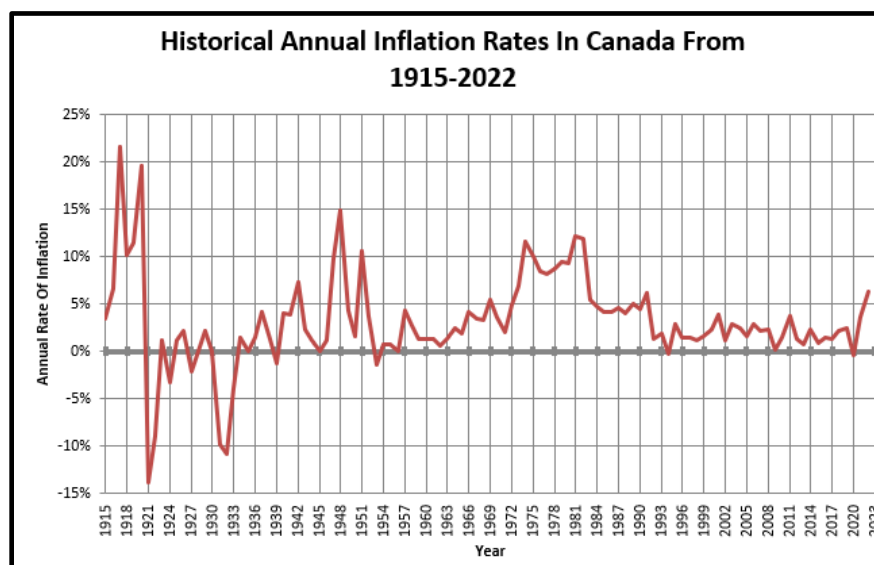
To answer the question, you have to express both incomes with reference to the same year. Either convert the 1955 income to its 2010 equivalent or vice versa. To help, what if you are told the rate of inflation from 1955 to 2010 averaged 3.91%³ per year? To convert, you might use either the percent change method or the real income method. Each of these poses problems, though:

- To use the percent change formula from Section 3.1, you would have to apply it 55 consecutive times! Clearly, this is impractical.
- To use the real income formula from Section 4.3, you would need the consumer price index numbers for both years. Locating CPI values can be time consuming, especially for years in the distant past. Another complication is that if you want to project future values (such as the equivalent income in, say, 2020 instead of 2010), CPI values do not exist for future years.

There has to be an easier way! In this section, you will see how to adapt the compound interest concepts and formulas to suit such applications as inflation, purchasing power, and even percent change.

Inflation

Inflation is the overall upward price movement of products in an economy. It is measured by positive change in the consumer price index. Historical inflation rates in Canada are illustrated in the figure to the right. Note that historically prices have always risen over the long term. In the short term, though, there have been periods during which prices moved downward. This is known as **deflation**, which is measured by negative change in the consumer price index. Such periods of deflation usually do not last very long; the longest period recorded was during the Great Depression. Most recently, deflation occurred for a few months in 2020 due to COVID-19.



¹ Statistics Canada, "Average Annual, Weekly and Hourly Earnings, Male and Female Wage-Earners, Manufacturing Industries, Canada, 1934 to 1969," adapted from Series E60–68, www.statcan.gc.ca/pub/11-516-x/section/E60_68-eng.csv, accessed July 28, 2010.

² Statistics Canada, "Earnings, Average Weekly, by Province and Territory," adapted from CANSIM table 281-0027 and Catalogue no. 72-002-X, <http://www40.statcan.gc.ca/l01/cst01/labr79-eng.htm>, accessed August 11, 2011.

³ Statistics Canada, "Consumer Price Indexes for Canada, Monthly, 1914–2022"; all values based in May of each year; www.bankofcanada.ca/rates/related/inflation-calculator/?page_moved=1, accessed June 13, 2022.

Inflation is most commonly expressed as an annual rate; therefore, you treat it mathematically as an annually compounded interest rate. This is the nominal interest rate (IY) with a compounding frequency of one, or $CY = 1$. Note that if deflation has occurred during the time period in question, the interest rate is a negative number. If you have a series of inflation rates, you treat this as a variable interest rate. If you are finding an average inflation rate for some time period, you treat this like finding the equivalent fixed interest rate.

In using the compound interest formulas, assign the present value to any beginning value in the problem at hand, while the future value is any ending value. The number of compounding periods (N) still reflects the number of compounds between the two values.



How It Works

You can use any of the formulas and techniques from Chapter 9 to work with inflation. The opening example of incomes in 1955 and 2010 will illustrate the processes.

Solving for Future Value. If the unknown variable is an ending value, apply Formula 9.3, solving for the future value. If only one inflation rate (a fixed rate) applies, this requires only one application of the formula. If the inflation rate changes over time, you apply the formula multiple times or use the quick method of calculation:

$$FV = PV \times (1 + i_1)^{N_1} \times (1 + i_2)^{N_2} \times \dots \times (1 + i_n)^{N_n}$$

In the example, you could move the 1955 income to 2010. In this case, the $PV = \$2,963$, $IY = 3.91\%$, $CY = 1$, and $N = 55$. With a fixed interest rate, apply Formula 9.3 and calculate $FV = \$2,963(1 + 0.0391)^{55} = \$24,427.87$. Therefore, the 2010 equivalent income of 1955 is approximately \$24,427. Note the actual 2010 income is about 81% higher, which would imply that Canadians have indeed become wealthier.

Solving for Present Value. If the unknown variable is a beginning value, apply Formula 9.3, rearranging for present value. Depending on whether the inflation rate is fixed or variable, solve using the same technique as for future value. In the example, you could move the 2010 income to 1955. In this case, the $FV = \$44,366$, $IY = 3.91\%$, $CY = 1$, and $N = 55$. With a fixed interest rate, apply Formula 9.3, rearranging for PV, and calculate $PV = \$44,366 \div (1 + 0.0391)^{55} = \$5,381.41$. Therefore, the 1955 equivalent income of 2010 is approximately \$5,381. Note this income is the same percentage (81%) higher than the actual 1955 income, as you found by the first method.

Solving for the Rate. If the average inflation rate is the unknown variable, then the beginning and ending values must be known. In the example, assume the average inflation rate is unknown, but the equivalent incomes of $PV = \$2,963$ in 1955 and $FV = \$24,427.87$ in 2010 are known. The $CY = 1$ and $N = 55$ years. Applying Formula 9.3 results in $\$24,427.87 = \$2,963(1 + i)^{55}$. Solving for i you get 3.91%. This is the average inflation rate (since $CY = 1$, then $i = IY$).

Solving for the Term. If the unknown variable is the amount of time between the beginning and ending values, once again apply Formula 9.3 and rearrange for N. In the example, assume the beginning and ending values of $PV = \$2,963$ in 1955 and $FV = \$24,427.87$ are known, but the year for the future value is unknown. The $IY = 3.91\%$ and $CY = 1$. Applying Formula 9.3 gives you $\$24,427.87 = \$2,963(1 + 0.0391)^N$. Solving for N gives 55 years. The starting year of 1955 + 55 years is the ending year of 2010, in which the equivalent income of \$24,427.87 applies.



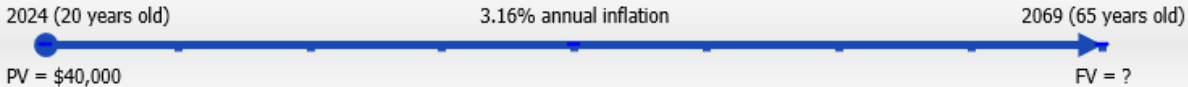
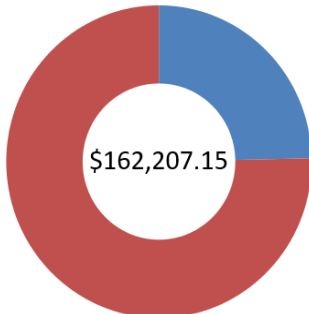
Important Notes

Your BAI Plus Calculator. The time value of money buttons are designed for financial calculations and require you to follow the cash flow sign convention at all times. Remember that this convention requires money leaving you to be entered as a negative number, and money being received by you to be entered as a positive number.

When you adapt this function to economic calculations such as inflation, no money is being invested or received—numbers are being moved across time. To obey the calculator requirement of the cash flow sign convention, ensure that the signs attached to the present (PV) and future values (FV) are opposites. The choice as to which is positive and which is negative is arbitrary and does not affect the outcome of the calculation. Ignore the cash flow sign displayed on any solutions.

Example 10.4A: Your Retirement Income

Many experts today claim that the average retiree in 2024 needs approximately \$40,000 in gross annual income to retire comfortably. Assume that you are 20 years old in 2024. Historically, the inflation rate has averaged 3.16%. If the average inflation rate continues in the future, what gross income will you require when you retire at age 65?

Plan	Calculate the ending value (FV) for your annual gross income when you retire at age 65.														
Understand	What You Already Know Step 1: The present value, fixed inflation rate, and time frame are known, as illustrated in the timeline. Years = 45 	How You Will Get There Step 2: With a CY = 1, the $i = IY$. Step 3: Apply Formula 9.2 to calculate the number of compounding periods: Step 4: Calculate the future value by applying Formula 9.3.													
	Step 2: $i = 3.16\%$ Step 3: $N = 1 \times 45 = 45$ Step 4: $FV = \\$40,000(1 + 0.0316)^{45} = \\$162,207.15$	Calculator Instructions <table><tr><td>N</td><td>I/Y</td><td>PV</td><td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>45</td><td>3.16</td><td>-40000</td><td>0</td><td>Answer: 160,798.0247</td><td>1</td><td>1</td></tr></table>	N	I/Y	PV	PMT	FV	P/Y	C/Y	45	3.16	-40000	0	Answer: 160,798.0247	1
N	I/Y	PV	PMT	FV	P/Y	C/Y									
45	3.16	-40000	0	Answer: 160,798.0247	1	1									
Present	When you retire in 2069, if inflation continues to average 3.16% annually, according to the experts your annual gross income must be \$162,207.15 for you to live comfortably. Your Income At Age 65  ■ Today's Retirement Income (2024) ==> \$40,000 ■ Cost of Living Adjustment From 2024 To 2069 @ 3.16% ==> \$122,207.15														

Purchasing Power

Recall from Section 4.3 that the purchasing power of a dollar is the amount of goods and services that can be exchanged for a dollar. Purchasing power has an inverse relationship to inflation. When inflation occurs and prices increase, your purchasing power decreases.

**The Formula**

In the formula introduced in Section 4.3, the denominator used the CPI to represent the change in product prices. In compound interest applications, you instead use the inflation rate, as shown in Formula 10.2.

PPD is Purchasing Power of A Dollar: A measure of the ability of your money to purchase products over time.

× 100 is Percent Conversion: The purchasing power of a dollar is expressed in percent format.

Formula 10.2 - Purchasing Power of a Dollar (Compound Interest Method):

$$\text{PPD} = \frac{\$1}{(1 + i)^N} \times 100$$

i is Inflation Rate: The fixed periodic (usually annual) inflation rate for the time period. If the rate is variable, substitute $(1 + i_1)^{N_1} \times (1 + i_2)^{N_2} \times \dots \times (1 + i_n)^{N_n}$ as needed into the denominator. If the inflation rate is annual, then $i = \text{IY}$.

N is Number of Compounding Periods: Most commonly, the compounding frequency (CY) equals 1 when you deal with inflation. Therefore, the total number of compounding periods usually equals the total years involved. If this is not true, then use Formula 9.2 to calculate N.



How It Works

Follow these steps to solve a purchasing power question using compound interest:

Step 1: Identify the inflation rate (IY), the compounding on the inflation rate (CY), and the term (Years). Normally, $i = \text{IY}$ and $N = \text{Years}$; however, apply Formulas 9.1 and 9.2 if you need to calculate i or N .

Step 2: Apply Formula 10.2, solving for the purchasing power of a dollar.

Using the income example, determine how an individual's purchasing power has changed from 1955 to 2010. Recall that the average inflation during the period was 3.91% per year.

Step 1: The IY = 3.91%, CY = 1, and Years = 55. The annual rate lets $i = 3.91\%$ and $N = 54$.

Step 2: From Formula 10.2, the $\text{PPD} = \frac{\$1}{(1 + 0.0391)^{55}} \times 100 = 12.1296\%$. As a rough interpretation, this means that if someone could purchase 100 items with \$100 in 1955, that same \$100 would only purchase about 12 of the same items in 2010.



Things To Watch Out For

Determine exactly what the question is asking you to calculate with respect to the purchasing power of a dollar.

- If the question asks, "What is the purchasing power of a dollar?" then your answer is the result of Formula 10.2.
- If the questions asks, "How has your purchasing power decreased?" then your answer is 100% minus the result of Formula 10.2.

It is critical to *Plan* when solving these questions so that your solution addresses the question asked.



Give It Some Thought

What is the purchasing power of your dollar if the prices on products:

1. Double?
2. Triple?
3. Quadruple?

$$3.1\% = 3.1/100 = 0.031$$

$$-0.26\% = -0.26/100 = -0.0026$$

$$0.96\% = 0.96/100 = 0.0096$$

:uoitnI

Example 10.4B: Purchasing Power

The historical inflation rates in Canada were 3.13% from June 2007 to June 2008, -0.26% from June 2008 to June 2009, and 0.96% from June 2009 to June 2010. Comparing June 2007 to June 2010, what is the purchasing power of a 2007 dollar in 2010?

Plan

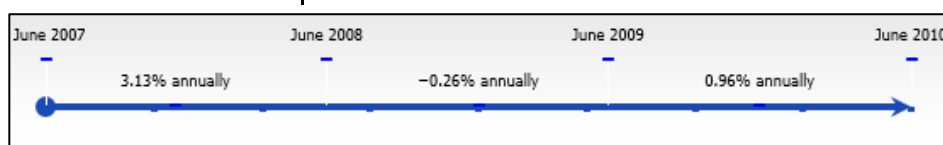
Calculate the purchasing power of a 2007 dollar (PPD) in 2010.

Understand**What You Already Know**

Step 1: The inflation rates and terms are known, as illustrated in the timeline.
CY (for each time segment) = 1,
Term (for each time segment) = 1 Year

How You Will Get There

Step 1 (continued): Identify the periodic interest rate (i) and the number of compounds (N) for each time segment. With CY = 1, then $i = IY$ and $N = \text{Years}$.
Step 2: Apply Formula 10.2, substituting the variable interest rate version in the denominator.

**Perform**

Step 1 (continued): $i_1 = 3.13\%$, $i_2 = -0.26\%$,
 $i_3 = 0.96\%$; $N_1, N_2, N_3 = 1$

$$\text{Step 2: PPD} = \frac{\$1}{(1 + 0.0313) \times (1 - 0.0026) \times (1 + 0.0096)} \times 100 = 96.2933\%$$

Calculator Instructions

Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	1	3.13	Answer: 0.969649	0	-1	1	1
2	✓	-0.26	Answer: 0.972177	✓	-0.969649	✓	✓
3	✓	0.96	Answer: 0.962933	✓	-0.972177	✓	✓

Present

The purchasing power of a June 2007 dollar in June 2010 is 96.2933%. If \$100 in June 2007 could purchase 100 items, in June 2010 \$100 could only purchase about 96 items.

Rates of Change

Recall from Section 3.1 that you could calculate a percent change between Old and New numbers. While this formula works well when you are interested in just a single percent change, it becomes time consuming and tedious when you work with a series of percent changes.

For example, assume the TSX has a value of 12,000. The TSX then drops by 4% each month for five months, and then rises by 4% each month for five months. What is the “New” value for the TSX? It is not 12,000! If you use the formula for percent change, you need a series of 10 calculations solving for “New” each time—one calculation for each month! Not much fun. Mathematically, in a series of percent changes, each change compounds on the previous one. Therefore, you can use compound interest formulas to work with any series of percent changes.

**The Formula**

You can solve any series of percent changes by applying an adapted version of Formula 9.3 for variable interest rates:

FV is Future Value or Maturity Value: The “New” value at the end of the series of percent changes.

PV is Present Value or Principal: The “Old” value before the series of percent changes.

N is Number of Times In A Row: This is the total number of times in a row that the particular percent change occurs. Similar to the i , this variable may have multiple values for each percent change as denoted by the subscripts from 1 through n .

Formula 9.3 for Variable Rates with Interpretations Adapted to Percent Change:

$$\mathbf{FV = PV \times (1 + i_1)^{N_1} \times (1 + i_2)^{N_2} \times \dots \times (1 + i_n)^{N_n}}$$

i is Percent Change: The percent change in decimal format. If there are different values for percent changes (such as with a variable interest rate), this variable takes on multiple values as denoted by the subscripts from 1 through n . If the percent change is a decrease, the variable takes on a negative value.

n is Number of Different Percent Change Values: The variables i and N take on as many values as there are different percent changes involved in the problem.



How It Works

Follow these steps to adapt Formula 9.3 for percent change applications:

Step 1: Assign either the “Old” value to PV or the “New” value to FV (depending on which you know).

Step 2: Identify your series of percent changes (i_1 through i_n) and how many times in a row each percent change value occurs (N_1 through N_n). Remember that decreases are negative values.

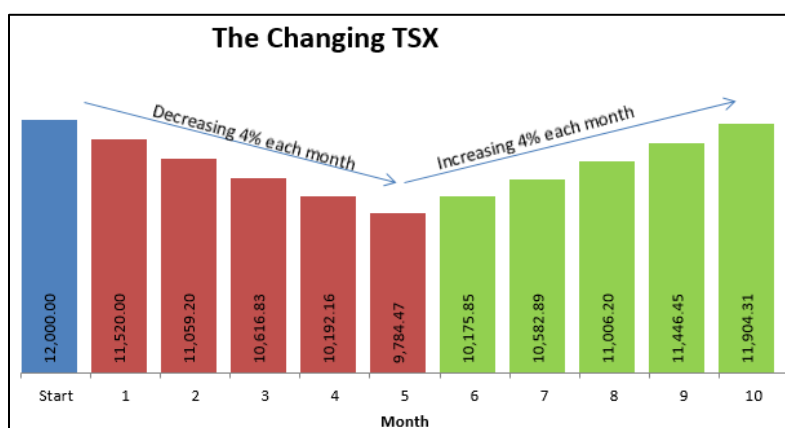
Step 3: Apply the adapted version of Formula 9.3, solving for either FV or PV.

As an example, find out the new value for the TSX based on a starting value of 12,000 with decreases of 4% for five months followed by increases of 4% for five months.

Step 1: The “Old” value is $PV = 12,000$.

Step 2: The first change is a decrease of 4%, or $i_1 = -4\%$. It occurs for five months in a row, thus $N_1 = 5$. Next, an increase of 4%, or $i_2 = 4\%$, also occurs five months in a row, thus $N_2 = 5$.

Step 3: Use Formula 9.3 to calculate $FV = 12,000 \times (1 - 0.04)^5 \times (1 + 0.04)^5 = 11,904.31$. The “New” value of the TSX after the 10 months of change is 11,904.31.





Important Notes

You can use your financial calculator for these nonfinancial calculations just as you did for inflation. You must follow the cash flow sign convention by ensuring that PV and FV have opposite signs. However, once again the signs do not have any meaning in these calculations.



Things To Watch Out For

When you work with variable inflation rates, purchasing power of a dollar, or rates of change, you can simplify the variable rates into a single fixed rate using a geometric average before applying the compound interest formula. For example, using the TSX example above, the geometric average of the 10 months of change is

$$\text{GAvg} = \left[((1 - 0.04)^5 \times (1 + 0.04)^5)^{1/10} - 1 \right] \times 100 = -0.080032\%$$

Using this calculated value as the periodic interest rate and substituting into Formula 9.3 produces

$$\text{FV} = 12,000 \times (1 - 0.00080032)^{10} = 11,904.31$$

Similarly, you could solve Example 10.5B with this approach. Some students have found this technique easier than working with the variable rates. Either way, whether you use variable rates or the geometric average as the fixed rate, both techniques produce the same solution.



Give It Some Thought

If a quantity decreases by $x\%$ and then increases by the same percent, why do you not arrive at the original quantity as your “New” value?

Solution: The initial decrease will make the base number smaller. If the new base number is then increased by the same percentage, that percentage is determined from a smaller base. It will not produce the same value as the decrease.

Example 10.4C: Changing Employment in an Urban Centre

An urban centre tracks employment in its Metro Core area. In 1991, approximately 45,000 employees in the Metro Core area were classified as “professional and commercial service” workers. From 1991 to 2001, the number of workers grew by about 4.5% per year. From 2001 to 2011, the growth rate was about 3.1% per year, and from 2011 to 2021 the growth rate was about 1.5% per year.⁴ Rounded to the nearest thousand, how many people were employed in the “professional and commercial services” field in the urban centre in 2021?

Plan	You are looking for the “New” value (FV) of the number of employees after the 30 years of percent changes.	
Understand	What You Already Know Step 1: The beginning value and structure of the sequence of percent changes are known, as shown in the timeline.	How You Will Get There Step 2: Capture the values of i_n and N_n for each consecutive percent change value. Note that the timeline has three time segments, so each variable has three values. Step 3: Apply Formula 9.3.

⁴ Example based upon City of Vancouver, “Employment Change in the Metro Core,” November 22, 2005, http://s3.amazonaws.com/zanran_storage/vancouver.ca/ContentPages/7116122.pdf, accessed August 20, 2013.

Perform	Step 2: $i_1 = 4.5\%$, $N_1 = 10$, $i_2 = 3.1\%$, $N_2 = 10$, $i_3 = 1.5\%$, $N_3 = 10$ Step 3: $FV = 45,000 \times (1 + 0.045)^{10} \times (1 + 0.031)^{10} \times (1 + 0.015)^{10}$ $FV = 45,000 \times 1.045^{10} \times 1.031^{10} \times 1.015^{10}$ $FV = 45,000 \times 1.552969 \times 1.357021 \times 1.160540 = 110,058 = 110,000$																																					
	Calculator Instructions <table border="1"> <thead> <tr> <th>Time Segment</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr> </thead> <tbody> <tr> <td>1</td><td>10</td><td>4.5</td><td>-45000</td><td>0</td><td>Answer: 69,883.62398</td><td>1</td><td>1</td></tr> <tr> <td>2</td><td>✓</td><td>3.1</td><td>-69,883.62398</td><td>✓</td><td>Answer: 94,833.56372</td><td>✓</td><td>✓</td></tr> <tr> <td>3</td><td>✓</td><td>1.5</td><td>-94,833.56372</td><td>✓</td><td>Answer: 110,058.2223</td><td>✓</td><td>✓</td></tr> </tbody> </table>							Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y	1	10	4.5	-45000	0	Answer: 69,883.62398	1	1	2	✓	3.1	-69,883.62398	✓	Answer: 94,833.56372	✓	✓	3	✓	1.5	-94,833.56372	✓	Answer: 110,058.2223	✓
Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y																															
1	10	4.5	-45000	0	Answer: 69,883.62398	1	1																															
2	✓	3.1	-69,883.62398	✓	Answer: 94,833.56372	✓	✓																															
3	✓	1.5	-94,833.56372	✓	Answer: 110,058.2223	✓	✓																															
Present	From 1991 to 2021, the number of employees in the "professional and commercial services" field in the urban centre grew from 45,000 to 110,000.																																					



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Chapter 10 Summary

Key Concepts Summary

Section 10.1: Application: Long-Term GICs (Keeping Your Money Safe When Investing)

- The characteristics and calculations involved with an interest payout GIC
- The characteristics and calculations involved with compound interest GICs
- The characteristics and calculations involved with escalator GICs

Section 10.2: Application: Long-Term Promissory Notes (IOUs)

- The sale of interest-bearing promissory notes
- The sale of noninterest-bearing promissory notes

Section 10.3: Application: Strip Bonds (Buy Low, Sell High)

- Characteristics of strip bonds
- Calculating the present value or purchase price of a strip bond
- Calculating the nominal yield on a strip bond

Section 10.4: Application: Inflation, Purchasing Power, and Rates of Change (Your Grandparents Used to Go to a Movie for a Quarter)

- Applying the concepts of compound interest to rates of inflation
- Applying the concepts of compound interest to purchasing power
- Applying the concepts of compound interest to rates of change

The Language of Business Mathematics

compound interest GIC A GIC that uses compound interest rates for which interest is periodically calculated and converted to the principal of the GIC for further compounding.

deflation The overall downward price movement of products in an economy, which is measured by negative change in the consumer price index.

escalator interest GIC A GIC that uses compound interest rates that usually remain constant during each of a series of time intervals, always rising stepwise throughout the term of the investment with any accrued interest being converted to principal.

inflation The overall upward price movement of products in an economy, which is measured by positive change in the consumer price index.

interest payout GIC A GIC where the interest is periodically paid out to the investor, but it is never added to the principal of the GIC. Because the interest does not actually compound, in essence the concepts of simple interest are used.

strip bond A marketable bond that has been stripped of all interest payments.

The Formulas You Need to Know

Symbols Used

I = Interest payment amount

i = Periodic interest rate

n = number of pieces of data or variables

N = Number of compounding periods

PPD = Purchasing power of a dollar

PV = Principal or present value

Formulas Introduced

Formula 10.1 Periodic Interest Amount: $I = PV \times i$ (Section 10.1)

Formula 10.2 Purchasing Power of a Dollar (Compound Interest Method): $PPD = \frac{\$1}{(1+i)^N} \times 100$ (Section 10.5)

Technology

Calculator

No new calculator functions were introduced in this chapter.

Chapter 11: Compound Interest: Annuities

(There Are Payments?)

Payments show up everywhere. You make payments on retail purchases; bank loans require payments; you even make payments to yourself into your RRSP! Payments fall into a few categories subject to important characteristics. Let's explore these categories with a few everyday examples.

- Last week you completed the purchase of your \$150,000 starter home and finished all of the paperwork on your mortgage with the bank. You snagged a great mortgage rate of 5% compounded semi-annually, locking it in for a five-year term with monthly payments of \$872.41 (for the next 25 years) starting one month after you move in.
- You excitedly cruise to the car dealership to pick up your used \$22,475 vehicle, on which you have a four-year lease. Pulling out your iPhone (for which you pay \$60 monthly), you ensure that you have enough funds in your chequing account to make the \$2,000 down payment today along with the first of your \$242.16 monthly lease payments at 2.9% compounded monthly.
- Driving away in your car, you stop in at Sleep Country Canada and purchase a queen size mattress for your new bedroom. You recall a TV ad promoting an \$895 queen mattress plus 12% taxes with no money down and 12 easy, interest-free monthly payments of \$83.53. It will be nice to have a comfortable bed to sleep in!

Did you notice some of the characteristics of these various types of payments?

- The mattress was low priced and was paid off in a short, one-year time frame, while the house was high priced and required a long, 25-year time frame to extinguish the debt.
- The lease on your car required a payment up front before you could drive it home, while your mortgage and mattress had payments that would start later.
- Your monthly iPhone payment recurs endlessly until you terminate the contract.

These examples also illustrate how businesses participate in consumer payments. Companies must understand financing for their consumers for a variety of reasons:

- Banks create mortgages for their clients so that consumers can make large purchases through affordable payments. These payment plans are structured to allow the bank to earn profits through the interest charged.
- Car dealerships use similar payment plans that earn interest on the payments in addition to the profits built into the products themselves. It is like double-dipping!
- Sleep Country Canada sells more mattresses by allowing consumers to spread their payments over the course of a year instead of requiring them to hand over one big lump sum, which many people may not be able to do.

And do not forget that regular payments are needed in business-to-business deals too, including mortgages, leases, insurance, utilities, and even product acquisitions.

This chapter explores the concept of making regular payments toward savings goals or debt extinguishment. First it introduces key payment concepts and reviews different types of payment plans. You will then calculate the future and present value of a stream of payments. You extend the application of the concepts further to solve for other variables such as the payment amount, the term, and the nominal interest rate. Chapter 12 discusses commonplace annuity applications such as planning your RRSP and buying a car.

Outline of Chapter Topics

- 11.1: Fundamentals of Annuities (What Do I Need to Know?)
- 11.2: Future Value of Annuities (Will I Have Enough?)
- 11.3: Present Value of Annuities (How Much Do I Need Now?)
- 11.4: Annuity Payment Amounts (What Commitment Am I Making?)
- 11.5: Number of Annuity Payments (Exactly How Long Are We Talking About?)
- 11.6: Annuity Interest Rates (How Does the Money Grow?)

11.1: Fundamentals of Annuities

(What Do I Need To Know?)

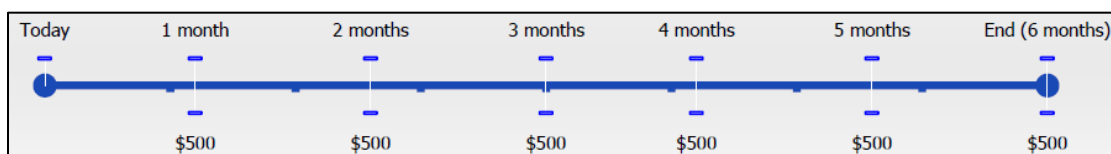
It is impossible to calculate payments accurately until you recognize a few of the key characteristics illustrated in the chapter introduction:

- All of the examples required a payment in the same amount on a regular basis, such as the \$872.41 every month for your mortgage.
- The timing of the payments varied. The car lease required the first payment up front (at the beginning of the month), while the mortgage and mattress purchase had the first payment due a month after the purchase (at the end of the month).
- The frequency of the payments and the frequency of the interest rate varied. The mortgage had monthly payments with interest being charged semi-annually, while the car lease had monthly payments and monthly interest.

Unlike single payments (covered in Chapter 9) for which there is only one formula, you solve series of payments by choosing the appropriate formula from four possibilities defined by the financial characteristics of the payments. This section defines the characteristics of four different types of payment series and then contrasts them to the Chapter 9 and 10 single-payment calculations. This section also develops a new, simplified structure for timelines to help you visualize a series of payments.

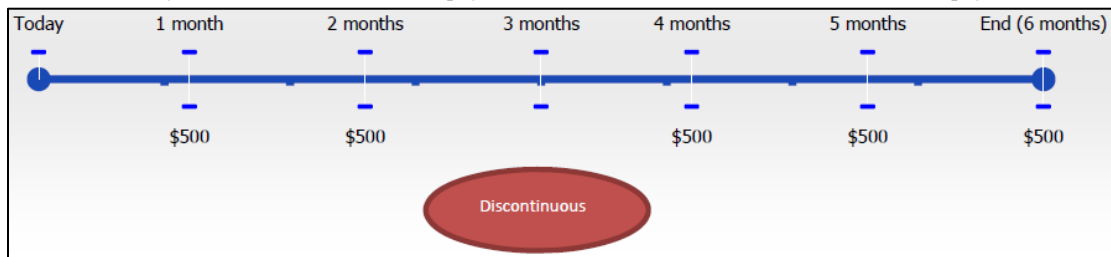
What Are Annuities?

An **annuity** is a continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation. An **annuity payment** is the dollar amount of the equal periodic payment in an annuity environment. The figure below illustrates a six-month annuity with monthly payments. Notice that the payments are continuous, equal, periodic, and occur over a fixed time frame. If any one of these four characteristics is not satisfied, then the financial transaction fails to meet the definition of a singular annuity and requires other techniques and formulas to solve.

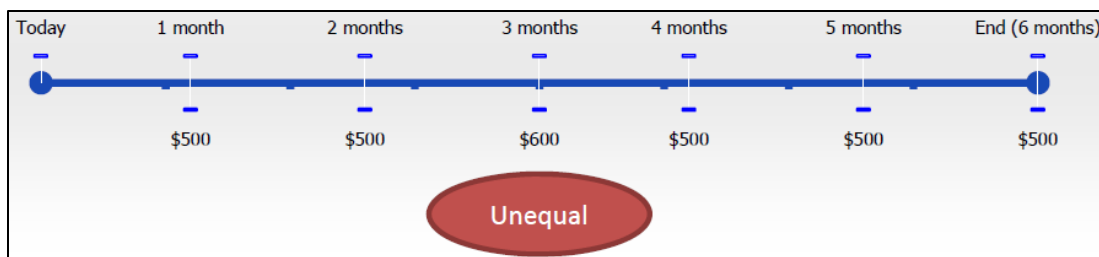


The examples below illustrate four timelines that look similar to the one above, but with one of the characteristics of an annuity violated. This means that none of the following in their entirety are considered an annuity:

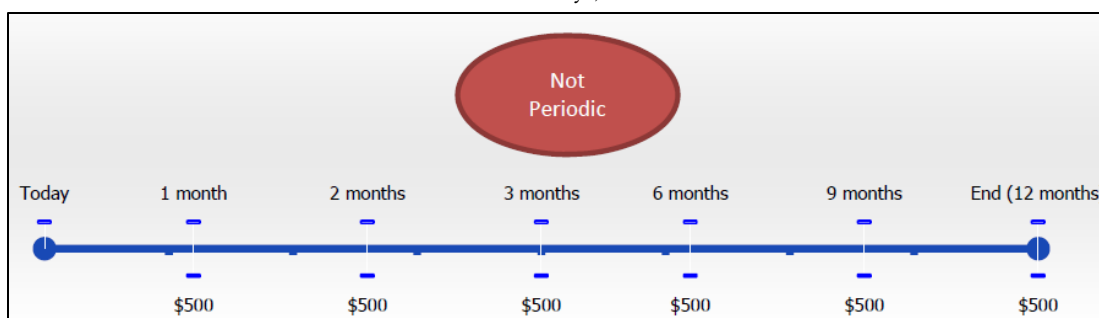
1. **Continuous.** Annuity payments are without interruption or breaks from the beginning through to the end of the annuity's term. In the figure above there are no breaks in the annuity since every month has an annuity payment. This next figure is not an annuity because the absence of a payment in the third month makes the series of payments discontinuous.



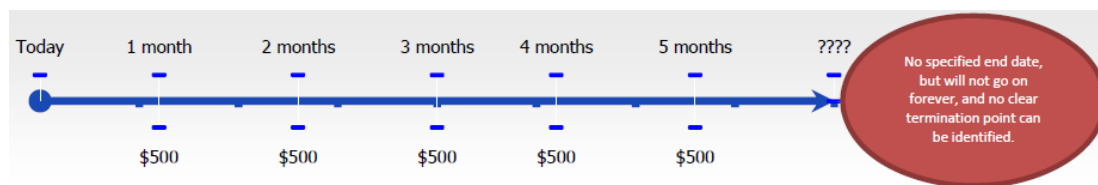
2. **Equal.** The annuity payments must be in the same amount every time from the beginning through to the end of the annuity's term. In the original figure, every monthly annuity is \$500. Notice that in this next figure the payment amount varies and includes values of both \$500 and \$600.



3. **Periodic.** The amount of time between each continuous and equal annuity payment is known as the **payment interval**. Hence, a monthly payment interval means payments have one month between them, whereas a semi-annual payment interval means payments have six months between them. Annuity payments must always have the same payment interval from the beginning through to the end of the annuity's term. In the original figure there is exactly one month between each equal and repeated payment in the annuity. Notice that in this next figure the payments, although equal and continuous, do not occur with the same amount of time between each one. In fact, the first three payments are made monthly while the last three payments are made quarterly. (You may notice that if the first three payments are treated separately from the last three payments, then each separate grouping represents an annuity and therefore there are two separate annuities. However, as a whole this is not an annuity.)



4. **Specified Period of Time.** The annuity payments must occur within an identifiable time frame that has a specified beginning and a specified end. The annuity's time frame can be (1) known with a defined starting date and defined ending date, such as the annuity illustrated in the original figure, which endures for six periods; (2) known but nonterminating, such as beginning today and continuing forever into the future (hence an infinite period of time); or (3) unknown but having a clear termination point, for example, monthly pension payments that begin when you retire and end when you die—a date that is obviously not known beforehand. In the next figure, the annuity has no defined ending date, does not continue into the future, and no clear termination point is identifiable.



In summary, the first figure is an annuity that adheres to all four characteristics and can be addressed via an annuity formula. The next four figures are not annuities and need other financial techniques or formulas to perform any necessary calculations.

Types of Annuities

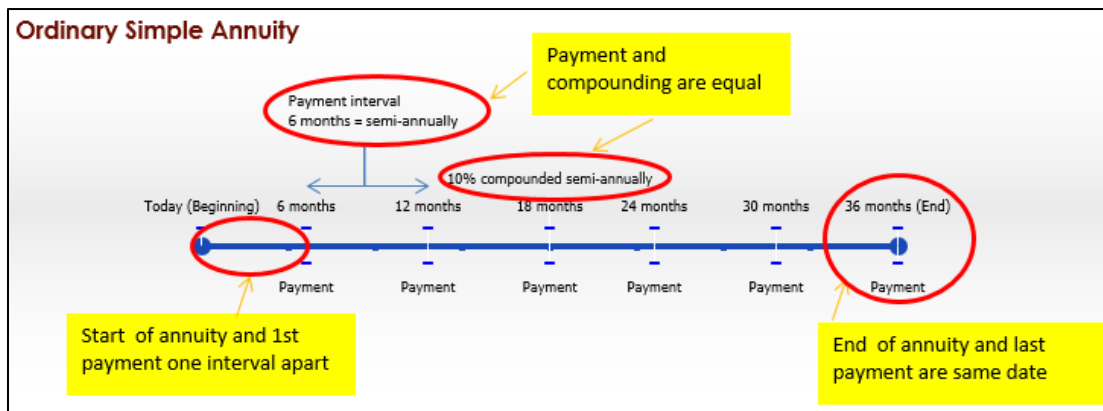
There are four types of annuities, which are based on the combination of two key characteristics: timing of payments and frequency. Let's explore these characteristics first, after which we will discuss the different annuity types.

1. **Timing of Payments.** An example best illustrates this characteristic. Assume that you take out a loan today with monthly payments. If you were to make your first annuity payment on the day you take out the loan, the amount of principal owing would be immediately reduced and you would accumulate a smaller amount of interest during your first month. This is called making your annuity payment at the *beginning* of a payment interval, and this payment is known as a **due**. However, if a month passes before you make your first monthly payment on the loan, your original principal

accumulates more interest than if the principal had already been reduced. This is called making your payment at the *end* of the payment interval, and this payment is known as an **ordinary** because it is the most common form of annuity payment. Depending on when you make your payment, different principal and interest amounts occur.

2. **Frequency.** The frequency of an annuity refers to a comparison between the payment frequency and the compounding frequency. A **payment frequency** is the number of annuity payments that would occur in a complete year. Recall from Chapter 9 that the *compounding frequency* is the number of compounds per complete year. If the payment frequency is the same as the compounding frequency, this is called a **simple annuity**. When interest is charged to the account monthly and payments are also made monthly, you determine principal and interest using simplified formulas. However, if the payment frequency and the compounding frequency are different, this is called a **general annuity**. If, for example, you make payments monthly while interest is compounded semi-annually, you have to use more complex formulas to determine principal and interest since the paying of principal and charging of interest do not occur simultaneously.

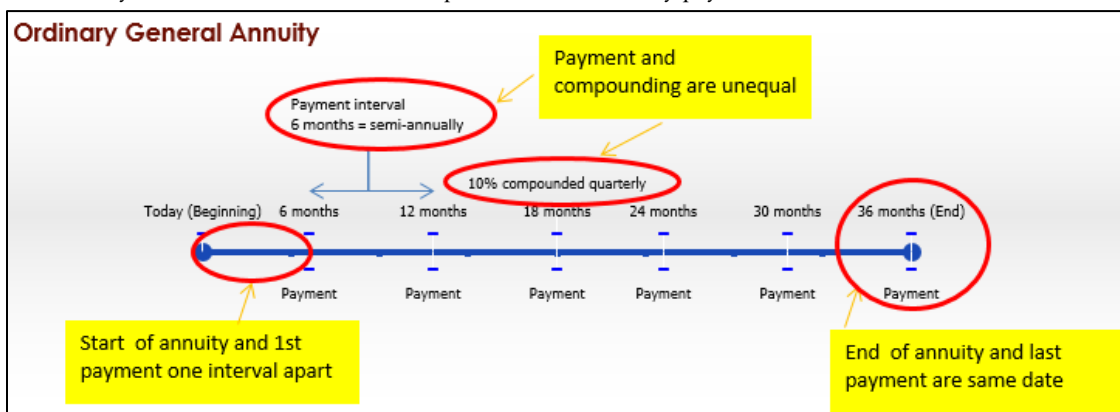
Putting these two characteristics together in their four combinations creates the four types of annuities. Each timeline in these figures assumes a transaction involving six semi-annual payments over a three-year time period.



Ordinary Simple Annuity An **ordinary simple annuity** has the following characteristics:

- Payments are made at the end of the payment intervals, and the payment and compounding frequencies are equal.
- The first payment occurs one interval after the beginning of the annuity.
- The last payment occurs on the same date as the end of the annuity.

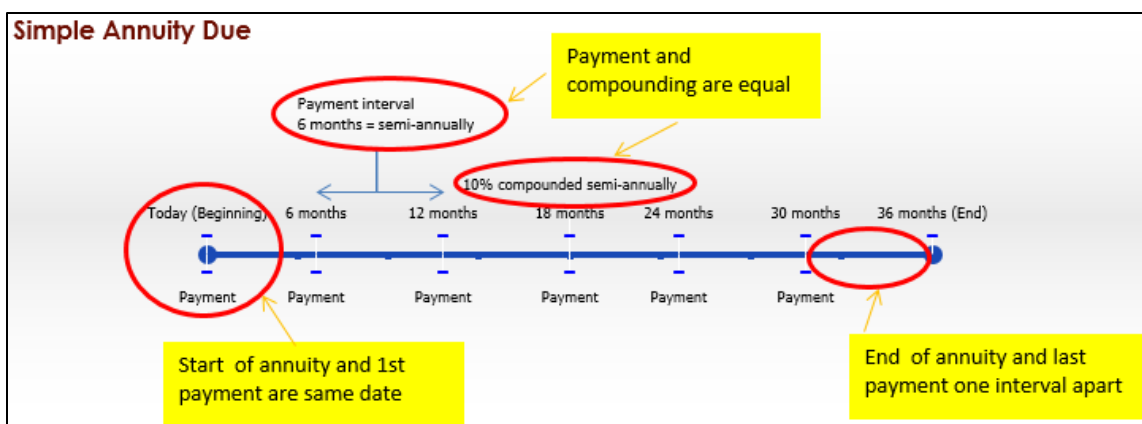
For example, most car loans are ordinary simple annuities where payments are made monthly and interest rates are compounded monthly. As well, car loans do not require the first monthly payment until the end of the first month.



Ordinary General Annuity An **ordinary general annuity** has the following characteristics:

- Payments are made at the end of the payment intervals, and the payment and compounding frequencies are unequal.
- The first payment occurs one interval after the beginning of the annuity.
- The last payment occurs on the same date as the end of the annuity.

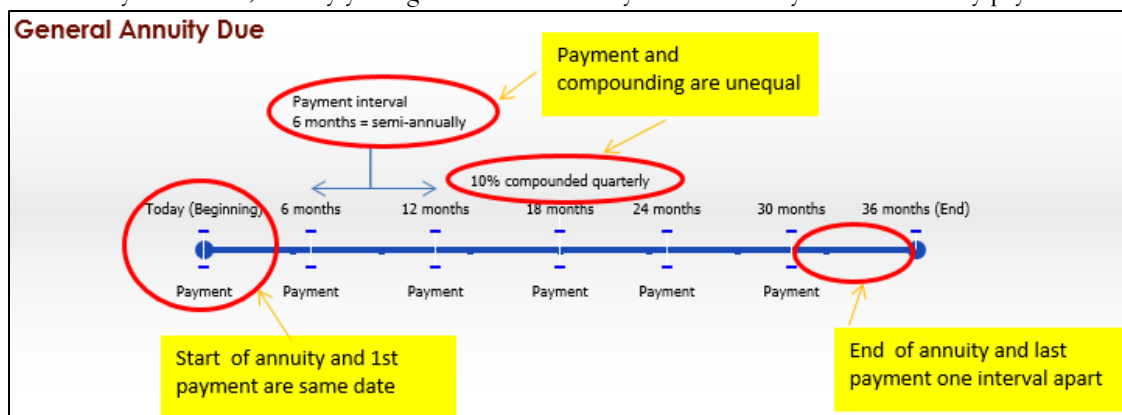
For example, most mortgages are ordinary general annuities, where payments are made monthly and interest rates are compounded semi-annually. As with car loans, your first monthly payment is not required until one month elapses.



Simple Annuity Due A **simple annuity due** has the following characteristics:

- Payments are made at the beginning of the payment intervals, and the payment and compounding frequencies are equal.
- The first payment occurs on the same date as the beginning of the annuity.
- The last payment occurs one payment interval before the end of the annuity.

For example, most car leases are simple annuities due, where payments are made monthly and interest rates are compounded monthly. However, the day you sign the lease is when you must make your first monthly payment.



General Annuity Due A **general annuity due** has the following characteristics:

- Payments are made at the beginning of the payment intervals, and the payment and compounding frequencies are unequal.
- The first payment occurs on the same date as the beginning of the annuity.
- The last payment occurs one payment interval before the end of the annuity.

For example, many investments, like your RRSP, are general annuities due where payments (contributions) are typically made monthly but the interest compounds in another manner, such as annually. As well, when most people start an RRSP they pay into it on the day they set it up, meaning that their RRSP commences with the first deposited payment.

The table below summarizes the four types of annuities and their characteristics for easy reference.

Annuity Type	Timing of Payments in a Payment Interval	Payment Frequency and Compounding Frequency	Start of Annuity and First Payment Same Date?	End of Annuity and Last Payment Same Date?
Ordinary Simple Annuity	End	Equal	No, first payment one interval later	Yes
Ordinary General Annuity	End	Unequal	No, first payment one interval later	Yes
Simple Annuity Due	Beginning	Equal	Yes	No, last payment one interval earlier
General Annuity Due	Beginning	Unequal	Yes	No, last payment one interval earlier

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Paths To Success

One of the most challenging aspects of annuities is recognizing whether the annuity you are working with is ordinary or due. This distinction plays a critical role in formula selection later in this chapter. To help you recognize the difference, the table below summarizes some key words along with common applications in which the annuity may appear.

Type	Key Words or Phrases	Common Applications
Ordinary	-...payments are at the end... -...payments do not start today... -...payments are later... -...first payment next interval...	- bank loans of any type - mortgages - bonds - Canada Pension Plan (CPP)
Due	-...payments are at the beginning... -...payments start today... -...payments are in advance... -...first payment today... -...payments start now...	- any kind of lease - any kind of rental - RRSPs (usually) - membership dues - insurance

Annuities versus Single Payments

To go from single payments in Chapter 9 to annuities in this chapter, you need to make several adaptations:

1. **Annuity Payment Amount (PMT).** This variable was not used in any of the formulas introduced in Chapters 9 or 10, though it was briefly introduced when Section 9.2 when demonstrating the calculator functions. Notice that in the previous two chapters this variable was set to zero for every question, since every payment was *not* part of an annuity. Annuity calculations require you to tie a value to this variable in the formulas and when you use technology such as the BAII+ calculator.
2. **Payment Frequency or Payments per Year (PY).** For single payments, this variable did not show up in any of the formulas from the previous two chapters. It too was introduced in Section 9.2 as a calculator button with the requirement that it automatically be set as a null variable matching the compounding frequency (CY). When you work with annuities, an actual value for PY is determined by the payment frequency. For simple annuities PY remains the same as CY, whereas the variables are different for general annuities.
3. **Cash Flow Sign Convention on the Calculator.** It now becomes critical to ensure the proper application of the cash flow sign convention on the calculator—failure to do so will result in an incorrect answer. For example, if you borrow money and then make annuity payments on it, you enter the present value (PV) as a positive (you received the money) while you enter the annuity payments as negatives (you paid the money to the bank). This results in future balances getting smaller and you owing less money. If you inadvertently enter the annuity payment as a positive number, this would mean you are borrowing more money from the bank so your future balance would increase and you would owe more money. These two answers are very different!
4. **Definition and Computation of N .** When you worked with single payments, N was defined as the total number of compounds throughout the term of the financial transaction. When you work with annuities, N is defined as the total number of payments throughout the term of the annuity. You calculate it using Formula 11.1.



The Formula

N is Total Number of Annuity Payments: The total number of payments from the start of the annuity to the end of the annuity, inclusive.

Years is Term of the Annuity: The length of the annuity from start to end, expressed as the number of years.

Formula 11.1 – Number of Annuity Payments:
$$N = PY \times \text{Years}$$

PY is Payments per Year or Payment Frequency: The number of annuity payments per complete year.



How It Works

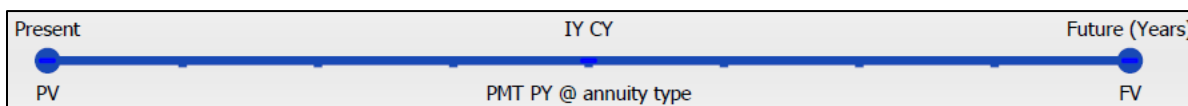
On a two-year loan with monthly payments and semi-annual compounding, the payment frequency is monthly, or 12 times per year. With a term of two years, that makes $N = 2 \times 12 = 24$ payments. Note that the calculation of N for an annuity does not involve the compounding frequency.

Adapting Timelines to Incorporate Annuities

Annuity questions can involve many payments. For example, in a typical 25-year mortgage with monthly payments, that would be $25 \times 12 = 300$ payments in total. How would you draw a timeline for these? Clearly, it would be impractical to draw 300 payments.

A good annuity timeline should illustrate the present value (PV), future value (FV), number of annuity payments (N), nominal interest rate (IY), compounding frequency (CY), annuity payment (PMT), and the payment frequency (PY). One of these variables will be the unknown.

As well, a good timeline requires a clear distinction between ordinary annuities and annuities due. END is used to represent ordinary annuities, since payments occur at the *end* of the payment interval. Similarly, BGN is used to represent annuities due, since payments occur at the *beginning* of the payment interval. The figure below illustrates the adapted annuity timeline format.



Sometimes a variable will change partway through the period of an annuity, in which case the timeline must be broken up into two or more segments. When you use this structure, in any time segment the annuity payment PMT is interpreted to have the same amount at the same payment interval continuously throughout the entire segment. The number of annuity payments N does not directly appear on the timeline since it is the result of a formula. However, its two components (Years and PY) are drawn on the timeline.

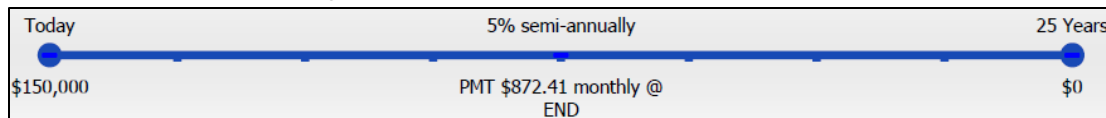


How It Works

A mortgage is used to illustrate this new format. For now, focus strictly on the variables and how to illustrate them in a timeline. Do not focus on any mortgage calculations yet.

- As per the chapter introduction, you purchased a \$150,000 home (PV) with a 25-year mortgage (Years). The mortgage rate is 5% (IY) compounded semi-annually (CY = 2), and the monthly payments (PY = 12) are \$872.41 (PMT).
- After 25 years you will own your house and therefore have no balance remaining. This sets the future value to \$0 (FV).
- Mortgages are ordinary in nature, meaning that payments occur at the end of the payment intervals (END).

The figure below places all of these mortgage elements into the new timeline format. Once you have drawn the timeline, you can determine the following:



- The number of payments (N), which you calculate using Formula 11.1. Since both PY and Years are known, you have $N = 25 \times 12 = 300$.
- Whether the annuity is simple or general, which depends on the PY below the timeline and CY above the timeline. If they match, the annuity is simple; if they differ, as in this example, it is general.



Things To Watch Out For

The word "payment" often confuses people because it has two interpretations. It could mean either "single payment," such as in Chapter 9, or "annuity payment," which is meant in this chapter. To correctly interpret this word, recall the characteristics of an annuity payment and determine if the situation at hand matches the criteria. Let's review two examples illustrating this point:

1. Suppose John owes Mary three payments of \$200 due two months, five months, and eleven months from now. In this case, the word "payment" means "single payment" because the payments are equal (all \$200) and the time periods are known, but they are neither periodic (payment intervals are unequal) nor continuous (since no periods exist, there is no continuity). Working with these payments requires you to apply the formulas and techniques from Chapter 9.
2. Suppose John owes Julia three payments of \$200 due three months, six months, and nine months from now. In this case, the word "payment" means "annuity payment" because the payments are equal (all \$200), periodic (every quarter), and continuous (occurring every quarter without interruption), and they have a specified time frame (over nine months). Working with these payments allows you to apply the formulas and techniques from this chapter.

Ultimately, when in doubt you can solve any question involving time value of money using the formulas and techniques from Chapter 9. The annuity formulas introduced in the next section just allow you to arrive at the same answer with a lot less calculation.



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11.2: Future Value of Annuities

(Will I Have Enough?)

Though your retirement is probably still a long way off, the earlier you start investing the more you can take advantage of the power of compounding interest to generate your savings.

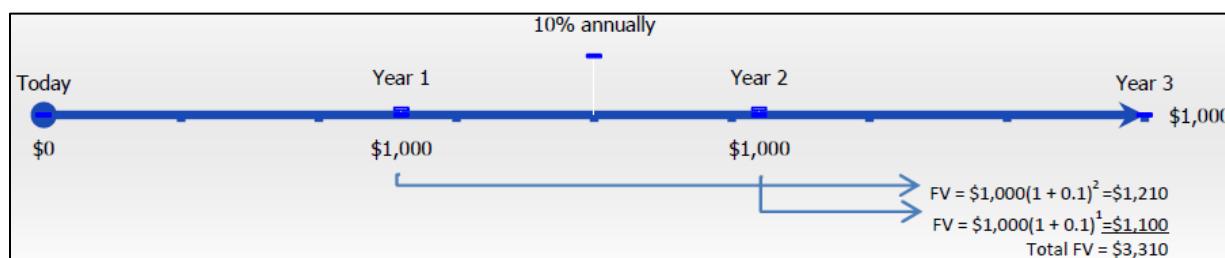
Did it startle you to learn in Section 10.5 that your annual gross retirement income might need to be \$160,000 when you turn 65 years old? How long will you live? The average Canadian life expectancy is around 80 years old; if you live to be that age, you will need 15 years of retirement income. Using a conservative interest rate of 5% compounded annually along with 3% annual inflation, this works out to saving up approximately \$2 million for when you retire. A daunting goal, isn't it? You might well wonder, "If I start saving \$300 per month today, will I have enough?"

Clearly it is important to know how much your annuities are worth in the future. This matters not only for investments but also for debts, since most businesses and individuals pay off their debts through annuity structures. After you have made several annuity payments, can you tell at any time how much you or your company still owes on an outstanding debt?

In the previous section you learned to recognize the fundamental characteristics of annuities, so now you can start to solve any annuity for any unknown variable. There are four annuity formulas. This section covers the first two, which calculate future values for both ordinary annuities and annuities due. These formulas accommodate both simple and general annuities.

Ordinary Annuities

The **future value of any annuity** equals the sum of all the future values for all of the annuity payments when they are moved to the end of the last payment interval. For example, assume you will make \$1,000 contributions at the end of every year for the next three years to an investment earning 10% compounded annually. This is an ordinary simple annuity since payments are at the end of the intervals, and the compounding and payment frequencies are the same. If you wanted to know how much money you have in your investment after the three years, the figure below illustrates how you would apply the fundamental concept of the time value of money to move each payment amount to the future date (the focal date) and sum the values to arrive at the future value.



While you could use this technique to solve all annuity situations, the computations become increasingly cumbersome as the number of payments increases. In the above example, what if the person instead made quarterly contributions of \$250? That is 12 payments over three years, resulting in 11 separate future value calculations. Or if they made monthly payments, the 36 payments over three years would result in 35 separate future value calculations! Clearly, solving this would be tedious and time consuming—not to mention prone to error. There must be a better way!



The Formula

The formula for the future value of an ordinary annuity is indeed easier and faster than performing a series of future value calculations for each of the payments. At first glance, though, the formula is pretty complex, so the various parts of the formula are first explored in some detail before we put them all together.

The annuity formula is a more complex version of the rate, portion, and base formula introduced in Chapter 2. Relating Formula 2.2 and the first payment from the figure above gives the following:

$$\text{Portion} = \text{Base} \times \text{Rate}$$

$$\$1,210 = \$1,000 \times (1 + 0.1)^2$$

The *portion* equals the future value and the *base* equals the annuity payment amount. The rate is expressed as a formula and written as $(1 + 0.1)^2$. However, notice that each payment in the figure has a different exponent to reflect the compounding required. This requires a mathematical adaptation of $(1 + i)^N$, which permits the determination of an equivalent rate representing all payments in a single calculation.

This equivalent rate turns out to be a pretty complex expression, which is examined in three parts: the percent change overall, the percent change with each payment, and their quotient.

1. Numerator: The Percent Change Overall: $\left[\left(1 + i \right)^{\frac{CY}{PY}} \right]^N - 1$. This part of the formula determines the percent change from the start of the annuity to the end of the annuity. It has three critical elements:

- a. **Interest Rate Conversion $\left(1 + i \right)^{\frac{CY}{PY}}$.** The rate of interest that occurs with each payment must be known. All annuity calculations require the compounding period to equal the payment interval. If this is not already the case, then you must convert the expressed interest rate into an equivalent interest rate.
 - For simple annuities, no conversion is necessary since the frequencies are the same: $CY = PY$. The exponent of $\frac{CY}{PY}$ always equals 1 and has no effect.
 - For general annuities, recall Formula 9.4 for calculating equivalent interest rates. Here the old compounding frequency forms the numerator (CY_{Old}) and the new compounding frequency (which matches the payment frequency) forms the denominator (CY_{New}) = PY . Thus $\frac{CY_{Old}}{CY_{New}}$ becomes $\frac{CY}{PY}$.
- b. **The Compounding N .** The exponent N compounds the periodic interest rate (which matches the payment interval) in accordance with the number of annuity payments made. For example, assume there are two end-of-year payments at 10% compounded semi-annually. If the interest rate is left semi-annually, there are four compounds over the two years, which does not match the payments. The interest rate is converted within the brackets from 10% compounded semi-annually to its equivalent 10.25% compounded annually rate. The end result is that interest will now compound twice over the two years, matching the number of payments.
- c. **Removing the Starting Point (-1) .** Since you added 1 to perform the compounding, mathematically you now need to remove the 1. The end result is that you now know (in decimal format) how much larger the future value is relative to its starting value.

2. Denominator: The Percent Change with Each Payment: $\left(1 + i \right)^{\frac{CY}{PY}} - 1$. The denominator in the formula shows the percent change in the account with each payment made. It too ensures that the denominator has a periodic rate matching the payment interval.

3. The Quotient By taking the numerator and dividing by the denominator, the percent change overall is divided by the percent change with each payment. This establishes a ratio between what is happening overall in the annuity versus what is happening with each transaction. Thus, this computation determines how much bigger the final value is relative to what happens with each payment. This ratio (the rate) is then multiplied against the annuity payment (the base) to get the final balance (the portion).

You are now in a position to see the whole formula by which you calculate the future value of any ordinary annuity. This single formula applies regardless of the number of payments that are made. Formula 11.2 summarizes the variables and calculations required.

FV_{ORD} is Future Value of an Ordinary Annuity: This is the amount of money in the account including all of the payments and all interest at some future point in time. If the future point is at the end of the annuity, the future value is the maturity value.

CY is Compounds Per Year or Compounding Frequency: The number of compounding periods per complete year.

N is Number of Annuity Payments: The total number of payments made during the time segment of the annuity. It results from Formula 11.1.

Formula 11.2 – Ordinary Annuity Future Value:

$$FV_{ORD} = PMT \left[\frac{(1+i)^{\frac{CY}{PY}N} - 1}{(1+i)^{\frac{CY}{PY}} - 1} \right]$$

PMT is Annuity Payment Amount: The amount of money that is invested or paid after each payment interval.

i is Periodic Interest Rate: This is the rate of interest that is used in converting the interest to principal. It results from Formula 9.1, $i = \frac{IY}{CY}$. As mentioned in the discussion preceding this formula, the compounding of the periodic interest rate used in the formula must always be based on the payment interval. The exponent $\frac{CY}{PY}$ ensures that this is always the case regardless of the annuity taking a simple or general structure.

PY is Payments Per Year or Payment Frequency: The number of payment intervals per complete year.



How It Works

There is a five-step process for calculating the future value of any ordinary annuity:

Step 1: Identify the annuity type. Draw a timeline to visualize the question.

Step 2: Identify the known variables, including PV, IY, CY, PMT, PY, and Years.

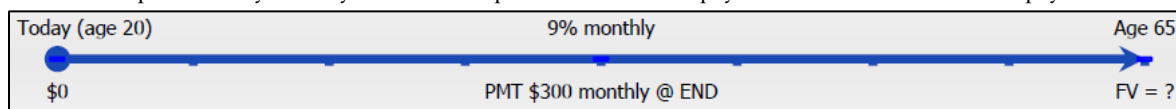
Step 3: Use Formula 9.1 to calculate i .

Step 4: If $PV = \$0$, proceed to step 5. If there is a nonzero value for PV, treat it like a single payment. Apply Formula 9.2 to determine N since this is not an annuity calculation. Move the present value to the end of the time segment using Formula 9.3.

Step 5: Use Formula 11.1 to calculate N for the annuity. Apply Formula 11.2 to calculate the future value. If you calculated a future value in step 4, combine the future values from steps 4 and 5 to arrive at the total future value.

Revisiting the RRSP scenario from the beginning of this section, assume you are 20 years old and invest \$300 at the end of every month for the next 45 years. You have not started an RRSP previously and have no opening balance. A fixed interest rate of 9% compounded monthly on the RRSP is possible.

Step 1: This is a simple ordinary annuity since the frequencies match and payments are at the end of the payment interval.



Step 2: The known variables are $PV = \$0$, $IY = 9\%$, $CY = 12$, $PMT = \$300$, $PY = 12$, and Years = 45.

Step 3: The periodic interest rate is $i = 9\% \div 12 = 0.75\%$.

Step 4: Since $PV = \$0$, skip this step.

Step 5: The number of payments is $N = 12 \times 45 = 540$. Applying Formula 11.2 gives the following:

$$FV_{\text{ORD}} = \$300 \left[\frac{\left[(1 + 0.0075)^{\frac{12}{12}} \right]^{540} - 1}{(1 + 0.0075)^{\frac{12}{12}} - 1} \right] = \$2,221,463.54$$

Hence, 540 payments of \$300 at 9% compounded monthly results in a total saving of \$2,221,463.54 by the age of retirement.



Important Notes

Calculating the Interest Amount. For investment annuities, if you are interested in knowing how much of the future value is principal and how much is interest, you can adapt Formula 8.3, where $I = S - P = FV - PV$.

- The FV is the solution to Formula 11.2.
- The PV is the principal (total contributions) calculated by taking $N \times PMT + PV$. In the figure above, $N \times PMT + PV = 540 \text{ payments} \times \$300 + \$0 = \$162,000$ in principal. Therefore, $I = \$2,221,463.54 - \$162,000 = \$2,059,463.54$, which is the interest earned.

Your BAI+ Calculator. Adapting your calculator skills to suit annuities requires three important changes:

1. You now have a value for PMT. Be sure to enter it with the correct cash flow sign convention. When you invest, the payment has the same sign as the PV. When you borrow, the sign of the payment is opposite that of PV.
2. The P/Y is no longer automatically set to the same value as C/Y. Enter your values for P/Y and C/Y separately. If the values are the same, as in the case of simple annuities, then taking advantage of the "Copy" feature on your calculator let's you avoid having to key the value in twice.
3. If a present value (PV) is involved, by formula you need to do two calculations using Formulas 9.3 and 11.2. If you input values for both PV and PMT, the calculator does these calculations simultaneously, requiring only one sequence to solve.



Things To Watch Out For

In many annuity situations there might appear to be more than one unknown variable. Usually the extra unknown variables are "unstated" variables that can reasonably be assumed. For example, in the RRSP illustration above, the statement "you have not started an RRSP previously and have no opening balance" could be omitted. If something were saved already, the number would need to be stated. As another example, it is normal to finish a loan with a zero balance. Therefore, in a loan situation you can safely assume that the future value is zero unless otherwise stated.

**TOP
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Paths To Success

The ability to recognize a simple annuity can allow you to simplify Formula 11.2. Since $CY = PY$, these two variables form a quotient of 1 for the exponent. For a simple annuity, you can simplify any appearances of the following algebraic expressions in any annuity formula (not just Formula 11.2) as follows:

$$(1 + i)^{\frac{CY}{PY}} \text{ in the numerator can be simplified to } (1 + i) \text{ and}$$

$$(1 + i)^{\frac{CY}{PY}} - 1 \text{ in the denominator can be simplified to just } i.$$

Thus, for simple annuities only, you simplify Formula 11.2 as

$$FV_{\text{ORD}} = PMT \left[\frac{(1 + i)^N - 1}{i} \right]$$



Give It Some Thought

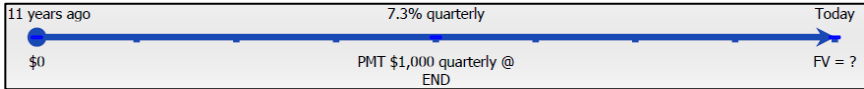
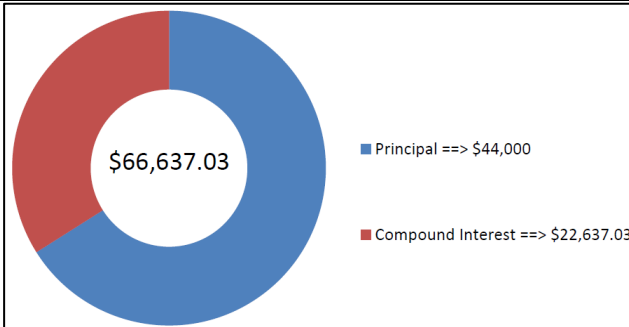
Assume you had planned to make 10 annuity payments to an investment. However, before you started paying in to the investment, you changed your mind, doubling your original payment amount while still making 10 payments. What happens to the maturity value of your new investment compared to that of your original plan? Will your new balance be exactly double, more than double, or less than double? Explain and justify your answer.

Solution:

Your new balance will be exactly double. Mathematically, you have taken PMT in Formula 11.2 and multiplied it by 2. That is the only difference between your original plan and your new plan. Therefore, the future value will double as well.

Example 11.2A: Future Value of an Investment Account

A financial adviser is reviewing one of her client's accounts. The client has been investing \$1,000 at the end of every quarter for the past 11 years in a fund that has averaged 7.3% compounded quarterly. How much money does the client have today in his account?

Plan	Step 1: The payments are at the end of the payment intervals, and both the compounding period and the payment intervals are the same. This is an ordinary simple annuity. Calculate its value at the end, which is its future value, or FV_{ORD} .															
Understand	What You Already Know Step 1 (continued): The timeline shows the client's account. Step 2: $PV = \\$0$, $IY = 7.3\%$, $CY = 4$, $PMT = \\$1,000$, $PY = 4$, Years = 11	How You Will Get There Step 3: Apply Formula 9.1. Step 4: Skip this step since $PV = \\$0$. Step 5: Apply Formula 11.1 and Formula 11.2.														
																
Perform	Step 3: $i = 7.3\% \div 4 = 1.825\%$ Step 5: $N = 4 \times 11 = 44$ payments $FV_{ORD} = \$1000 \left[\frac{\left[(1 + 0.01825)^{\frac{4}{4}} \right]^{44} - 1}{(1 + 0.01825)^{\frac{4}{4}} - 1} \right] = \$66,637.03$	Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>44</td><td>7.3</td><td>0</td><td>-1000</td><td>Answer: 66,637.03449</td><td>4</td><td>4</td></tr></table>	N	I/Y	PV	PMT	FV	P/Y	C/Y	44	7.3	0	-1000	Answer: 66,637.03449	4	4
	N	I/Y	PV	PMT	FV	P/Y	C/Y									
44	7.3	0	-1000	Answer: 66,637.03449	4	4										
Present	The figure shows how much principal and interest make up the final balance. After 11 years of \$1,000 quarterly contributions, the client has \$66,637.03 in the account.															

Example 11.2B: Future Value of a Savings Annuity

A savings annuity already contains \$10,000. If an additional \$250 is invested at the end of every month at 9% compounded semi-annually for a term of 20 years, what will be the maturity value of the investment?

Plan	Step 1: The payments are at the end of the payment intervals, and the compounding period and payment intervals are different. This is an ordinary general annuity. Calculate its value at the end, which is its future value, or FV_{ORD} .	
Understand	What You Already Know Step 1 (continued): The timeline appears below. Step 2: $PV = \\$10,000$, $IY = 9\%$, $CY = 2$, $PMT = \\$250$, $PY = 12$, Years = 20	How You Will Get There Step 3: Apply Formula 9.1. Step 4: Apply Formula 9.2 and Formula 9.3. Step 5: Apply Formula 11.1 and Formula 11.2. The final future value is the sum of the answers to step 4 (FV) and step 5 (FV_{ORD}).

Perform	Step 3: $i = 9\% \div 2 = 4.5\%$ Step 4: $N = 2 \times 20 = 40$ compounds $FV = \\$10,000(1 + 0.045)^{40} = \\$58,163.64538$ Step 5: $N = 12 \times 20 = 240$ payments $FV_{ORD} = \$250 \left[\frac{(1 + 0.045)^{\frac{2}{12} \times 240} - 1}{(1 + 0.045)^{\frac{2}{12}} - 1} \right] = \$163,529.9492$ Total FV = $\\$58,163.64538 + \\$163,529.9492 = \\$221,693.59$
Present	The figure shows how much principal and interest make up the final balance. The savings annuity will have a balance of \$221,693.59 after the 20 years.



Important Notes

If any of the variables, including IY, CY, PMT, or PY, change between the start and end point of the annuity, or if any additional single payment deposit or withdrawal is made, a new time segment is created and must be treated separately. There will then be multiple time segments that require you to work left to right by repeating steps 3 through 5 in the procedure. The future value at the end of one time segment becomes the present value in the next time segment. Example 11.2C illustrates this concept.



Things To Watch Out For

Pay extra attention when the variable that changes between time segments is the payment frequency (PY). When inputted into a BAII+ calculator, the PY automatically copies across to the compounding frequency (CY). Unless your CY also changed to the same frequency, this means that you must scroll down to the CY window and re-enter the correct value for this variable, even if it didn't change.

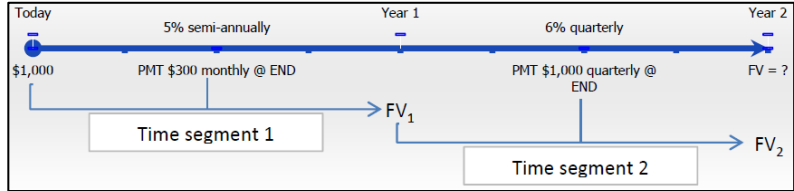
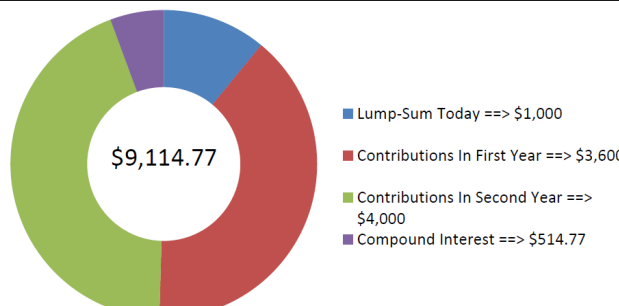
**TOP
SECRET**

Paths To Success

When working with multiple time segments, it is important that you always start your computations on the side opposite the unknown variable. For future value calculations, this means you start on the left-hand side of your timeline; for present value calculations, start on the right-hand side.

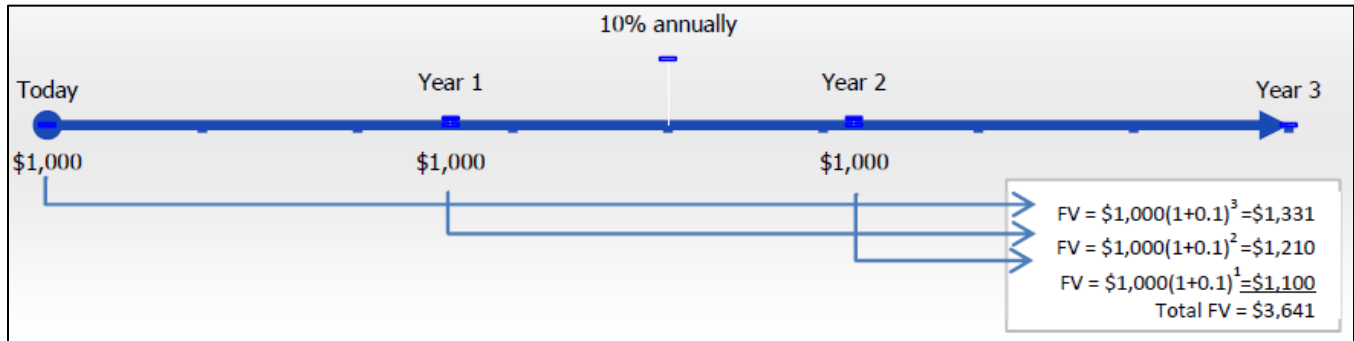
Example 11.2C: Saving Up for a Vacation

Genevieve has decided to start saving up for a vacation in two years, when she graduates from university. She already has \$1,000 saved today. For the first year, she plans on making end-of-month contributions of \$300 and then switching to end-of-quarter contributions of \$1,000 in the second year. If the account can earn 5% compounded semi-annually in the first year and 6% compounded quarterly in the second year, how much money will she have saved when she graduates?

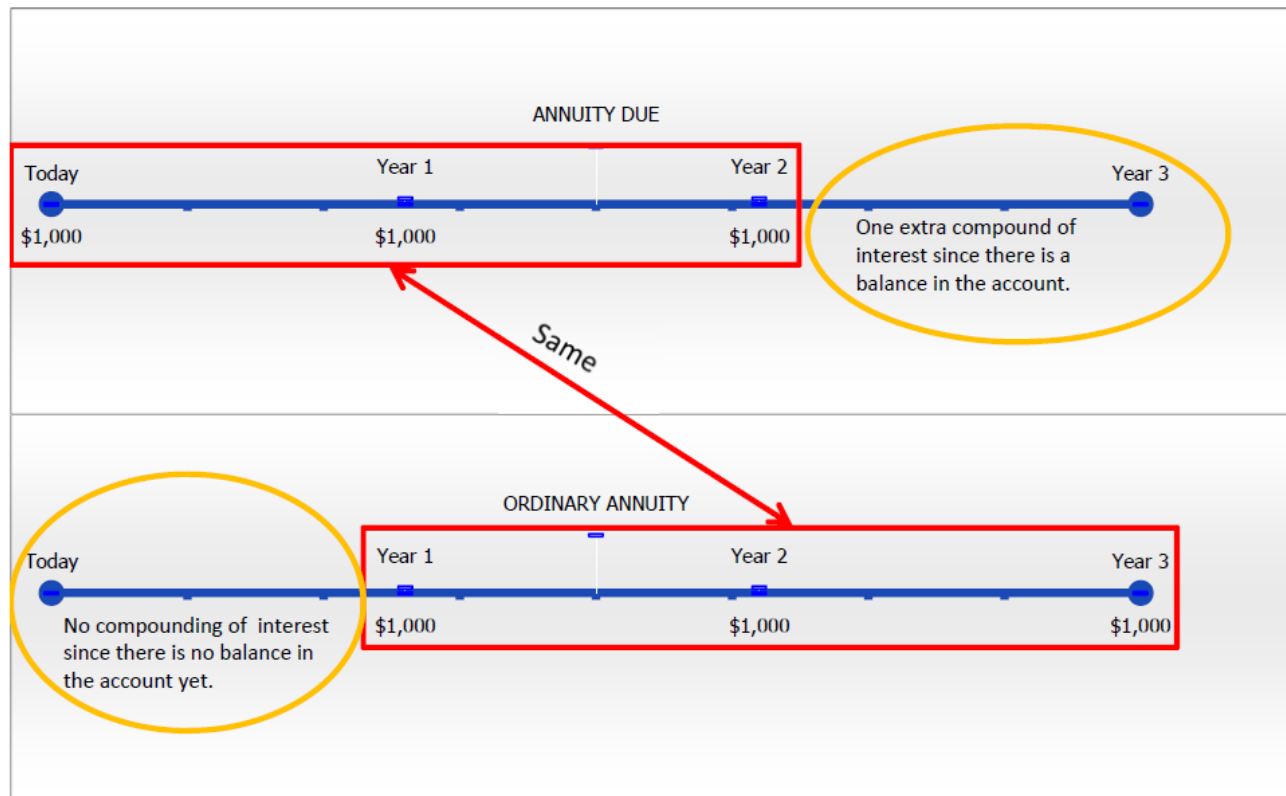
Plan	Step 1: There is a change of variables after one year. As a result, you need a Year 1 time segment and a Year 2 time segment. In both segments, payments are at the end of the period. In Year 1, the compounding period and payment intervals are different. In Year 2, the compounding period and payment intervals are the same. This is an ordinary general annuity followed by an ordinary simple annuity. You aim to calculate the future value, or FV_{ORD} .																										
Understand	What You Already Know Step 1 (continued): The timeline for her vacation savings appears below. Step 2: Time segment 1: $PV = \\$1,000$, $IY = 5\%$, $CY = 2$, $PMT = \\$300$, $PY = 12$, Years = 1 Time segment 2: $PV = FV_1$ of time segment 1, $IY = 6\%$, $CY = 4$, $PMT = \\$1,000$, $PY = 4$, Years = 1 How You Will Get There For <i>each</i> time segment repeat the following steps: Step 3: Apply Formula 9.1. Step 4: Apply Formula 9.2 and Formula 9.3. Step 5: Apply Formula 11.1 and Formula 11.2. The total future value in any time segment is the sum of the answers to step 4 (FV) and step 5 (FV_{ORD}). 																										
Perform	For the first time segment: Step 3: $i = 5\% \div 2 = 2.5\%$ Step 4: $N = 2 \times 1 = 2$ compounds, $FV_1 = \\$1,000(1 + 0.025)^2 = \\$1,050.625$ Step 5: $N = 12 \times 1 = 12$ payments $FV_{ORD1} = \\$300 \left[\frac{\left[(1 + 0.025)^{\frac{2}{12}} - 1 \right]}{(1 + 0.025)^{\frac{2}{12}} - 1} \right]^{12} = \\$3,682.786451$ Total $FV_1 = \\$1,050.625 + \\$3,682.786451 = \\$4,733.411451$ For the second time segment: Step 3: $i = 6\% \div 4 = 1.5\%$ Step 4: $N = 4 \times 1 = 4$ compounds, $FV_2 = \\$4,733.411451(1 + 0.015)^4 = \\$5,023.870384$ Step 5: $N = 4 \times 1 = 4$ payments $FV_{ORD2} = \\$1,000 \left[\frac{\left[(1 + 0.015)^{\frac{4}{4}} - 1 \right]}{(1 + 0.015)^{\frac{4}{4}} - 1} \right] = \\$4,090.903375$ Total $FV_2 = \\$5,023.870384 + \\$4,090.903375 = \\$9,114.77$ Calculator Instructions <table><tr><th>Time Segment</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>1</td><td>12</td><td>5</td><td>-1000</td><td>-300</td><td>Answer: 4,733.411451</td><td>12</td><td>2</td></tr><tr><td>2</td><td>4</td><td>6</td><td>-4,733.411451</td><td>-1000</td><td>Answer: 9,114.773759</td><td>4</td><td>4</td></tr></table>		Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y	1	12	5	-1000	-300	Answer: 4,733.411451	12	2	2	4	6	-4,733.411451	-1000	Answer: 9,114.773759	4	4	
Time Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y																				
1	12	5	-1000	-300	Answer: 4,733.411451	12	2																				
2	4	6	-4,733.411451	-1000	Answer: 9,114.773759	4	4																				
Present	The figure shows how much principal and interest make up the final balance. When Genevieve graduates she will have saved \$9,114.77 toward her vacation. 																										

Annuities Due

An annuity due occurs when payments are made at the beginning of the payment interval. To understand the difference this makes to the future value, let's recalculate the RRSP example from earlier in this section, but treat it as an annuity due. You want to know the future value of making \$1,000 annual contributions at the beginning of every payment interval for the next three years to an investment earning 10% compounded annually. This forms a simple annuity due. The figure below illustrates how you apply the fundamental concept of the time value of money to move each payment amount to the future date (the focal date) and sum the values to arrive at the future value.



Note that you do not end up with the same balance of \$3,310 achieved under the ordinary annuity. Instead, you have a larger balance of \$3,641. Why is this? Placing the two types of annuities next to each other in the next figure demonstrates the key difference between the two examples.



Both annuities have an identical sequence of three \$1,000 payments. However, in the ordinary annuity no interest is earned during the first payment interval since the principal is zero and the payment does not occur until the end of the interval. On the other hand, in the annuity due an extra compound of interest is earned in the last payment interval because of the existing principal at the end of the second year. If you take the ordinary annuity's final balance of \$3,310 and give it one extra compound you have $FV = 3,310(1 + 0.1) = 3,641$. In summary, the key difference between the two types of annuities is that an annuity due always receives one extra compound of interest.



The Formula

Adapting the ordinary annuity future value formula to suit the extra compound creates Formula 11.3. Note that all the variables in the formula remain the same; however, the subscript on the FV symbol is changed to recognize the difference in the calculation required.

FV_{DUE} is Future Value of an Annuity Due: The payments and interest together at a future point in time.

CY is Compounds Per Year or Compounding Frequency: The number of compounding periods per complete year.

N is Number of Annuity Payments: The total number of payments made during the time segment of the annuity. It results from Formula 11.1.

Annuity Due Future Value:
$$FV_{DUE} = PMT \left[\frac{(1+i)^{\frac{CY}{PY}N} - 1}{(1+i)^{\frac{CY}{PY}} - 1} \right] \times (1+i)^{\frac{CY}{PY}}$$

PMT is Annuity Payment Amount: The amount of money that is invested or paid at the beginning of each payment interval.

PY is Payments Per Year or Payment Frequency: The number of payment intervals per complete year.

i is Periodic Interest Rate: The rate of interest used in converting the interest to principal according to Formula 9.1. The interest rate compounding period must always match the payment interval, which is ensured by the $\frac{CY}{PY}$ exponent.



How It Works

The steps required to solve for the future value of an annuity due are almost identical to those you use for the ordinary annuity. The only difference lies in step 5, where you use Formula 11.3 instead of Formula 11.2. Examples 11.2D and 11.2E illustrate the adaptation.



Important Notes

To adapt your calculator to an annuity due, you must toggle the payment timing setting from END to BGN. The calculator default is END, which is the ordinary annuity. The payment timing setting is found on the second shelf above the PMT key (because it is related to the PMT!). To toggle the setting, complete the following sequence:

1. 2nd BGN (the current payment timing of END or BGN is displayed)

2. 2nd SET (it toggles to the other setting)
3. 2nd Quit (to get out of the window)

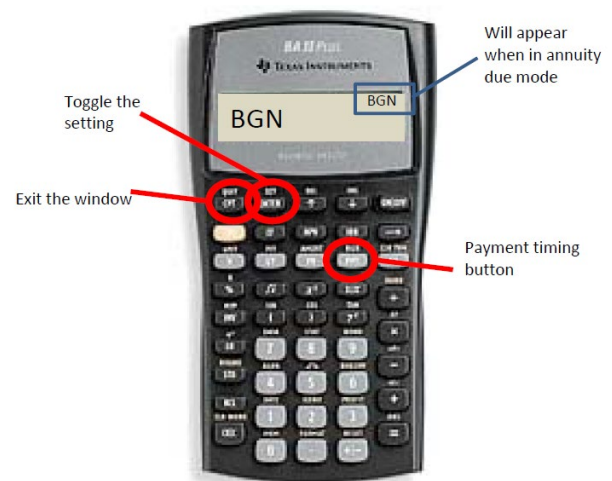
When the calculator is in annuity due mode, a tiny BGN appears in the upper right-hand corner of your calculator. To return the calculator to ordinary mode, repeat the above sequence.



Give It Some Thought

Under equal conditions,

1. For any investment, which will always have the higher future value: an ordinary annuity or an annuity due? Explain.
2. For any debt, which will always have a higher future value: an ordinary annuity or an annuity due? Explain.

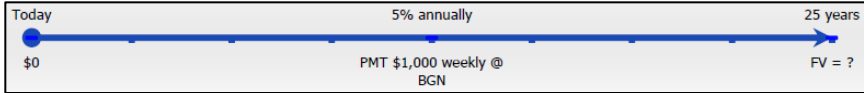
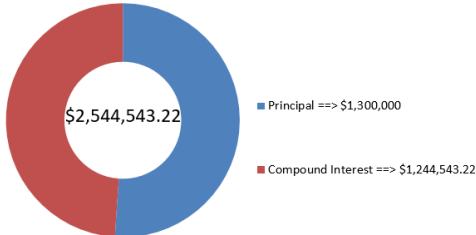


Solutions:

1. The annuity due will have the higher future value, since it always has one extra compound compared to an ordinary annuity.
2. The ordinary annuity will have the higher future value, since the principal in the first payment interval is higher and therefore more interest accrues than in the annuity due.

Example 11.2D: Lottery Winnings

An instant scratch n' win ticket offers players a chance to win \$1,000 per week for the next 25 years starting immediately upon validation. If a winner was to invest all of his money into an account earning 5% compounded annually, how much money would he have at the end of his 25-year term? Assume each year has exactly 52 weeks.

Plan	Step 1: The payments start immediately, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Calculate its value at the end, which is its future value, or FV_{DUE} .																		
Understand	What You Already Know Step 1 (continued): The timeline for the lottery savings is below. Step 2: $PV = \\$0$, $IY = 5\%$, $CY = 1$, $PMT = \\$1,000$, $PY = 52$, Years = 25 	How You Will Get There Step 3: Apply Formula 9.1. Step 4: Skip this step since there is no PV. Step 5: Apply Formula 11.1 and Formula 11.3.																	
Perform	Step 3: $i = 5\% \div 1 = 5\%$ Step 5: $N = 52 \times 25 = 1,300$ payments $FV_{DUE} = \$1,000 \left[\frac{\left[(1 + 0.05)^{\frac{1}{52}} \right]^{1300} - 1}{(1 + 0.05)^{\frac{1}{52}} - 1} \times (1 + 0.05)^{\frac{1}{52}} \right]$ $= \$2,544,543.22$	Calculator Instructions <table><tr><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>BGN</td><td>1300</td><td>5</td><td>0</td><td>-1000</td><td>Answer: 2,544,543.218</td><td>52</td><td>1</td></tr></table>		Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	BGN	1300	5	0	-1000	Answer: 2,544,543.218	52	1
Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y												
BGN	1300	5	0	-1000	Answer: 2,544,543.218	52	1												
Present	The figure shows how much principal and interest make up the final balance. If the winner was to invest all of his lottery prize money, he would have \$2,544,543.22 after 25 years. 																		

Example 11.2E: Saving into a Trust Fund with a Variable Change

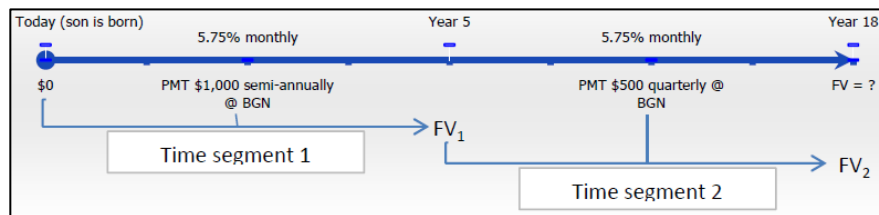
When Roberto's son was born, Roberto started making payments of \$1,000 at the beginning of every six months to a trust fund earning 5.75% compounded monthly. After five years, he changed his contributions and started depositing \$500 at the beginning of every quarter. How much money will be in his son's trust fund when his son turns 18?

Plan Step 1: There is a change of variables after five years. As a result, you need two time segments. In both segments, payments are at the beginning of the period and the compounding periods and payment intervals are different. Therefore, Roberto has two consecutive general annuities due. Combined, calculate the future value, or FV_{DUE} .

Understand

What You Already Know
 Step 1 (continued): The timeline for the trust fund is below.
 Step 2: Time segment 1: $PV = \$0$, $IY = 5.75\%$, $CY = 12$, $PMT = \$1,000$, $PY = 2$, Years = 5
 Time segment 2: $PV = FV_1$ of time segment 1, $IY = 5.75\%$, $CY = 12$, $PMT = \$500$, $PY = 4$, Years = 13

How You Will Get There
 For each time segment, repeat the following steps:
 Step 3: Apply Formula 9.1.
 Step 4: If there is a PV, apply Formula 9.2 and Formula 9.3.
 Step 5: Apply Formula 11.1 and Formula 11.3.
 The total future value in any time segment is the sum of the answers to step 4 (FV) and step 5 (FV_{ORD}).



Perform

Step 3: $i = 5.75\% \div 12 = 0.4719\bar{6}\%$
 Step 4: No PV, so skip this step.
 Step 5: $N = 2 \times 5 = 10$ payments

$$FV_{DUE1} = \$1,000 \left[\frac{\left[(1 + 0.004719\bar{6})^{\frac{12}{2}} \right]^{10} - 1}{(1 + 0.004719\bar{6})^{\frac{12}{2}} - 1} \times (1 + 0.004719\bar{6})^{\frac{12}{2}} \right] = \$11,748.47466 = FV_1$$

For the second time segment:
 Step 3: i remains unchanged = $0.4719\bar{6}\%$
 Step 4: $N = 12 \times 13 = 156$ compounds, $FV_2 = \$11,748.47466(1 + 0.004719\bar{6})^{156} = \$24,765.17$
 Step 5: $N = 4 \times 13 = 52$ payments

$$FV_{DUE2} = \$500 \left[\frac{\left[(1 + 0.004719\bar{6})^{\frac{12}{4}} \right]^{52} - 1}{(1 + 0.004719\bar{6})^{\frac{12}{4}} - 1} \times (1 + 0.004719\bar{6})^{\frac{12}{4}} \right] = \$38,907.21529$$

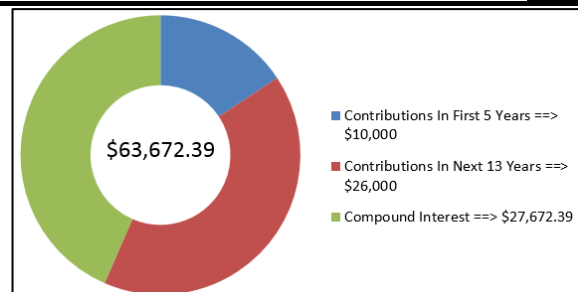
Total $FV_2 = \$24,765.17 + \$38,907.21529 = \$63,672.39$

Calculator Instructions

Time Segment	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	BGN	10	5.75	0	-1000	Answer: 11,748.47466	2	12
2	✓	52	✓	-11,748.47466	-500	Answer: 63,672.38529	4	12

Present

The figure shows how much principal and interest make up the final balance. When Roberto's son turns 18, the trust fund will have a balance of \$63,672.39.





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11.3: Present Value of Annuities

(How Much Do I Need Now?)

Have you ever noticed that the prices of expensive products are generally not advertised? Instead, companies that market these high-ticket products promote the payment amounts of the annuity, not the actual sticker price. In fact, locating the actual price for these products in the advertisements usually requires a magnifying glass. For example, a sport utility vehicle was recently advertised for only \$193 biweekly. That does not sound so bad, and perhaps you might be tempted to head on down to the car dealership to buy one of those vehicles. However, the fine print indicates that you need to make 182 payments, which then total almost \$32,000. Why is the vehicle advertised this way? Numerically, \$193 sounds a lot better than \$32,000!

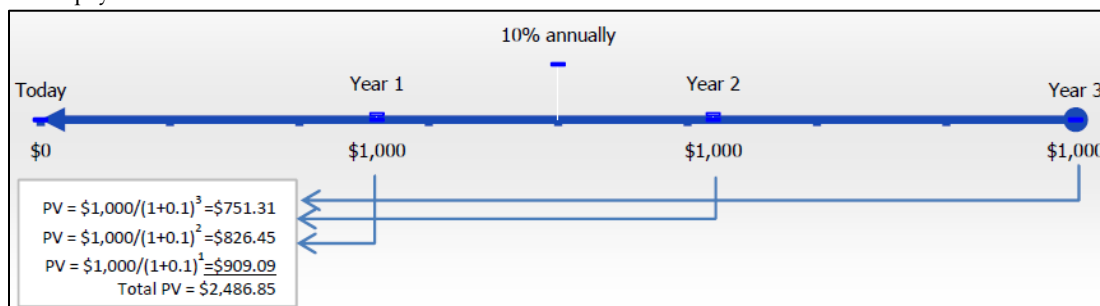
In business, whether you are setting up consumers with payment plans or buying and selling loan contracts, you need to calculate present values. As a consumer, you encounter present value calculations in many ways:

- How to take advertised payment amounts and convert them to the actual price that you must pay.
- Meeting financial goals by planning your RRSP, which requires knowing how much money you need at the start.

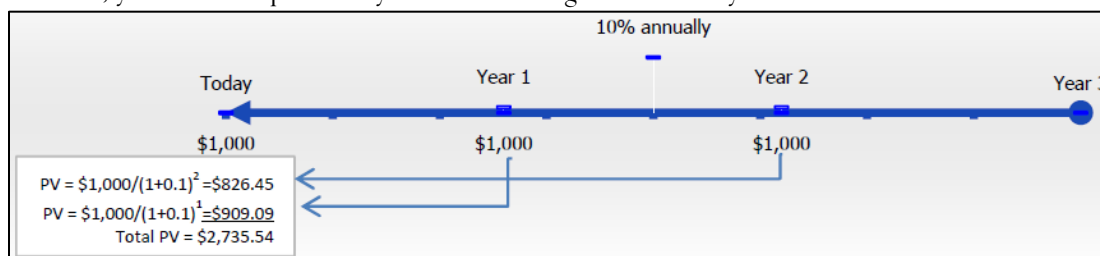
This section develops present value formulas for both ordinary annuities and annuities due. Like future value calculations, these formulas accommodate both simple and general annuities as needed. From investments, we will then extend annuity calculations to loans as well.

Ordinary Annuities and Annuities Due

The **present value of any annuity** is equal to the sum of all of the present values of all of the annuity payments when they are moved to the beginning of the first payment interval. For example, assume you will receive \$1,000 annual payments at the end of every payment interval for the next three years from an investment earning 10% compounded annually. How much money needs to be in the annuity at the start to make this happen? In this case, you have an ordinary simple annuity. The figure below illustrates the fundamental concept of the time value of money and shows the calculations in moving all of the payments to the focal date at the start of the timeline.

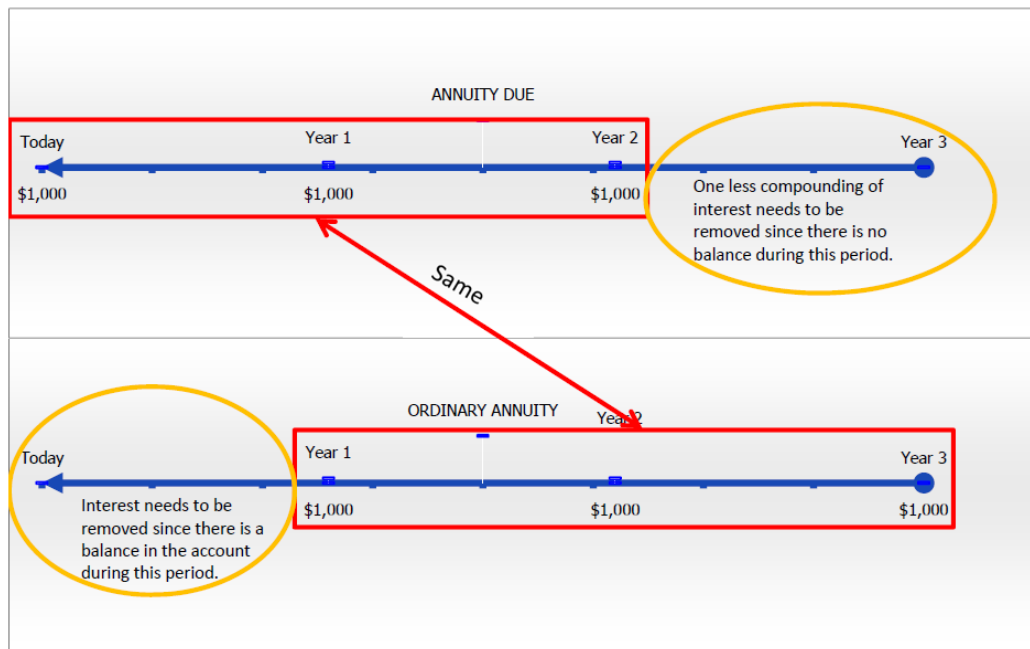


Observe that all three payments are present valued to your focal date, requiring an investment of \$2,486.85 today. In contrast, what happens to your timeline and calculations if those payments are made at the beginning of every payment interval? In this case, you have a simple annuity due. The next figure illustrates your timeline and calculations.



Observe that only two of the three payments need to be present valued to your focal date since the first payment is already on the focal date. The total investment for an annuity due is higher at \$2,735.54 because the first payment is withdrawn immediately, so a smaller principal earns less interest than does the ordinary annuity.

The next figure below contrasts the two annuity types. Working from right to left on the timeline, the key difference is that the annuity due has one less compound of interest to remove. Its first time segment (from the right) contains a zero balance, while the ordinary annuity does contain a balance that needs to have interest discounted. Note that if you take the annuity due and remove one extra compound of interest, you arrive at $\$2,735.54 \div (1 + 0.1) = \$2,486.85$, which is the present value of the corresponding ordinary annuity.



The Formula

As with future value calculations, calculating present values by manually moving each payment to its present value is extremely time consuming when there are more than a few payments. Similarly, annuity formulas allow you to move all payments simultaneously in a single calculation. The formulas for ordinary annuities and annuities due are presented together.

PV_{ORD} and PV_{DUE} are Present Value of an Ordinary and Due Annuity respectively: This is the amount of money in the account including all of the payments less the interest at an earlier point in time. If the earlier point is the start of the annuity, then the present value is the principal.

N is Number of Annuity Payments: The total number of payments in the annuity time segment via Formula 11.1.

Formula 11.4 - Ordinary Annuity Present Value:

$$PMT \left[\frac{1 - \left[\frac{1}{(1 + i)^{\frac{CY}{PY}}} \right]^N}{(1 + i)^{\frac{CY}{PY}} - 1} \right] = PV_{ORD}$$

Formula 11.5 - Annuity Due Present Value:

$$PV_{DUE} = PMT \left[\frac{1 - \left[\frac{1}{(1 + i)^{\frac{CY}{PY}}} \right]^N}{(1 + i)^{\frac{CY}{PY}} - 1} \right] \times (1 + i)^{\frac{CY}{PY}}$$

PMT is Annuity Payment Amount: The amount of money that is invested or paid at each payment interval.

i is Periodic Interest Rate: The interest rate used in converting the interest to principal via Formula 9.1. The interest rate compounding period must always match the payment interval, which is ensured by the $\frac{CY}{PY}$ exponent.

CY and PY are Compound / Payment Frequency respectively: The number of compounding periods and payments respectively per complete year.

The following observations are made about these two formulas:

1. The formulas adapt to both simple and general annuities. In the case of simple annuities, the compounding frequency already matches the payment frequency, so it requires no conversion; numerically, the exponent of $\frac{CY}{PY}$ produces a quotient of 1 and removes itself from the formula. In the case of general annuities, the exponent converts the compounding frequency of the interest rate to match the payment frequency.
2. Identical in structure to the future value formulas, these two formulas for present value are also complex equations involving rate, portion, and base. Here the present value (PV_{ORD} or PV_{DUE}) = portion, PMT = base, and everything else = rate. The numerator, $1 - \left[\frac{1}{(1+i)^{\frac{CY}{PY}}} \right]^N$, produces the overall percent decrease in the annuity; the denominator, $(1+i)^{\frac{CY}{PY}} - 1$, produces the percent change with each payment; the division of these two percent changes creates a ratio by which present value relates to the annuity payment itself. To illustrate, assume that $i = 5\%$, $N = 2$, and $CY = PY = 1$. Substituting these numbers into the formula gives the following:

$$\frac{1 - \left[\frac{1}{(1+i)^{\frac{CY}{PY}}} \right]^N}{(1+i)^{\frac{CY}{PY}} - 1} = \frac{1 - \left[\frac{1}{(1+.05)^1} \right]^2}{(1+.05)^1 - 1} = \frac{0.092970}{0.05} = 1.859410$$

From these numbers, it is evident that the present value amount represents a decrease of 9.2970% (the numerator) overall. With each payment, the decrease is 5% (the denominator). Therefore, the ratio of the present value overall to each payment (the PMT) is 1.859410. If the annuity payment amount is $PMT = \$1,000$, then the value of the $PV_{ORD} = \$1,000(1.859410) = \$1,859.41$.

3. The only difference from an ordinary annuity is the multiplication of the extra term $(1+i)^{\frac{CY}{PY}}$. The multiplication results in the removal of one fewer compounds of interest than the ordinary annuity.



How It Works

Follow these steps to calculate the present value of any ordinary annuity or annuity due:

Step 1: Identify the annuity type. Draw a timeline to visualize the question.

Step 2: Identify the variables that you know, including FV , iy , CY , PMT , PY , and Years.

Step 3: Use Formula 9.1 to calculate i .

Step 4: If $FV = \$0$, proceed to step 5. If there is a nonzero value for FV , treat it like a single payment. Apply Formula 9.2 to determine N since this is not an annuity calculation. Move the future value to the beginning of the time segment using Formula 9.3, rearranging for PV .

Step 5: Use Formula 11.1 to calculate N . Apply either Formula 11.4 or Formula 11.5 based on the annuity type. If you calculated a present value in step 4, combine the present values from steps 4 and 5 to arrive at the total present value.



Important Notes

Calculating the Interest Amount. If you are interested in knowing how much interest was removed in the calculation of the present value, adapt Formula 8.3, where $I = S - P = FV - PV$. The present value (PV) is the solution to either Formula 11.4 or Formula 11.5. The FV in Formula 8.3 is expanded to include the sum of all future monies, so it is replaced by $N \times PMT + FV$. Therefore, you rewrite Formula 8.3 as $I = (N \times PMT + FV) - PV$.

Your BAII+ Calculator. If a single payment future value (FV) is involved in a present value calculation, then you require two formula calculations using Formula 9.3 and either Formula 11.4 or 11.5. The calculator performs both of these calculations simultaneously if you input values obeying the cash flow sign convention for both FV and PMT .



Give It Some Thought

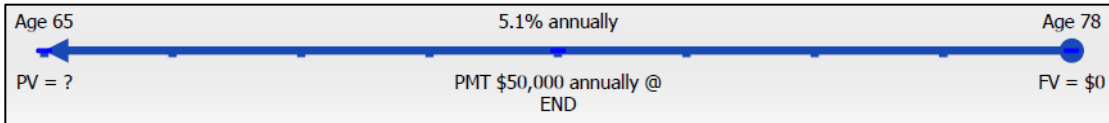
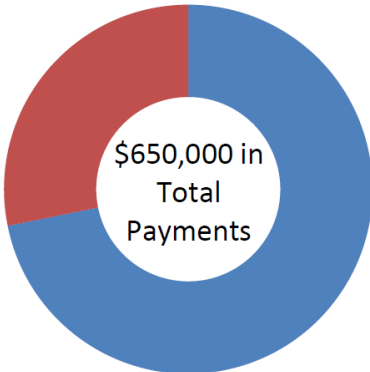
For two equal investment annuities, will the present value of an ordinary annuity and annuity due be the same or different?

Solution:

They will be different. The annuity due always has the larger present value since it removes one fewer compound of interest than the ordinary annuity.

Example 11.3A: Amount Needed at Time of Retirement

Rodriguez is planning on having an annual gross income of \$50,000 at the end of every year when he retires at age 65. He is planning for the account to be emptied by age 78, which is the average life expectancy for a Canadian man. If the account earns 5.1% compounded annually, what amount of funds needs to be in the account when he retires?

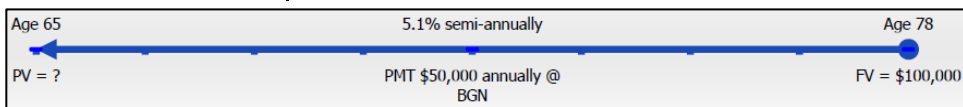
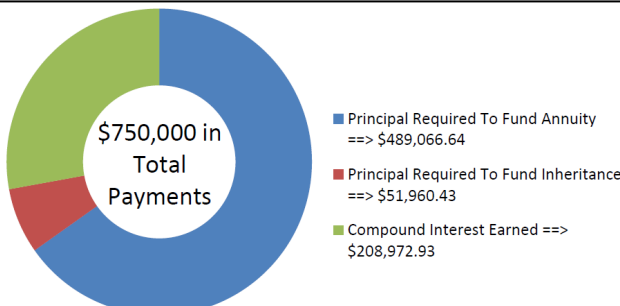
Plan	Step 1: The payments are at the end of the payment intervals, and the compounding period and payment intervals are the same. This is, therefore, an ordinary simple annuity. Calculate its value at the start, which is its present value, or PV_{ORD} .																
Understand	What You Already Know Step 1 (continued): The timeline for the client's account appears below. Step 2: $FV = \\$0$, $IY = 5.1\%$, $CY = 1$, $PMT = \\$50,000$, $PY = 1$, Years = 13	How You Will Get There Step 3: Apply Formula 9.1. Step 4: Since $FV = \\$0$, skip this step. Step 5: Apply Formula 11.1 and Formula 11.4.															
Perform	Step 3: $i = 5.1\% \div 1 = 5.1\%$ Step 5: $N = 1 \times 13 = 13$ payments $PV_{ORD} = \$50,000 \frac{1 - \left[\frac{1}{(1 + 0.051)^{\frac{1}{1}}} \right]^{13}}{(1 + 0.051)^{\frac{1}{1}} - 1} = \$466,863.69$	Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>13</td><td>5.1</td><td>Answer: -466,863.694</td><td>50000</td><td>0</td><td>1</td><td>1</td></tr></table>	N	I/Y	PV	PMT	FV	P/Y	C/Y	13	5.1	Answer: -466,863.694	50000	0	1	1	
N	I/Y	PV	PMT	FV	P/Y	C/Y											
13	5.1	Answer: -466,863.694	50000	0	1	1											
Present	The figure shows how much principal and interest make up the payments. Rodriguez will need to have \$466,863.69 in his account when he turns 65 if he wants to receive 13 years of \$50,000 payments.  \$650,000 in Total Payments ■ Principal (Required At Age 65) ==> \$466,863.69 ■ Compound Interest Earned (Until Age 78) ==> \$181,136.31																

Example 11.3B: Leaving an Inheritance

Recalculate Example 11.3A applying three changes:

1. Rodriguez wants to leave a \$100,000 inheritance for his children (assuming he dies at age 78).
2. Payments are at the beginning of the year.
3. His interest rate is 5.1% compounded semi-annually.

Calculate the present value and the amount of interest.

Plan	Step 1: The payments are made at the beginning of the payment intervals, and the compounding period (semi-annually) and payment intervals (annually) are different. This is now a general annuity due. Calculate its value at the start, which is its present value, or PV_{DUE} .																			
Understand	What You Already Know Step 1 (continued): The timeline for the client's account appears below. Step 2: $FV = \$100,000$, $IY = 5.1\%$, $CY = 2$, $PMT = \$50,000$, $PY = 1$, $Years = 13$	How You Will Get There Step 3: Apply Formula 9.1. Step 4: Since there is a future value, apply Formula 9.2 and Formula 9.3. Step 5: Apply Formula 11.1 and Formula 11.5. Step 6: To calculate the interest, apply and adapt Formula 8.3, where $FV = N \times PMT + FV$ and $I = FV - PV$																		
																				
Perform	Step 3: $i = 5.1\% \div 2 = 2.55\%$ Step 4: $N = 2 \times 13 = 26$ compounds $\$100,000 = PV(1 + 0.0255)^{26}$ $PV = \$100,000 \div 1.0255^{26} = \$51,960.42776$ Step 5: $N = 1 \times 13 = 13$ payments $PV_{DUE} = \$50,000 \left[\frac{1 - \left[\frac{1}{(1 + 0.0255)^{\frac{2}{1}}} \right]^{13}}{(1 + 0.0255)^{\frac{2}{1}} - 1} \right] \times (1 + 0.0255)^{\frac{2}{1}}$ $= \$489,066.6372$ $PV = \$51,960.42776 + \$489,066.6372 = \$541,027.07$ Step 6: $I = (13 \times \$50,000 + \$100,000) - \$541,027.07$ $I = \$750,000 - \$541,027.07 = \$208,972.93$		Calculator Instructions <table border="1" data-bbox="985 844 1453 909"><tr><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th></tr><tr><td>BGN</td><td>13</td><td>5.1</td><td>Answer: -541,027.065</td></tr></table> <table border="1" data-bbox="985 940 1323 1008"><tr><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>50000</td><td>100000</td><td>1</td><td>2</td></tr></table>		Mode	N	I/Y	PV	BGN	13	5.1	Answer: -541,027.065	PMT	FV	P/Y	C/Y	50000	100000	1	2
	Mode	N	I/Y	PV																
BGN	13	5.1	Answer: -541,027.065																	
PMT	FV	P/Y	C/Y																	
50000	100000	1	2																	
Present	The figure shows how much principal and interest make up the payments. Rodriguez will require more money, needing to have \$541,027.07 in his account when he turns 65 if he wants to receive 13 years of \$50,000 payments while leaving a \$100,000 inheritance for his children. His account will earn \$208,972.93 over the time frame. 																			

**Important Notes**

If any of the variables, including IY , CY , PMT , or PY , change between the start and end point of the annuity, or if any additional single payment deposit or withdrawal is made, this creates a new time segment that you must treat separately. There will then be multiple time segments that require you to work from right to left by repeating steps 3 through 5 in the procedure. Remember that the present value at the beginning of one time segment becomes the future value at the end of the next time segment. Example 11.3C illustrates this concept.

Example 11.3C: Adjusting for Inflation

Continuing with the previous two examples, Rodriguez realizes that during his retirement he needs to make some type of adjustment to his annual gross income to account for the rising cost of living. Consequently, he will take \$50,000 at the beginning of each year for six years, then increase it to \$60,000 for the balance. Assume his interest rate is still 5.1% semi-annually and that he still wants to leave a \$100,000 inheritance for his children. How much money needs to be in his retirement fund at age 65?

Plan

Step 1: There is a change of variables after six years. As a result, you need two time segments. In both segments, payments are made at the beginning of the period, and the compounding periods and payment intervals are different. These are two consecutive general annuities due. You need to calculate the resulting present value, or PV_{DUE} .

Understand**What You Already Know**

Step 1 (continued): The timeline for the client's account appears below.

Step 2: Time segment 1 (starting from the right side): $FV = \$100,000$, $IY = 5.1\%$, $CY = 2$, $PMT = \$60,000$, $PY = 1$, Years = 7
 Time segment 2: $FV = PV_1$ of time segment 1, $IY = 5.1\%$, $CY = 2$, $PMT = \$50,000$, $PY = 1$, Years = 6

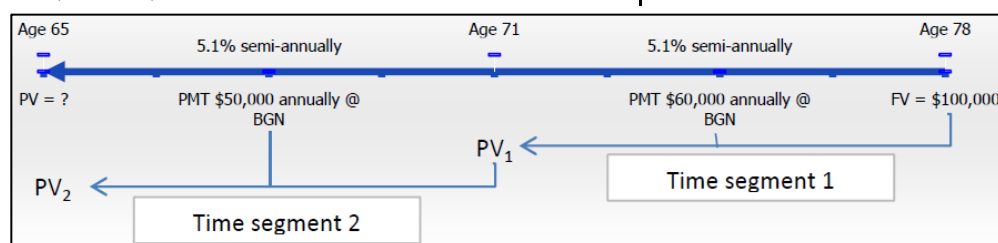
How You Will Get There

For each time segment repeat the following steps:

Step 3: Apply Formula 9.1.

Step 4: Apply Formula 9.2 and Formula 9.3.

Step 5: Apply Formula 11.1 and Formula 11.5.

**Perform**

For the first time segment (starting at the right side):

Step 3: $i = 5.1\% \div 2 = 2.55\%$

Step 4: $N = 2 \times 7 = 14$ compounds, $\$100,000 = PV(1 + 0.0255)^{14}$ $PV_1 = \$100,000 \div 1.0255^{14} = \$70,291.15736$

Step 5: $N = 1 \times 7 = 7$ payments

$$PV_{DUE1} = \$60,000 \left[\frac{1 - \left[\frac{1}{(1 + 0.0255)^{\frac{2}{1}}} \right]^7}{(1 + 0.0255)^{\frac{2}{1}} - 1} \right] \times (1 + 0.0255)^{\frac{2}{1}} = \$362,940.8778$$

Total $PV_1 = \$70,291.15736 + \$362,940.8778 = \$433,232.0352$

For the second time segment:

Step 3: i remains the same = 2.55%

Step 4: $N = 2 \times 6 = 12$ compounds, $\$433,232.0352 = PV(1 + 0.0255)^{12}$

$$PV_2 = \$433,232.0352 \div 1.0255^{12} = \$320,252.5426$$

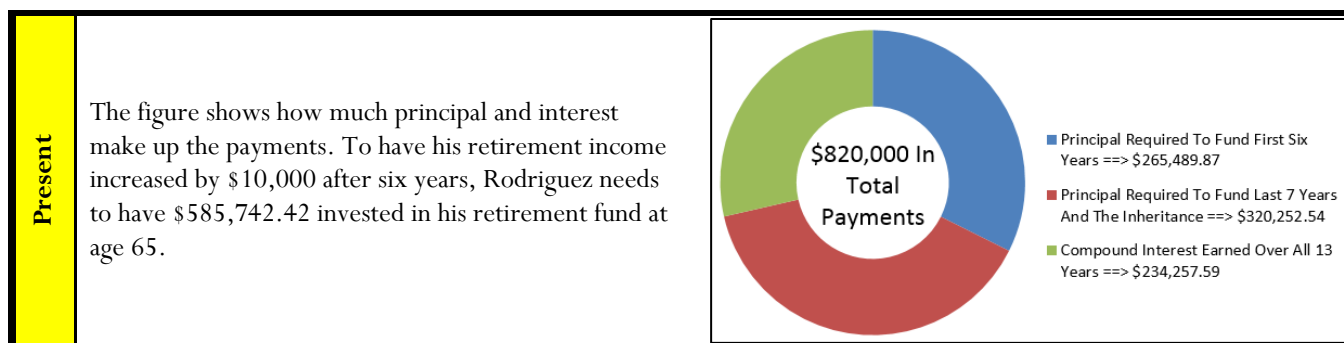
Step 5: $N = 1 \times 6 = 6$ payments

$$PV_{DUE2} = \$50,000 \left[\frac{1 - \left[\frac{1}{(1 + 0.0255)^{\frac{2}{1}}} \right]^6}{(1 + 0.0255)^{\frac{2}{1}} - 1} \right] \times (1 + 0.0255)^{\frac{2}{1}} = \$265,489.8749$$

Total $PV_2 = \$320,252.5426 + \$265,489.8749 = \$585,742.42$

Calculator Instructions

Time Segment	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	BGN	7	5.1	Answer: -433,232.0352	60000	100000	1	2
2	✓	6	✓	Answer: -585,742.4175	50000	433,232.0352	✓	✓



Working with Loans

At the start of this chapter, you purchased your first home and started your \$150,000 mortgage at 5% compounded semi-annually. Assume your mortgage has a five-year term. Before that term expires, you have to start shopping around at different financial institutions to find the best rate for your next mortgage term. These other financial institutions will have one burning question: How much do you still owe on your mortgage and, thus, how much do you need to borrow from them?

Up to this point, this chapter has addressed only the concept of investment annuities. But what about debts? All annuity concepts also apply to the borrowing of money. When you work with loans, both future value and present value calculations may be required, which is why this topic has been delayed to this point.

As a consumer, you are probably most interested in the balance owing on any of your debts at any given point. Today's technology has made it easy to know your current balance by visiting your online bank account; however, the bank account does not assist you in identifying your future balance at a given point in time. To figure this out, you need annuity calculations.

Similarly, businesses apply annuity calculations all the time. Ongoing financial reporting has to be accurate. To provide insight into the company's true financial health, balance sheets need to reflect not only monies payable or receivable today, but also all future cash flows such as those arising from annuities. The purchase and sale of business contracts, such as the sale of a consumer payment plan to a financial institution, requires working with future payments and discounting those payments to the contract's date of sale.

In this section, you will calculate loan balances at any given point in time throughout the loan's term. As well, you will explore how to buy and sell loan contracts.

The Balance Owing on Any Loan Contract

To determine accurately the balance owing on any loan at any point in time, always start with the loan's starting principal and then deduct the payments made. This means a future value calculation using the loan's interest rate.

Some may ask why they can't figure out the loan balance by starting at the end of the loan (where the future value is zero, since no balance remains) and calculating a present value of the outstanding payments? The answer is because the annuity payment PMT is almost always a rounded number (this characteristic is explored in greater depth in Section 11.4, when you learn how to calculate the annuity payment). Every payment therefore has a slight discrepancy from its true value, which accumulates with each subsequent payment. For example, assume you calculate your payment mathematically as \$500.0045. Since you can't pay more than two decimals, your actual payments are \$500.00. The \$0.0045 is dropped, which means every payment you make is \$0.0045 short of what is mathematically required. If you make 100 such payments, the \$0.0045 error accumulates to \$0.45 plus interest. This means your very last payment must be increased by \$0.45 or more to pay off your loan. Thus, the last payment is a different amount than every other payment.

To complicate matters further, the last payment amount may be unknown and incalculable, particularly if interest rates are variable. You can't calculate a present value from an unknown number nor can you use an annuity formula where a payment is in a different amount. Chapter 13 provides much more detail about these concepts of loan payments, loan balances, and final payment differences. For now, you can conclude that an accurate calculation of a loan balance is achieved through a future value annuity formula.



The Formula

Loans are most commonly ordinary annuities requiring the application of Formula 11.2 (ordinary annuity future value) to calculate the future balance, FV_{ORD} . This is the basic assumption in performing loan calculations unless otherwise specified. In the rare instance of a loan structured as an annuity due, you apply Formula 11.3 (annuity due future value) to calculate the future value, FV_{DUE} .

Calculating the total amount of interest paid on a loan (in whole or for any time segment) once again requires the adaptation of Formula 8.3 ($I = S - P = FV - PV$), where:

- The future value (FV) term in the formula represents the total principal and interest combined. In loan annuities, the annuity payment incorporates both of these elements. As well, any future principal remaining at the end of the loan, or a future balance outstanding, must also be factored into the calculation. Hence, the FV at any time interval in the formula is expanded to include both of these elements and replaced by $N \times PMT + FV$.
- The present value (PV) in the formula is what you started with. It is the opening amount of the loan.

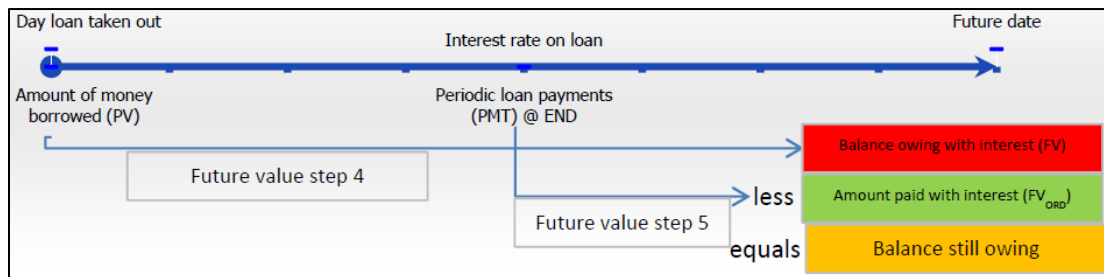
Therefore, the adaptation of Formula 8.3 remains the same as previously discussed and is written as:

$$I = (N \times PMT + FV) - PV.$$



How It Works

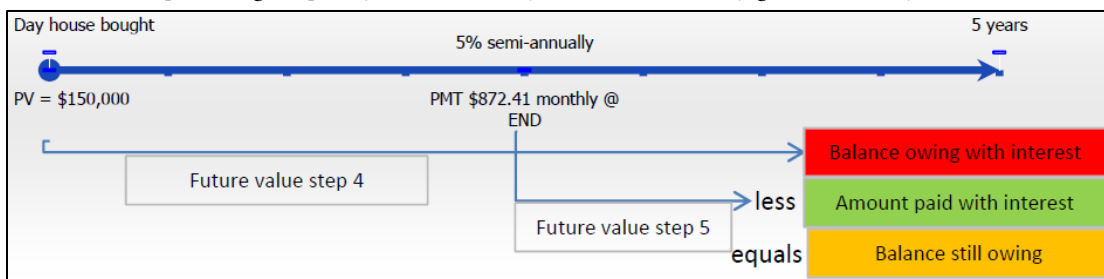
Solving for a future loan balance is a future value annuity calculation. Therefore, you use the same steps as discussed in Section 11.2. However, you need to modify your interpretation of these steps for loan balances. The figure below helps you understand these differences.



- The principal of the loan forms the present value, or PV. In step 4, when you move the present value to the future, your answer (FV) represents the total amount owing on the loan with interest as if no payments had been made.
- In step 5, the future value of the annuity (FV_{ORD}) represents the total amount paid against the loan with interest. With both the FV and FV_{ORD} on the same focal date, the fundamental concept of the time value of money allows you to then take the FV and subtract the FV_{ORD} to produce the balance owing on the loan.

Let's calculate what you still owe on your \$150,000 new home after five years of making \$872.41 monthly payments at 5% compounded semi-annually:

Step 1: The structure of your mortgage is depicted in Figure 11.32. Since payments are at the end of the period, and the payment interval and compounding frequency are different, you have an ordinary general annuity.



Step 2: The known variables are $PV = \$150,000$, $iy = 5\%$, $CY = 2$, $PMT = \$872.41$, $PY = 12$, and $Years = 5$.

Step 3: The periodic interest rate is $i = 5\% \div 2 = 2.5\%$.

Step 4: $N = 2 \times 5 = 10$ compounds, $FV = \$150,000(1 + 0.025)^{10} = \$192,012.6816$

Step 5: $N = 12 \times 5 = 60$ payments

$$FV_{ORD} = \$872.41 \left[\frac{\left[(1 + 0.025)^{\frac{2}{12}} \right]^{60} - 1}{(1 + 0.025)^{\frac{2}{12}} - 1} \right] = \$59,251.59215$$

Therefore, if the unpaid loan is worth \$192,012.6816 and your payments are worth \$59,251.59215, then your balance still owing is \$192,012.6816 - \$59,251.59215 = \$132,761.09. The amount of interest paid is $I = (60 \times \$872.41 + \$132,761.09) - \$150,000.00 = \$35,105.69$. With payments totalling $\$872.41 \times 60 = \$52,344.60$, that means only \$17,238.91 actually went towards the principal!



Important Notes

Present Value Method of Arriving at a Balance Owning. In the rare circumstance where the final payment is *exactly* equal to all other annuity payments, you can arrive at the balance owing through a present value annuity calculation. However, this strict condition *must* be met. In this instance, since you are starting at the end of the loan, the future value is always zero, so to bring all payments back to the focal date you only need Formula 11.4.

Important Notes: Your BAI+ Calculator. Proper application of the cash flow sign convention for the present value and annuity payment will automatically result in a future value that nets out the loan principal and the payments. Assuming you are the borrower, you enter the present value (PV) as a positive number since you are receiving the money. You enter the annuity payment (PMT) as a negative number since you are paying the money. When you calculate the future value (FV), it displays a negative number, indicating that it is a balance owing.



Give It Some Thought

On any interest-bearing loan at any point in time, will the principal be reduced by an amount equal to the payments made, more than the payments made, or less than the payments made?

Solution:

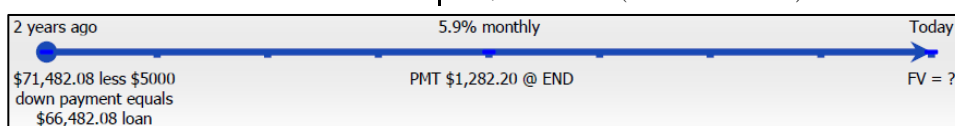
The principal will be reduced by an amount less than the payments. A portion of the payments always goes toward the interest that is being charged on the loan.

Example 11.3D: Balance Owning on a New Truck

Two years ago, Jillian purchased a new pickup truck for \$71,482.08 with a \$5,000 down payment and the remainder financed through her dealership at 5.9% compounded monthly. She has been making monthly payments of \$1,282.20. What is her balance owing today? How much interest has she paid to date?

Plan Step 1: The payments are made at the end of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is a simple ordinary annuity. Calculate its value two years after its start, which is its future value, or FV_{ORD} . Once you know the FV_{ORD} , you can determine the amount of interest, or I .

Understand What You Already Know Step 1 (continued): The timeline for the savings annuity appears below. Step 2: $PV = \\$71,482.08 - \\$5,000 = \\$66,482.08$, $IY = 5.9\%$, $CY = 12$, $PMT = \\$1,282.20$, $PY = 12$, $Years = 2$	How You Will Get There Step 3: Apply Formula 9.1. Step 4: Apply Formula 9.2 and Formula 9.3. Step 5: Apply Formula 11.1 and Formula 11.2. The final future value is the difference between the answers to step 4 and step 5. Step 6: To calculate the interest, apply the adapted Formula 8.3, where $I = (N \times PMT + FV) - PV$.



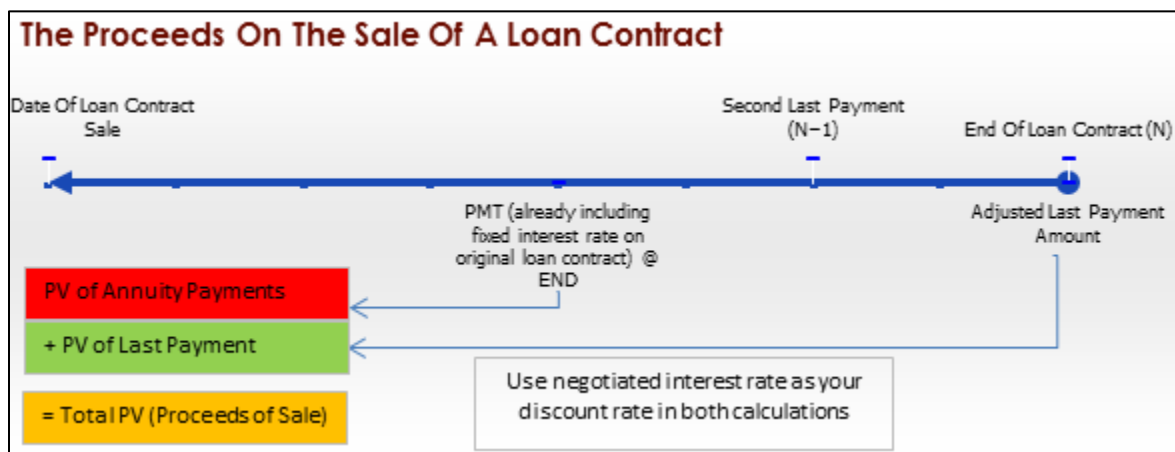
Perform	Step 3: $i = 5.9\% \div 12 = 0.491\overline{6}\%$ Step 4: $N = 12 \times 2 = 24$ compounds $FV = \\$66,482.08(1 + 0.00491\overline{6})^{24} = \\$74,786.94231$ Step 5: $N = 12 \times 2 = 24$ payments $FV_{ORD} = \$1,282.20 \left[\frac{\left[(1 + 0.00491\overline{6})^{\frac{12}{12}} \right]^{24} - 1}{(1 + 0.00491\overline{6})^{\frac{12}{12}} - 1} \right] = \$32,577.13179$ $FV = \\$74,786.94231 - \\$32,577.13179 = \\$42,209.81$ Step 6: $I = (24 \times \\$1,282.20 + \\$42,209.81) - \\$66,482.08$ $= \\$72,982.61 - \\$66,482.08 = \\$6,500.53$	Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th></tr><tr><td>24</td><td>5.9</td><td>66482.08</td><td>-1282.2</td></tr></table> <table><tr><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Answer: -42,209.81052</td><td>12</td><td>12</td></tr></table>	N	I/Y	PV	PMT	24	5.9	66482.08	-1282.2	FV	P/Y	C/Y	Answer: -42,209.81052	12	12
N	I/Y	PV	PMT													
24	5.9	66482.08	-1282.2													
FV	P/Y	C/Y														
Answer: -42,209.81052	12	12														
Present	The figure shows how much principal and interest make up the payments. After two years of making monthly payments, Jillian has a balance owing on the pickup truck of \$42,209.81. Altogether, she has made \$30,772.80 in payments, of which \$6,500.53 went toward the interest on her loan.	\$30,772.80 in Total Payments - Truck Loan Interest Paid ==> \$6,500.53 - Truck Loan Principal Paid ==> \$24,272.27														

Selling a Loan Contract

It is common for loan contracts to be sold from retailers to financial institutions. For example, when a consumer makes a purchase from Sleep Country Canada on its payment plan, the financing is actually performed through its financial partner. In most retail situations, this would then mean that Sleep Country Canada receives the money right away by selling the contract to its partner, whereas the financial partner then assumes the financial responsibility of collecting the payments.

When a finance company purchases a loan contract from another organization, it is essentially investing in the future payments of the loan contract. The two companies commonly agree on a profitable interest rate for the finance company and use it to determine the amount, known as the sale price, paid by the finance company to the other organization to purchase the contract. This textbook covers only fixed interest rate calculations with known final payment amounts.

Previously, it was discussed how the last payment in a loan almost always differs from every other payment in the annuity because of the rounding discrepancy in the annuity payment amount. Thus, the selling of a loan contract needs to calculate the present value of all remaining annuity payments (except for the last one) plus the present value of the adjusted single final payment as shown in this figure.





The Formula

As evident in the figure, two calculations are required. The first involves a present value annuity calculation using Formula 11.4. Note that the annuity stops one payment short of the end of the loan contract, so you need to use $N - 1$ rather than N . The second calculation involves a present-value single payment calculation at a fixed rate using Formula 9.3 rearranged for PV. Thus, no new formulas are required to complete this calculation.



How It Works

The steps involved in selling any loan contract are almost identical to any present value annuity calculation with only minor differences as noted below.

Step 1: No changes.

Step 2: Identify the known variables, including PMT, PY, and Years, along with the newly negotiated IY and CY. Also identify the amount of the last payment, which is the FV.

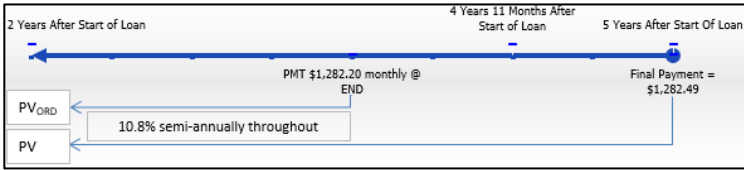
Step 3: Use Formula 9.1 to calculate i .

Step 4: The last payment is the FV, which you treat like a single payment. Apply Formula 9.2 to determine N since this step is not an annuity calculation. Move the future value to the beginning of the time segment using Formula 9.3, rearranging for PV.

Step 5: Use Formula 11.1 to calculate N and subtract 1 to remove the final payment (since it is accounted for in step 4). Apply Formula 11.4 (or Formula 11.5 if it is an annuity due) to calculate the present value. Add both of the present values from steps 4 and 5 together to arrive at the total present value, which is known as the total proceeds of the sale.

Example 11.3E: Dealership Sells the Truck Contract

Continuing with Jillian's pickup truck purchase, recall that Jillian's monthly payments are fixed at \$1,282.20 for five years. Assume that after two years the dealership wants to sell the contract to another finance company, which agrees to a discount rate of 10.8% compounded semi-annually. Jillian's final payment is known at \$1,282.49. What are the proceeds of the sale?

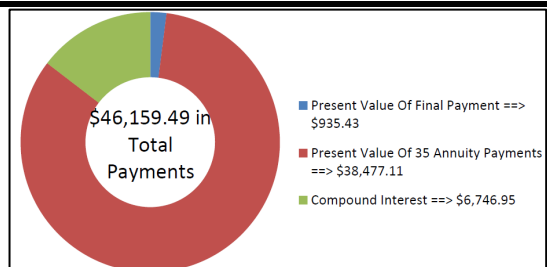
Plan	Step 1: The payments are made at the end of the payment intervals, and the compounding period (semi-annually) and payment intervals (monthly) are different. Therefore, this is an ordinary general annuity. Calculate its value on the date of sale, which is its present value, or PV_{DUE} , plus the present value of the final payment, or PV.
Understand	What You Already Know Step 1 (continued): The timeline for the sale of the loan contract appears below. Step 2: $FV = \\$1,282.49$, $IY = 10.8\%$, $CY = 2$, $PMT = \\$1,282.20$, $PY = 12$, $\text{Years} = 3$ How You Will Get There Step 3: Apply Formula 9.1. Step 4: For the final payment amount, apply Formula 9.2 and Formula 9.3, rearranging for PV. Step 5: For the annuity, apply Formula 11.1 (taking away the last payment) and Formula 11.4. 
Perform	Step 3: $i = 10.8\% \div 2 = 5.4\%$ Step 4: $N = 2 \times 3 = 6$ compounds, $\\$1,282.49 = PV(1 + 0.054)^6$ $PV = \\$1,282.49 \div 1.054^6 = \\$935.427906$ Step 5: $N = 12 \times 3 - 1 = 35$ payments $PV_{\text{ORD}} = \$1,282.20 \left[\frac{1 - \frac{1}{(1+0.054)^{35}}}{(1+0.054)^{\frac{12}{2}} - 1} \right] = \$38,477.10711$ $PV = \$935.427906 + \$38,477.10711 = \$39,412.54$

Calculator Instructions

Element	N	I/Y	PV	PMT	FV	P/Y	C/Y
Final Payment	6	10.8	Answer: -935.427906	0	1282.49	2	2
Annuity	35	✓	Answer: -38,477.10711	1282.2	0	12	2

Present

The figure shows the present value and interest amounts in the transaction. The finance company will pay \$39,412.54 for the contract. In return, it receives 35 payments of \$1,282.20 and one payment of \$1,282.49 for a nominal total of \$46,159.49.

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11.4: Annuity Payment Amounts

(What Commitment Am I Making?)

Whether you are acquiring merchandise and property or saving up toward some future goal, you will deal with annuity payments. When you graduate college and land that promising entry-level position with your employer, a lot of demands are going to get placed on your limited income. If you do not already own a place of your own, perhaps you will get one. This means the purchase of a starter home for which you will make monthly mortgage payments. To fill that home, acquiring some furniture and electronics might take you to The Brick, Sleep Country Canada, Best Buy, or The Home Depot. Then you may be staggered by all the home maintenance items you need. If you make a lot of purchases all at once, you will probably take advantage of various payment plans. These place even more demands on your monthly income. Do not forget that you will need some wheels too. You can either lease or purchase a car. Great, another payment to make! Finally, you remember what your math instructor taught you about the importance of getting started early on your RRSP, so you should begin making those monthly contributions soon, too.

Will you have enough income to cover all of your payments? Unlike the many consumers who must rely on retailers and banks to figure out their payments, your study of annuity payments will allow you to calculate the amounts yourself.

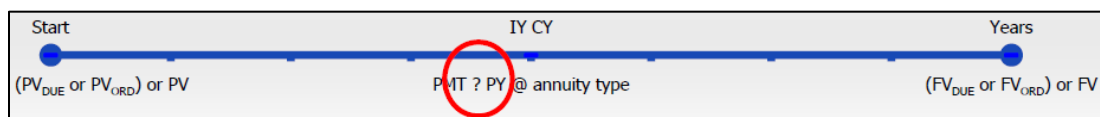
Businesses also make annuity payments for a wide variety of purposes. Whether saving up for future corporate goals or acquiring products and property, businesses have regular bills, too. Marketers develop payment plans for their consumers. Financial agents make investments involving periodic payments. Companies issue marketable bonds that require regular interest payments to investors. Human resource personnel look after employee benefits, including RRSP contributions and pension plan payments. Production departments need expensive machinery, so they must keep payment plans within operating budgets. No matter your choice of profession, as a business manager you will encounter annuity payment calculations.

Ordinary Annuities and Annuities Due

You need to calculate an annuity payment in many situations:

- Figuring out loan or mortgage payments
- Determining membership or product payment plans
- Calculating lease payments
- Determining the periodic payment necessary to achieve a savings goal
- Determining the maximum payment that an investment annuity can sustain over a period of time

A typical timeline for solving for an annuity payment amount appears in the next figure. All variables except for the payment amount (PMT) are known. While the figure shows all the possible variables and their location on the timeline, the following should be noted at each end of the timeline:



- If the PV_{DUE} or PV_{ORD} is known on the left, then FV is the only variable that may appear on the right. FV_{DUE} , FV_{ORD} , and PV are variables that will not appear on the timeline.
- If the FV_{DUE} or FV_{ORD} is known on the right, then PV is the only variable that may appear on the left. PV_{DUE} , PV_{ORD} , and FV are variables that will not appear on the timeline.



The Formula

Recall that the annuity payment amount, PMT , is one of the variables in Formulas 11.2, 11.3, 11.4, and 11.5. Calculating this amount then requires you to substitute the known variables and rearrange the formula for PMT . The most difficult part of this process is figuring out which of the four formulas to use. Your selection depends on the two factors summarized in this table and discussed afterwards.

Formula To Use	Annuity Payment Timing	Annuity Value Known
11.2	End	FV_{ORD}
11.4	End	PV_{ORD}
11.3	Beginning	FV_{DUE}
11.5	Beginning	PV_{DUE}

1. Are the annuity payments to be made at the beginning or at the end of the payment interval? In other words, do you have an annuity due or an ordinary annuity?
2. Do you know the amount that the annuity starts or ends with? In other words, do you know the present value or future value of the annuity?



How It Works

Follow these steps to solve for any annuity payment amount:

Step 1: Identify the annuity type. Draw a timeline to visualize the question.

Step 2: Identify the variables that you know, including IY, CY, PY, and Years. You must also identify a value for one of PV_{ORD} , PV_{DUE} , FV_{ORD} , or FV_{DUE} . You may or may not have a value for FV or PV.

Step 3: Use Formula 9.1 to calculate i .

Step 4: If a single payment PV or FV is known, move it to the other end of the time segment using Formula 9.3. To determine N , apply Formula 9.2 since this is for a single payment, not an annuity. When you move the amount to the same focal date as the present or future value of the annuity, either add this number to the annuity value or subtract it as the situation demands. Example 11.4C later in this section will illustrate this practice.

Step 5: Use Formula 11.1 to calculate N . Apply the correct annuity payment formula that matches your annuity type and known present or future value. Select from Formulas 11.2, 11.3, 11.4, or 11.5 then rearrange for the annuity payment amount, PMT. If you performed step 4 above, be sure to use the adjusted future or present value of your annuity in the formula.



Important Notes

When calculating loan payments, recall that the last payment is slightly different from all other payments. Chapter 13 explores how to calculate the last payment precisely. For the purposes of this section, treat the last payment like any other payment and assume it is equal to all the other payments when you make any statement about loan payment amounts or totals.



Things To Watch Out For

Should you grow or shrink a balance? Give some thought to the relationships between the present value, annuity payments, and future value.

1. If somebody is contributing to an investment, the future value of the annuity should be larger than the total of all annuity payments.
2. If somebody is receiving from an investment or making debt payments, the total of all annuity payments should exceed the present value.



Give It Some Thought

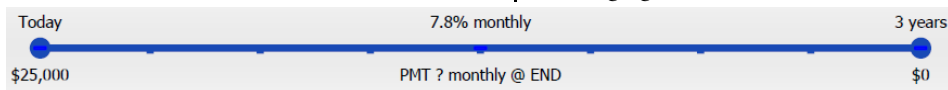
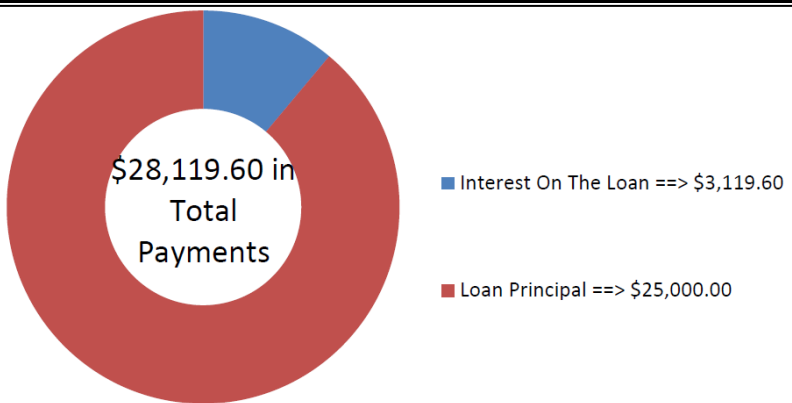
1. An investor has been making \$1,000 annual contributions to his account for five years. In this problem, should the future value be more than, less than, or equal to \$5,000?
2. A debtor needs to make \$1,000 annual payments on her loan for five years to extinguish her debt. In this problem, should the present value be more than, less than, or equal to \$5,000?

3. A retirement fund can make five annual payments of \$1,000 before being extinguished. In this problem, should the present value be more than, less than, or equal to \$5,000?

Solutions:
 1. The future value will be more than \$5,000. The payments earn interest.
 2. The present value will be less than \$5,000. The payments represent the principal and interest together.
 3. The present value will be less than \$5,000. The principal earns interest while paying out.

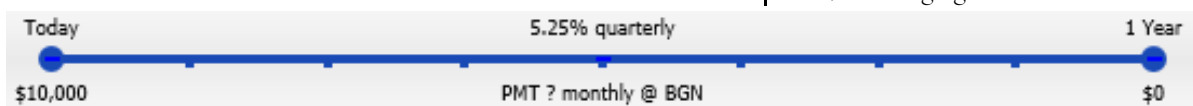
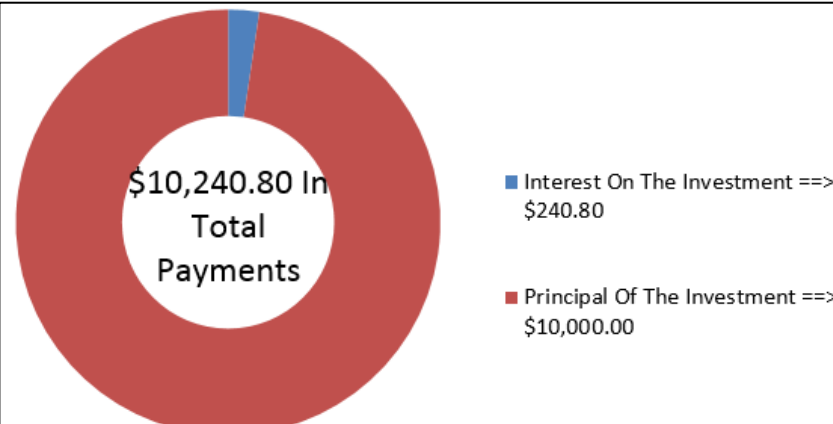
Example 11.4A: Payments on a Loan

Morgan wants to consolidate a lot of smaller debts into a single three-year loan for \$25,000. If the loan is charged interest at 7.8% compounded monthly, what is her payment amount at the end of every month?

Plan	Step 1: The payments are made at the end of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is an ordinary simple annuity. Calculate the monthly amount of her loan payment, or PMT.															
Understand	What You Already Know Step 1 (continued): The timeline for the loan appears below. Step 2: $PV_{ORD} = \\$25,000$, $IY = 7.8\%$, $CY = 12$, $PY = 12$, Years = 3, $FV = \\$0$ 	How You Will Get There Step 3: Apply Formula 9.1. Step 4: Since $FV = \\$0$, skip this step. Step 5: Apply Formula 11.1 and Formula 11.4, rearranging for PMT.														
Perform	Step 3: $i = 7.8\% \div 12 = 0.65\%$ Step 5: $N = 12 \times 3 = 36$ payments $\$25,000 = PMT \left[\frac{1 - \left[\frac{1}{(1 + 0.0065)^{12}} \right]^{36}}{(1 + 0.0065)^{12} - 1} \right]$ $PMT = \frac{\$25,000}{32.005957} = \$781.10$	Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>36</td><td>7.8</td><td>25000</td><td>Answer: -781.104587</td><td>0</td><td>12</td><td>12</td></tr></table>	N	I/Y	PV	PMT	FV	P/Y	C/Y	36	7.8	25000	Answer: -781.104587	0	12	12
N	I/Y	PV	PMT	FV	P/Y	C/Y										
36	7.8	25000	Answer: -781.104587	0	12	12										
Present	The figure shows how much principal and interest make up the payments. To pay off her consolidated loan, Morgan's month-end payments for the next three years will be \$781.10.															

Example 11.4B: Funding a Backpack Trip across Europe

Franco has placed \$10,000 into an investment fund with the goal of receiving equal amounts at the beginning of every month for the next year while he backpacks across Europe. If the investment fund can earn 5.25% compounded quarterly, how much money can Franco expect to receive each month?

Plan	Step 1: The payments are at the beginning of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Calculate the monthly amount he can receive, or PMT.																	
Understand	What You Already Know Step 1 (continued): The timeline for the vacation money appears below. Step 2: $PV_{DUE} = \\$10,000$, $IY = 5.25\%$, $CY = 4$, $PY = 12$, Years=1, $FV = \\$0$ 	How You Will Get There Step 3: Apply Formula 9.1. Step 4: Since $FV = \\$0$, skip this step. Step 5: Apply Formula 11.1 and Formula 11.3, rearranging for PMT.																
	Step 3: $i = 5.25\% \div 4 = 1.3125\%$ Step 5: $N = 12 \times 1 = 12$ payments $\$10,000 = PMT \left[\frac{1 - \left[\frac{1}{(1 + 0.013125)^{\frac{4}{12}}} \right]^{12}}{(1 + 0.013125)^{\frac{4}{12}} - 1} \right] \times (1 + 0.013125)^{\frac{4}{12}}$ $PMT = \frac{\$10,000}{11.717849} = \$853.40$	Calculator Instructions <table><tr><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>BGN</td><td>12</td><td>5.25</td><td>-10000</td><td>Answer: 853.398928</td><td>0</td><td>12</td><td>4</td></tr></table>		Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	BGN	12	5.25	-10000	Answer: 853.398928	0	12
Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y											
BGN	12	5.25	-10000	Answer: 853.398928	0	12	4											
Perform																		
Present																		
	The figure shows how much principal and interest make up the payments. While backpacking across Europe, Franco will have his annuity pay him \$853.40 at the beginning of every month.																	

Example 11.4C: Planning RRSP Contributions

Kingsley's financial adviser has determined that when he reaches age 65, he will need \$1.7 million in his RRSP to fund his retirement. Kingsley is currently 22 years old and has saved up \$10,000 already. His adviser thinks that his RRSP will average 9% compounded annually throughout the years. To meet his RRSP goal, how much does Kingsley need to invest every month starting today?

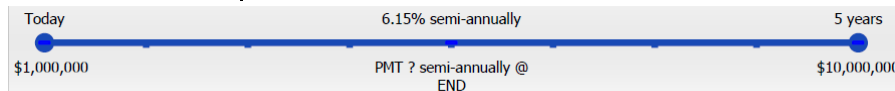
Plan	Step 1: The payments are made at the beginning of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Calculate the monthly amount Kingsley needs to contribute, or PMT.	
Understand	What You Already Know Step 1 (continued): The timeline for Kingsley's RRSP contributions appears below. Step 2: $PV = \\$10,000$, $FV_{DUE} = \\$1,700,000$, $IY = 9\%$, $CY = 1$, $PY = 12$, Years = 43	How You Will Get There Step 3: Apply Formula 9.1. Step 4: You need to move the present value to Kingsley's age 65, the same date as FV_{DUE}. Apply Formula 9.2 and Formula 9.3. The FV is money the annuity does not have to save, so you subtract it from FV_{DUE} to arrive at the amount the annuity must generate. Step 5: Apply Formula 11.1 and Formula 11.3, rearranging for PMT, using the adjusted FV_{DUE}.

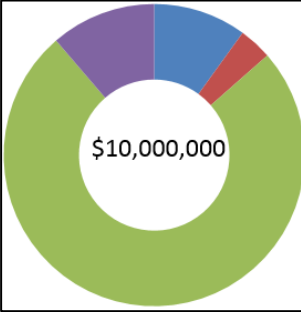
Perform	Step 3: $i = 9\% \div 1 = 9\%$ Step 4: $N = 1 \times 43 = 43$ compounds, $FV = \\$10,000(1 + 0.09)^{43} = \\$406,761.0984$ New $FV_{DUE} = \\$1,700,000 - \\$406,761.0984 = \\$1,293,238.902$ Step 5: $N = 12 \times 43 = 516$ payments $\$1,293,238.902 = PMT \left[\frac{\left[(1 + 0.09)^{\frac{1}{12}} \right]^{516} - 1}{(1 + 0.09)^{\frac{1}{12}} - 1} \right] \times (1 + 0.09)^{\frac{1}{12}}$ $PMT = \frac{\$1,293,238.902}{5,544.647665} = \$233.24$
Present	The figure shows how much principal and interest make up the final balance. To meet his retirement goals, Kingsley needs to invest \$233.24 at the beginning of every month for the next 43 years. In doing so, he will achieve a \$1.7 million balance in his account at age 65. (Note: Similar to loan payments, the last payment in actuality is required to be a slightly higher amount since the annuity payment was rounded downwards. However, the last payment is treated equally at this time for purposes of all calculations.)

Example 11.4D: Purchasing New Production Line Machinery

The production department just informed the finance department that in five years' time the robotic systems on the production line will need to be replaced. The estimated cost of the replacement is \$10 million. To prepare for this purchase, the finance department immediately deposits \$1,000,000 into a savings annuity earning 6.15% compounded semi-annually, and it plans to make semi-annual contributions starting in six months. How large do those contributions need to be?

Plan	Step 1: The payments are made at the end of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is an ordinary simple annuity. Calculate the monthly amount the finance department needs to contribute, or PMT.
Understand	What You Already Know Step 1 (continued): The timeline for the machinery fund appears below. Step 2: $PV = \\$1,000,000$, $FV_{ORD} = \\$10,000,000$, $IY = 6.15\%$, $CY = 2$, $PY = 2$, $Years = 5$ How You Will Get There Step 3: Apply Formula 9.1. Step 4: The present value must be moved to the five-year date, the same date as FV_{ORD}. Apply Formula 9.2 and Formula 9.3. This FV is money the annuity does not have to save, so it is subtracted from FV_{ORD} to arrive at the amount the annuity must generate. Step 5: Apply Formula 11.1 and Formula 11.2, rearranging for PMT, using the adjusted FV_{ORD}.



Perform	Step 3: $i = 6.15\% \div 2 = 3.075\%$ Step 4: $N = 2 \times 5 = 10$ compounds, $FV = \$1,000,000(1+0.03075)^{10} = \$1,353,734.306$ New $FV_{ORD} = \$10,000,000 - \$1,353,734.306 = \$8,646,265.694$ Step 5: $N = 2 \times 5 = 10$ payments $\$8,646,265.694 = PMT \left[\frac{\left[(1 + 0.03075)^{\frac{2}{2}} \right]^{10} - 1}{(1 + 0.03075)^{\frac{2}{2}} - 1} \right]$ $PMT = \frac{\$8,646,265.694}{11.503554} = \$751,616.87$	Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th></tr><tr><td>10</td><td>6.15</td><td>-1000000</td></tr></table> <table><tr><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Answer: -751,616.8656</td><td>10000000</td><td>2</td><td>2</td></tr></table>	N	I/Y	PV	10	6.15	-1000000	PMT	FV	P/Y	C/Y	Answer: -751,616.8656	10000000	2	2
	N	I/Y	PV													
10	6.15	-1000000														
PMT	FV	P/Y	C/Y													
Answer: -751,616.8656	10000000	2	2													
Present	The figure shows how much principal and interest make up the final balance. To have adequate funding for the production line machinery replacement five years from now, the finance department needs to deposit \$751,616.87 into the fund every six months. (Note: Similar to loan payments, the last payment in actuality is required to be a slightly lower amount since the annuity payment was rounded upwards. However, the last payment is treated equally at this time for purposes of all calculations.)	 - Initial Lump-Sum Deposit ==> \$1,000,000 - Interest On Lump-Sum ==> \$396,761.10 10 Contributions of \$751,616.87 ==> \$7,516,168.70 - Interest On Annuity ==> \$1,130,096.99														



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11.5: Number of Annuity Payments

(Exactly How Long Are We Talking About?)

How long do you require to fulfill the goal of your annuity? It all depends on your annuity payment, interest rate, and the amount of money involved.

When saving up for future goals, many people and businesses simply determine how much they can afford to invest each time period and then try to be patient until they meet their savings goal. What they do not know is how long it will take them. If you can put \$75 per month into your vacation fund, how long will it take to save up the \$1,000 needed for a spring break vacation in Puerto Vallarta?

Credit cards require a small minimum payment each month. A lot of Canadians view this minimum payment as a benefit and pay it without understanding what their decision implies. But interest rates on credit cards are around 21% compounded daily! So you can hardly afford to delay in paying off your credit card debt. Yet if you always make only the minimum monthly payment, extinguishing a \$5,000 balance will take approximately 50 years!

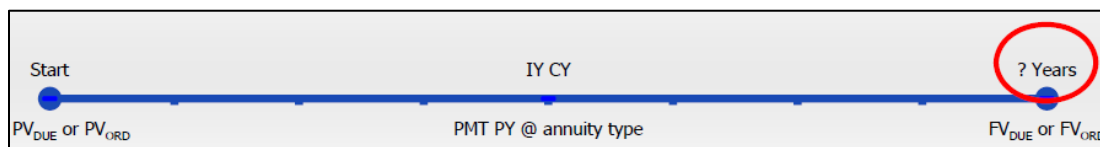
Evidently the number of annuity payments is critical to financial transactions. In this section, you calculate the term of annuities by figuring out the number of annuity payments required.

Ordinary Annuities and Annuities Due

You must calculate the number of annuity payments in a variety of scenarios:

- Savings planning
- Debt extinguishment
- Sustaining withdrawals from an investment

A typical timeline used in calculating the term of an annuity appears below. Recall that N is not illustrated in the timeline but is calculated through Formula 11.1 using Years and PY, which do appear on the timeline. The payment frequency, PY, is a choice determined by the parties involved in the transaction and therefore is always known. That leaves Years, or the term of the annuity, as the unknown variable on the timeline. If you calculate N then you can also calculate Years using a rearrangement of Formula 11.1.



At either end of the timeline, only one of PV_{DUE} , PV_{ORD} , FV_{DUE} , or FV_{ORD} will be known.



The Formula

Recall that the number of annuity payments, N , is one of the variables in Formulas 11.2, 11.3, 11.4, and 11.5. Calculating this amount then requires you to substitute the known variables and rearrange the formula to solve for N . Since N is located in the exponent, the rearrangement and isolation demands the usage of natural logarithms (see Section 2.6).

The most difficult part is figuring out which formula you need to use. Choose the formula using the same decision criteria explained in Section 11.4 (under “The Formula”).



How It Works

Follow these steps, to solve for the number of annuity payments or the annuity term:

Step 1: Identify the annuity type. Draw a timeline to visualize the question.

Step 2: Identify the variables that always appear, including PMT, IY, CY, and PY. You must also identify one of the known values of PV_{ORD} , PV_{DUE} , FV_{ORD} , or FV_{DUE} .

Step 3: Use Formula 9.1 to calculate i .

Step 4: Substitute into the correct annuity payment formula that matches your annuity type and known present or future value. Select from Formulas 11.2, 11.3, 11.4, or 11.5. Rearrange and solve for N .

Step 5: To convert N back to a more commonly expressed format, such as years and months, take Formula 11.1 and rearrange it for Years. If Years is a decimal number, recall the steps required for converting to common language from "Integer Compounding Periods" on page xx in Section 9.7.



Important Notes

Dealing with Decimals in N . At the end of step 4, the calculated value of N may have decimals. Start by applying the same rounding rules involving the computation of N for single payments, mentally rounding the decimal off to the third decimal place.

1. If the three decimals are zeroes, then the decimals are most likely a result of a rounded FV, PV, or PMT, so treat the N like an integer (ignoring the decimals).
2. If the three decimals are not all zeroes, then N must always be *rounded up* to the next integer regardless of the decimal value. You must never round it down, because the calculated value of N represents the minimum payments required. The interpretation and implications of this rounding are as follows:
 - **Payments on a Debt.** Assume your loan payments are \$100 and you calculate $N = 9.4$, which rounds up to 10 payments. The payments must exactly reduce the loan to a balance of zero. The first nine payments are all \$100. The last payment is not a complete payment, hence the 0.4 in the calculation. The precise calculation of this last payment amount is explained in Chapter 13; however, at this point it is sufficient to treat the decimal as an approximate percentage of what the final payment might be. The 0.4 is approximately 40% of \$100, or \$40. As a result, there are nine \$100 payments, and one final payment approximating \$40. Keep in mind that this approximation is rough at best and should only be used to get a feel for what the final payment amount could be. The important point here is that it requires 10 payments to pay off the loan completely, regardless of the amount of the payment.
 - **Payments toward an Investment.** Keeping the numbers the same as above, your investment payments are \$100 with $N = 9.4$. Again, this is 10 payments. If you only make nine full payments of \$100, your investment will be short of your intended future value. However, if you let it grow and make the tenth full payment, you will have more than your investment goal. What the $N = 9.4$ is telling you is that a final tenth payment is needed roughly 40% of the way through the next payment interval that is approximately 0.4 of the regular payment amount. However, most investors don't care about having too much money saved and exceeding their savings goal, so usually a full tenth payment is made.



Things To Watch Out For

Two mistakes are common in calculations of the number of annuity payments:

1. **Misinterpreting N .** Confusing the compounding frequency (CY) and payment frequency (PY) results in the misinterpretation of N and the incorrect calculation of the term. Since N represents the number of annuity payments, to assign its meaning you need to look at the PY, not the CY. Then use the value of PY in Formula 11.1 when expressing N in more common terms. For example, if $N = 8$, $PY = 4$, and $CY = 2$, a common mistake is to say that there are eight semi-annual payments (using the CY); ensure you look instead at the PY, identifying the eight quarterly payments, which means the term is two years.
2. **Confusing the Term and the Last Payment on Annuities Due.** One of the characteristics of an annuity due is that the last payment occurs one payment interval before the end of the term of the annuity. When you calculate N , you have calculated the term of the annuity. The last payment occurs $N - 1$ intervals from the start of the annuity. For example, assume payments are monthly and you calculate $N = 12$ months. This means the term of the annuity is 12 months. The last payment on the annuity due is $12 - 1 = 11$ months from the start. In your "Plan" for any problem, be sure to recognize whether you are looking for the end of the term or when the last payment is made. This problem does not occur for ordinary annuities since the last payment and the end of the term are on the same date.



Give It Some Thought

Holding all other variables constant, what happens mathematically to the number of payments (the precise calculated value of N produced by rearranging any of the annuity formulas) if the following occur:

1. The variable interest rate increases (and the payment itself remains unchanged)?
2. The annuity payments are doubled?

Solutions:

1. The number of payments increases if the variable interest rate increases since more interest is charged but the payment has not increased to cover the higher interest charges.
2. The number of payments is normally cut at least by half if the annuity payments are doubled since with each payment the principal becomes smaller at an accelerated rate and less interest is charged.

Example 11.5A: How Long Until Retirement Savings Are Depleted?

Samia has \$500,000 accumulated in her retirement savings when she decides to retire at age 60. If she wants to receive beginning-of-month payments of \$3,000 and her retirement annuity can earn 5.2% compounded monthly, how old is Samia when the fund is depleted?

Plan	Step 1: The payments are at the beginning of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is a simple annuity due. In looking for how old Samia will be when the fund is depleted, calculate the number of annuity payments, or N , that her retirement annuity can sustain.																		
Understand	What You Already Know Step 1 (continued): The timeline for the retirement annuity appears below. Step 2: $PV_{DUE} = \\$500,000$, $IY = 5.2\%$, $CY = 12$, $PMT = \\$3,000$, $PY = 12$, $FV = 0$	How You Will Get There Step 3: Apply Formula 9.1. Step 4: Substitute into Formula 11.5, rearranging for N. Step 5: Substitute into Formula 11.1 and solve for Years. Add this term to Samia's current age to figure out how old she is when the annuity is depleted.																	
Perform	Step 3: $i = 5.2\% \div 12 = 0.4\overline{3}\%$ Step 4: $\\$500,000 = \\$3,000 \left[\frac{1 - \left[\frac{1}{(1 + 0.004\overline{3})^{12}} \right]^N}{(1 + 0.004\overline{3})^{12} - 1} \right] \times (1 + 0.004\overline{3})^{12}$ $165.947560 = \frac{1 - 0.995685^N}{0.004\overline{3}} \qquad 0.995685^N = 0.280893$$N \times \ln(0.995685) = \ln(0.280893) \qquad N = \frac{-1.269778}{-0.004323}$$N = 293.660180, \text{ rounding up to 294 monthly payments consisting of 293 regular payments plus one additional smaller payment.}$ Step 5: $294 = 12 \times \text{Years} \qquad \text{Years} = \frac{294}{12} = 24.5 = 24 \text{ years, 6 months}$	Calculator Instructions <table><tr><td>Mode</td><td>N</td><td>I/Y</td></tr><tr><td>BGN</td><td>Answer: 293.660180</td><td>5.2</td></tr></table> <table><tr><td>PV</td><td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>-500000</td><td>3000</td><td>0</td><td>12</td><td>12</td></tr></table>	Mode	N	I/Y	BGN	Answer: 293.660180	5.2	PV	PMT	FV	P/Y	C/Y	-500000	3000	0	12	12	
Mode	N	I/Y																	
BGN	Answer: 293.660180	5.2																	
PV	PMT	FV	P/Y	C/Y															
-500000	3000	0	12	12															
Present	The figure shows how much principal and interest make up the payments. If Samia is currently 60 years old and the annuity endures for 24 years and six months, then she will be 84.5 years old when the annuity is depleted. Note that Samia will receive 293 payments of \$3,000 along with a smaller final payment that is approximated by taking $66.018\% \times \\$3,000 = \\$1,980.54$.																		

Example 11.5B: How Long to Pay Off Your Car?

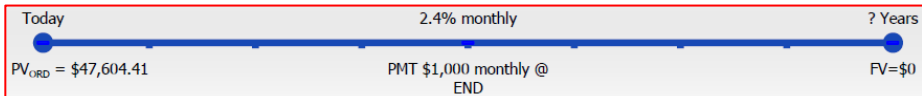
Brendan is purchasing a vehicle. Including all options, accessories, and fees, the total amount he needs to finance is \$47,604.41 at the dealer's special interest financing of 2.4% compounded monthly. If he makes payments of \$1,000 at the end of every month, how long will it take to pay off his car loan?

Plan Step 1: The payments are at the end of the payment intervals with a monthly compounding period and monthly payment intervals. Therefore, this is an ordinary simple annuity. Calculate the number of monthly payments, or N , to figure out the length of time required to pay off the loan.

Understand

What You Already Know
 Step 1 (continued): The timeline for the car payments appears below.
 Step 2: $PV_{ORD} = \$47,604.41$, $IY = 2.4\%$, $CY = 4$, $PMT = \$1,000$, $PY = 12$, $FV = 0$

How You Will Get There
 Step 3: Apply Formula 9.1.
 Step 4: Substitute into Formula 11.4, rearranging for N .
 Step 5: Substitute into Formula 11.1 and solve for Years.



Perform

Step 3: $i = 2.4\% \div 12 = 0.2\%$

Step 4: $\$47,604.41 = \$1,000 \left[\frac{1 - \left[\frac{1}{(1+0.002)^{\frac{12}{12}}} \right]^N}{(1+0.002)^{\frac{12}{12}} - 1} \right]$

$0.095208 = 1 - 0.998003^N$
 $0.998003^N = 0.904791 \quad N \times \ln(0.998003) = \ln(0.904791)$
 $N = \frac{-0.100051}{-0.001998} = 50.075560$, rounded up to 51 monthly payments

Step 5: $51 = 12 \times \text{Years} \quad \text{Years} = \frac{51}{12} = 4.25 = 4 \text{ years, 3 months}$

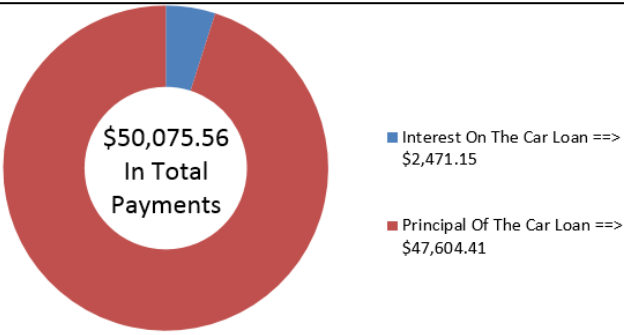
Calculator Instructions

N	I/Y	PV
Answer: 50.075560	2.4	47604.41

PMT	FV	P/Y	C/Y
-1000	0	12	12

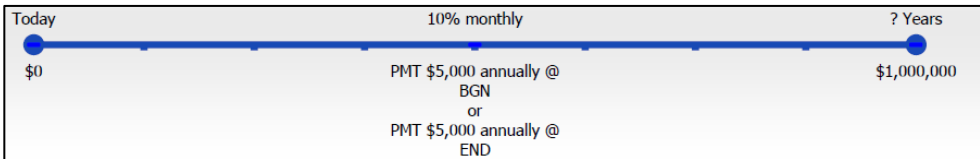
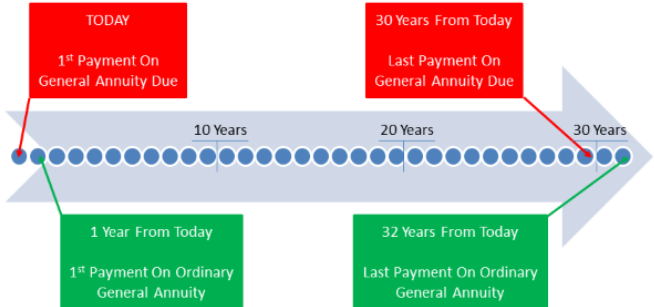
Present

The figure shows how much principal and interest make up the payments. To own his vehicle, Brendan will make payments for four-and-a-quarter years. This consists of 50 payments of \$1,000 and a smaller final payment that is approximated by taking $7.556\% \times \$1,000 = \$75.56$.


Example 11.5C: The Importance of the Annuity Type

Trevor wants to save up \$1,000,000. He will contribute \$5,000 annually to an investment earning 10% compounded monthly. What is the time difference between his last payments (not the end of the annuities) if he makes his contributions at the end of the year instead of at the beginning of the year?

Plan Step 1: In this question you are being asked to compare two annuities that differ in their payment intervals and their compounding periods. One annuity makes contributions at the beginning of the interval, while the other makes contributions at the end. Therefore, you must contrast one general annuity due with one ordinary general annuity. To determine the time difference, calculate N for each annuity and compare when the last payment is made.

Understand	What You Already Know Step 1 (continued): A combined timeline for the two annuities appears below. Step 2: Ordinary general annuity: $FV_{ORD} = \\$1,000,000$, $IY = 10\%$, $CY = 12$, $PMT = \\$5,000$, $PY = 1$ General annuity due: $FV_{DUE} = \\$1,000,000$, $IY = 10\%$, $CY = 12$, $PMT = \\$5,000$, $PY = 1$ How You Will Get There Step 3: Apply Formula 9.1. Step 4: Apply Formulas 11.2 and 11.3, rearranging for N. Step 5: If payments are annual, then $N = \text{Years}$. However, for the general annuity due the last payment is made $N - 1$ years from today. The difference between the two payments can then be calculated. 																															
Perform	Step 3: $i = 10\% \div 12 = 0.8\bar{3}\%$ Step 4: <table><tr><th>Ordinary General Annuity</th><th>General Annuity Due</th></tr><tr><td>$\\$1,000,000 = \\$5,000 \left[\frac{(1 + 0.008\bar{3})^{\frac{12}{1}N} - 1}{(1 + 0.008\bar{3})^{\frac{12}{1}} - 1} \right]$$20.942613 = 1.104713^N - 1$$21.942613 = 1.104713^N$$\ln(21.942613) = N \times \ln(1.104713)$$\frac{\ln(21.942613)}{\ln(1.104713)} = N$$N = 31.012812, \text{ or } 32 \text{ years}$</td><td>$\\$1,000,000 = \\$5,000 \left[\frac{(1 + 0.008\bar{3})^{\frac{12}{1}N} - 1}{(1 + 0.008\bar{3})^{\frac{12}{1}} - 1} \right] \times (1 + 0.008\bar{3})^{\frac{12}{1}}$$18.957514 = 1.104713^N - 1$$19.957514 = 1.104713^N$$\ln(19.957514) = N \times \ln(1.104713)$$\frac{\ln(19.957514)}{\ln(1.104713)} = N$$N = 30.060618, \text{ or } 31 \text{ years}$</td></tr></table> Step 5: The ordinary general annuity last payment is 32 years from today. The general annuity due has a term of 31 years, but the last payment is $31 - 1 = 30$ years from today. The difference between the last payments is $32 - 30 = 2$ years sooner. (See the figure below and count the dots!) Calculator Instructions <table><tr><th>Type</th><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Ordinary</td><td></td><td>Answer: 30.060618</td><td>10</td><td>0</td><td>-5000</td><td>1000000</td><td>1</td><td>12</td></tr><tr><td>Due</td><td>BGN</td><td>Answer: 31.012812</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr></table> In order for Trevor to reach his goal, if he were to make his \$5,000 contributions at the beginning of the year (that is, under the annuity due) instead of the end of the year (under the ordinary annuity), his last payment would be two years sooner. Do not confuse this with the terms of the two annuities, which end only one year apart (31 years from now and 32 years from now). 	Ordinary General Annuity	General Annuity Due	$\$1,000,000 = \$5,000 \left[\frac{(1 + 0.008\bar{3})^{\frac{12}{1}N} - 1}{(1 + 0.008\bar{3})^{\frac{12}{1}} - 1} \right]$ $20.942613 = 1.104713^N - 1$ $21.942613 = 1.104713^N$ $\ln(21.942613) = N \times \ln(1.104713)$ $\frac{\ln(21.942613)}{\ln(1.104713)} = N$ $N = 31.012812, \text{ or } 32 \text{ years}$	$\$1,000,000 = \$5,000 \left[\frac{(1 + 0.008\bar{3})^{\frac{12}{1}N} - 1}{(1 + 0.008\bar{3})^{\frac{12}{1}} - 1} \right] \times (1 + 0.008\bar{3})^{\frac{12}{1}}$ $18.957514 = 1.104713^N - 1$ $19.957514 = 1.104713^N$ $\ln(19.957514) = N \times \ln(1.104713)$ $\frac{\ln(19.957514)}{\ln(1.104713)} = N$ $N = 30.060618, \text{ or } 31 \text{ years}$	Type	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	Ordinary		Answer: 30.060618	10	0	-5000	1000000	1	12	Due	BGN	Answer: 31.012812	✓	✓	✓	✓	✓	✓
Ordinary General Annuity	General Annuity Due																															
$\$1,000,000 = \$5,000 \left[\frac{(1 + 0.008\bar{3})^{\frac{12}{1}N} - 1}{(1 + 0.008\bar{3})^{\frac{12}{1}} - 1} \right]$ $20.942613 = 1.104713^N - 1$ $21.942613 = 1.104713^N$ $\ln(21.942613) = N \times \ln(1.104713)$ $\frac{\ln(21.942613)}{\ln(1.104713)} = N$ $N = 31.012812, \text{ or } 32 \text{ years}$	$\$1,000,000 = \$5,000 \left[\frac{(1 + 0.008\bar{3})^{\frac{12}{1}N} - 1}{(1 + 0.008\bar{3})^{\frac{12}{1}} - 1} \right] \times (1 + 0.008\bar{3})^{\frac{12}{1}}$ $18.957514 = 1.104713^N - 1$ $19.957514 = 1.104713^N$ $\ln(19.957514) = N \times \ln(1.104713)$ $\frac{\ln(19.957514)}{\ln(1.104713)} = N$ $N = 30.060618, \text{ or } 31 \text{ years}$																															
Type	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y																								
Ordinary		Answer: 30.060618	10	0	-5000	1000000	1	12																								
Due	BGN	Answer: 31.012812	✓	✓	✓	✓	✓	✓																								
Present																																



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11.6: Annuity Interest Rates

(How Does The Money Grow?)

Canada One RV advertises that you can purchase a mid-size trailer, valued at \$32,999, for \$181.84 biweekly over the next 10 years. Toronto Hyundai offers a "Smokin' Hot Deal" for a used \$9,900 Hyundai with \$69 biweekly payments for seven years. Going online, you visit the Express Toll Route website to look at paying your tolls on this tollway and see that you can pay \$21.50 annually or \$3.25 per month. Checking your vehicle registration, you see on the insurance website that you can pay your motorcycle insurance in a single payment of \$2,667 or five beginning-of-month payments of \$538. The question in all of these situations is, should you pay all at once or in instalments?

Notice in all of these cases that the total of the payments is higher than the purchase price of the item. For example, the toll route is billing \$3.25 per month, totalling \$39.00 per year! From a strictly financial perspective, the difference from the \$21.50 can be treated like an amount of interest charged for not paying the full annual amount up front. But what interest rate are you paying?

In investing, you should choose the highest effective rate that carries a level of risk you can accept. If you make fixed contributions to an investment, what interest rate must you obtain to meet your future goal? On debts, do you know what you are being charged? That is your hard-earned money you are spending, so perhaps you can find a better offer at a lower rate from someone else? In business, debt interest is an expense that affects the pricing of products and is deductible from income tax. To calculate the interest, the business needs to know the interest rate it is being charged.

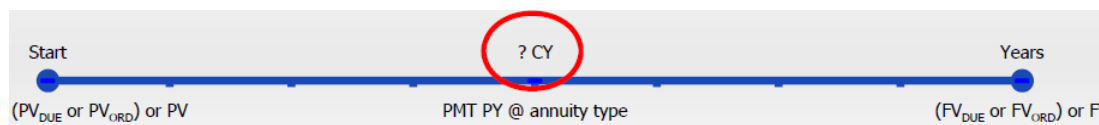
You have many reasons to calculate an annuity's interest rate. In this last section on annuities, you will make this calculation for both ordinary annuities and annuities due.

Ordinary Annuities and Annuities Due

You must calculate the interest rate on an annuity in a variety of situations:

- To determine the interest rate being charged on any debt
- To determine the interest rate that an investment is earning
- To calculate the required interest rate for savings to reach a goal within a certain time period
- To calculate the required interest rate needed for a series of payments to be sustained over a certain time period

A typical timeline for the unknown interest rate on an annuity appears in the figure below. All variables must be known except for the nominal interest rate, iY . At either end of the timeline:



- If you know PV_{DUE} or PV_{ORD} , there might be an FV on the other side as well. FV_{DUE} , FV_{ORD} , and PV will not appear.
- If you know FV_{DUE} or FV_{ORD} , there might be a PV on the other side. PV_{DUE} , PV_{ORD} , and FV will not appear.



The Formula

A most interesting circumstance arises when you attempt to solve any of the future value or present value annuity formulas, both ordinary and due, for the interest rate. Formula 11.2 is reprinted below for illustration; however, the same point holds for Formulas 11.3 to 11.5.

$$\text{Formula 11.2: } FV_{ORD} = PMT \left[\frac{(1+i)^{\frac{CY}{PY}N} - 1}{(1+i)^{\frac{CY}{PY}} - 1} \right]$$

In this formula, the FV_{ORD} , PMT, or N each appears only once. This allows you to easily manipulate the formula to solve for these variables, as we have done in previous sections. However, the periodic interest rate, i , appears in the formula

twice. This normally is not a problem because you can apply basic rules of algebra to isolate and combine like terms, but not so in this instance.

To illustrate the difficulty, assign $PMT = \$1$ and make $CY = PY$. That makes the equation look like this:

$$FV_{ORD} = \frac{(1+i)^N - 1}{i}$$

If you rearrange the formula you have

$$iFV_{ORD} + 1 = (1+i)^N$$

The i on the left-hand side is free to be isolated, but the i on the right-hand side is part of the base for an exponent. To cancel the N exponent, you need to raise both sides of the equation to the exponent of $1/N$, arriving at the following:

$$(iFV_{ORD} + 1)^{\frac{1}{N}} = 1 + i$$

Note the result: Now the left-hand side has i being part of a base for an exponent, while the i on the right-hand side is free to be isolated. Not much of an improvement, is it? To fix this, you will need to take both sides of the equation to the exponent of N , which puts you back in your original dilemma. It is an endless loop with no solution.

The bottom line is that there is no algebraic way to isolate the i in this equation. Thus, no rearrangement of Formulas 11.2 to 11.5 to isolate this variable is possible. So how do you solve for it? There are two ways:

1. **Trial and Error.** In this manual method, you simply keep plugging in various values for i and attempt to home in on the actual value for the variable. Perhaps you start with $i = 1\%$ and find out that your FV_{ORD} is too low. Then you try $i = 2\%$ and find out that your FV_{ORD} is too high. Now you know that i is between 1% and 2% , so perhaps you try 1.5% . You repeat this process until you end up with a precise enough number. As you can imagine, this process is extremely time consuming, tedious, and prone to errors, so no one performs this process by hand any more.
2. **Use Technology.** Whether you use calculators or spreadsheets, these tools are pre-programmed with the trial-and-error technique and can perform millions of iterations in less than a second to find the solution. All problems in this section, therefore, assume you have access to technology, so no algebraic solutions appear in the formula calculations.



How It Works

Follow these steps to solve for any nominal interest rate:

Step 1: Identify the annuity type. Draw a timeline to visualize the question.

Step 2: Identify the variables that you know, including CY , PMT , PY , and Years. You must also identify a value for one of PV_{ORD} , PV_{DUE} , FV_{ORD} , or FV_{DUE} . You may or may not have a value for FV or PV .

Step 3: Use Formula 11.1 to calculate N . Input all six of the known variables into your technology and solve for the interest rate.



Important Notes

Solving for the nominal interest rate on your BAII+ requires you to enter all six of the other variables excluding the I/Y . Then press $CPT I/Y$ to solve the problem. Because of the trial-and-error technique, it may take your calculator a few seconds to compute the answer. If your screen goes blank and your calculator hesitates, this is normal! The values you enter in the present value (PV), future value (FV), and annuity payment (PMT) must adhere to the cash flow sign convention. Remember that on the BAII+, the I/Y represents the nominal interest rate, which is compounded according to the value you entered into the C/Y variable.



Things To Watch Out For

When solving for the interest rate, pay careful attention to which form of interest rate is being sought:

1. Periodic (i): What is the interest rate per month?
2. Nominal (IY with $CY \neq 1$): What is the interest rate compounded monthly?
3. Effective (IY with $CY = 1$): What is the annually compounded or effective rate of interest?

TOP SECRET

Paths To Success

If you want to know both the nominal and effective interest rate in any situation, there are two methods to produce the answers. The first step in both methods is to solve the problem for the nominal interest rate using the appropriate value for the compounding frequency (CY). Then to calculate the effective rate you can do one of the following:

1. Change the CY to a value of one ($CY = 1$) and recalculate the IY. This book prefers this method as it is the easier and less error-prone method.
2. Take your calculated nominal interest rate and convert it to an effective interest rate by using Formula 9.4 (interest rate conversion) or your calculator (I Conv function). This technique produces the same basic solution; however, it has a higher chance of error and may lack precision if not enough decimals are used.



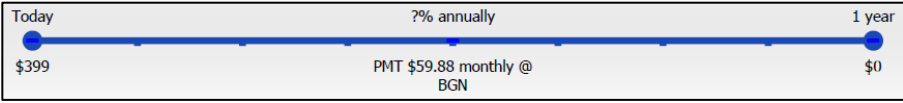
Give It Some Thought

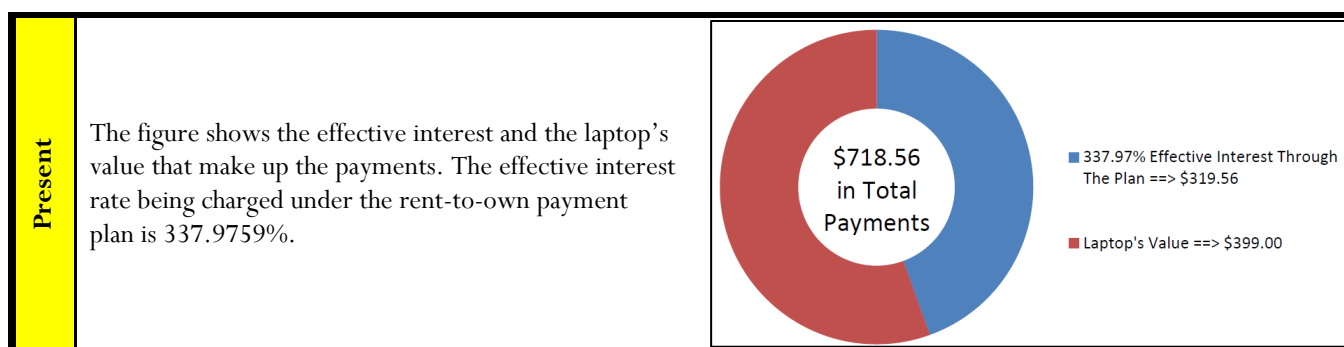
Is it possible in financial situations for the calculated value of the nominal interest rate, IY, to be negative?

Solution: Yes, it is possible. Many investments such as stocks fluctuate based on market conditions. It is possible for an investment to lose its value after it is purchased, which produces a negative rate of return.

Example 11.6A: Rent to Own

Smartchoice, a rent-to-own store, offers a base model laptop for a cash n' carry price of \$399. Alternatively, under its rent-to-own plan you could make \$59.88 monthly payments in advance and own the laptop after one year. What interest rate is effectively being charged on the rent-to-own plan?

Plan	Step 1: The payments are made at the beginning of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Solve for the effective interest rate, which is the nominal interest rate, IY, that has a compounding frequency of one.																	
Understand	What You Already Know Step 1 (continued): The timeline for the retirement annuity appears below. Step 2: $PV_{DUE} = \\$399$, $CY = 1$, $PMT = \\$59.88$, $PY = 12$, Years = 1	How You Will Get There Step 3: Apply Formula 11.1 and Formula 11.5. Enter the information into the calculator to solve for IY (which automatically applies Formula 9.1 to convert i to IY).																
																		
Perform	Step 3: $N = 12 \times 1 = 12$ payments $\$399 = \$59.88 \left[\frac{1 - \left[\frac{1}{(1+i)^{\frac{1}{12}}} \right]^{12}}{(1+i)^{\frac{1}{12}} - 1} \right] \times (1+i)^{\frac{1}{12}}$ See calculator instructions for the solution.	Calculator Instructions <table><tr><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>BGN</td><td>12</td><td>Answer: 337.975924</td><td>399</td><td>-59.88</td><td>0</td><td>12</td><td>1</td></tr></table>	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	BGN	12	Answer: 337.975924	399	-59.88	0	12	1
	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y										
BGN	12	Answer: 337.975924	399	-59.88	0	12	1											



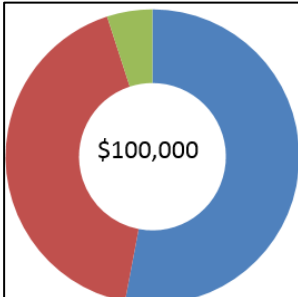
Example 11.6B: Interest Rate Required to Allow for Capital Project Savings

Cubonic Industries deposits \$30,000 at the end of every quarter to save up \$550,000 for a capital project in four years. To achieve its goal, what nominal interest rate compounded quarterly does Cubonic Industries require on its investment? What is the effective rate?

Plan	Step 1: There are two questions here. The first question about the nominal interest rate involves an ordinary simple annuity. Solve this for IY when CY = 4. The second question about the effective rate involves an ordinary general annuity. Solve this for IY when CY = 1.																									
Understand	What You Already Know Step 1 (continued): The timeline for both questions appears below. Step 2: Ordinary simple annuity: $FV_{ORD} = \\$550,000$, CY = 4, PMT = \$30,000, PY = 4, Years = 4 Ordinary general annuity: All the same except CY = 1	How You Will Get There Step 3: Apply Formula 11.1 and Formula 11.2. Ordinary simple annuity: Enter the information into the calculator and solve for IY. Ordinary general annuity: Change the CY to 1 (for the effective rate) and recalculate IY. Note that for both annuities the calculator automatically applies Formula 9.1 to convert i to IY.																								
Perform	Step 3: $N = 4 \times 4 = 16$ payments																									
	<table><tr><th>Ordinary Simple Annuity</th><th>Ordinary General Annuity</th></tr><tr><td>$\\$550,000 = \\$30,000 \frac{\left[\left(1 + i\right)^{\frac{4}{4} \cdot 16} - 1 \right]}{\left(1 + i\right)^{\frac{4}{4}} - 1}$</td><td>$\\$550,000 = \\$30,000 \frac{\left[\left(1 + i\right)^{\frac{1}{4} \cdot 16} - 1 \right]}{\left(1 + i\right)^{\frac{1}{4}} - 1}$</td></tr></table>	Ordinary Simple Annuity	Ordinary General Annuity	$\$550,000 = \$30,000 \frac{\left[\left(1 + i\right)^{\frac{4}{4} \cdot 16} - 1 \right]}{\left(1 + i\right)^{\frac{4}{4}} - 1}$	$\$550,000 = \$30,000 \frac{\left[\left(1 + i\right)^{\frac{1}{4} \cdot 16} - 1 \right]}{\left(1 + i\right)^{\frac{1}{4}} - 1}$																					
Ordinary Simple Annuity	Ordinary General Annuity																									
$\$550,000 = \$30,000 \frac{\left[\left(1 + i\right)^{\frac{4}{4} \cdot 16} - 1 \right]}{\left(1 + i\right)^{\frac{4}{4}} - 1}$	$\$550,000 = \$30,000 \frac{\left[\left(1 + i\right)^{\frac{1}{4} \cdot 16} - 1 \right]}{\left(1 + i\right)^{\frac{1}{4}} - 1}$																									
	See the calculator instructions below for the solution to each question.																									
	Calculator Instructions <table><tr><th>Type</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Simple</td><td>16</td><td>Answer: 7.145908</td><td>0</td><td>-30000</td><td>550000</td><td>4</td><td>4</td></tr><tr><td>General</td><td>✓</td><td>Answer: 7.339689</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td>1</td></tr></table>		Type	N	I/Y	PV	PMT	FV	P/Y	C/Y	Simple	16	Answer: 7.145908	0	-30000	550000	4	4	General	✓	Answer: 7.339689	✓	✓	✓	✓	1
Type	N	I/Y	PV	PMT	FV	P/Y	C/Y																			
Simple	16	Answer: 7.145908	0	-30000	550000	4	4																			
General	✓	Answer: 7.339689	✓	✓	✓	✓	1																			
Present	The figure shows how much principal and interest make up the final balance. For Cubonic Industries to achieve its savings goal, the savings annuity must earn 7.1459% compounded quarterly, or 7.3397% effectively.																									
	■ Interest At 7.1459% Quarterly or 7.3397% Effectively ==> \$70,000 ■ Total Of 16 Payments To Annuity ==> \$480,000																									

Example 11.6C: Interest Rate Required to Achieve RRSP Goal

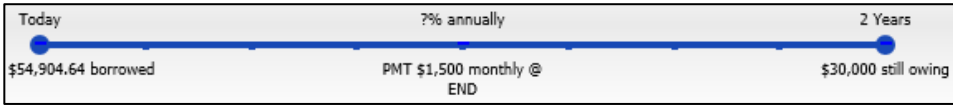
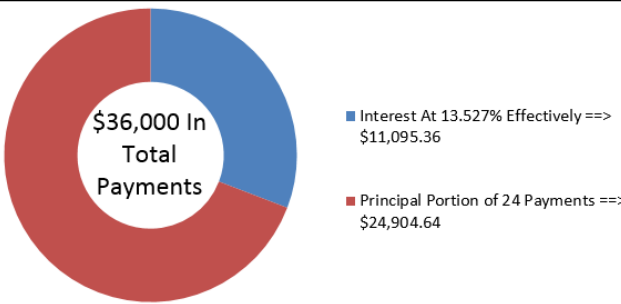
Amadeus has already saved \$5,000 in his RRSP today. Suppose he continues to make \$250 contributions at the beginning of each month for the next 14 years. For him to achieve his goal of having \$100,000, what monthly nominal rate of return must his investment earn?

Plan	Step 1: The payments are made at the beginning of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is a simple annuity due. Solve for the monthly nominal interest rate, iY .																	
Understand	What You Already Know Step 1 (continued): The timeline for RRSP contributions appears below. Step 2: $FV_{DUE} = \\$100,000$, $PV = \\$5,000$, $CY = 12$, $PMT = \\$250$, $PY = 12$, Years = 14	How You Will Get There Step 3: Apply Formula 11.1 and the calculator simultaneously solves Formulas 9.3 and 11.3. The annuity is simple, so N is the same number for both the number of payments and compounds. Enter the information into the calculator and solve for iY. 																
Perform	Step 3: $N = 12 \times 14 = 168$ payments $\$100,000 - \$5,000(1 + i)^{168} = \$250 \left[\frac{\left[(1 + i)^{\frac{12}{12}} - 1 \right]^{168}}{(1 + i)^{\frac{12}{12}} - 1} \right] \times (1 + i)^{\frac{12}{12}}$ See the calculator instructions for the solution.	Calculator Instructions <table><tr><td>Mode</td><td>N</td><td>I/Y</td></tr><tr><td>BGN</td><td>168</td><td>Answer: 8.808808</td></tr></table> <table><tr><td>PV</td><td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>-5000</td><td>-250</td><td>100000</td><td>12</td><td>12</td></tr></table>	Mode	N	I/Y	BGN	168	Answer: 8.808808	PV	PMT	FV	P/Y	C/Y	-5000	-250	100000	12	12
Mode	N	I/Y																
BGN	168	Answer: 8.808808																
PV	PMT	FV	P/Y	C/Y														
-5000	-250	100000	12	12														
Present	The figure shows how much principal and interest make up the final balance. The nominal rate of interest that Amadeus must earn on his investment is 8.8088% compounded monthly.  ■ Interest At 8.8088% Monthly ==> \$53,000 ■ Total Of 168 Contributions Of \$250 ==> \$42,000 ■ Original Amount in RRSP at Start ==> \$5,000																	

Example 11.6D: Ordinary General Annuity with Present Value and a Balance Owing in Future

Gemma is looking to purchase a new SUV for \$54,904.64 including all fees and sales taxes. She can afford to pay no more than \$1,500 at the end of every month, and she wants to have the balance owing reduced to \$30,000 after two years, when she can pay off the vehicle with her trust fund. What is the maximum effective rate of interest she could be charged on the car loan to meet her goals?

Plan	Step 1: The payments are made at the end of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is an ordinary general annuity. Solve for the effective interest rate, iY .

Understand	What You Already Know Step 1 (continued): The timeline for the car loan appears below. Step 2: $PV_{ORD} = \\$54,904.64$, $FV = \\$30,000$, $CY = 1$, $PMT = \\$1,500$, $PY = 12$, Years = 2	How You Will Get There Step 3: Apply Formula 11.1, and the calculator simultaneously solves Formulas 9.3 and 11.4. *Note that since the annuity is general, the N compounds and N payments are different numbers. Putting these two elements into the same equation requires that N take on a single value equalling the number of payments (the annuity requirement). In this regard, the periodic interest rate used in Formula 9.3 must be converted to match the payment frequency. You accomplish this using the same $\frac{CY}{PY}$ exponent from the annuities formulas. Enter the information into the calculator and solve for IY.														
																
Perform	Step 3: $N = 12 \times 2 = 24$ payments $\$54,904.64 - \frac{\$30,000}{\left[(1+i)^{\frac{1}{12}}\right]^{24}} = \$1,500 \left[\frac{1 - \left[\frac{1}{(1+i)^{\frac{1}{12}}}\right]^{24}}{(1+i)^{\frac{1}{12}} - 1} \right]$ See the calculator instructions for the solution.	Calculator Instructions <table border="1" data-bbox="924 653 1362 724"><tr><th>N</th><th>I/Y</th><th>PV</th></tr><tr><td>24</td><td>Answer: 13.527019</td><td>54904.64</td></tr></table> <table border="1" data-bbox="924 756 1299 829"><tr><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>-1500</td><td>-30000</td><td>12</td><td>1</td></tr></table>	N	I/Y	PV	24	Answer: 13.527019	54904.64	PMT	FV	P/Y	C/Y	-1500	-30000	12	1
N	I/Y	PV														
24	Answer: 13.527019	54904.64														
PMT	FV	P/Y	C/Y													
-1500	-30000	12	1													
Present	The figure shows how much principal and interest make up the payments. Gemma will be able to purchase the car if she can obtain a car loan that has an effective interest rate lower than 13.527%.	 ■ Interest At 13.527% Effectively ==> \$11,095.36 ■ Principal Portion of 24 Payments ==> \$24,904.64														



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Chapter 11 Summary

Key Concepts Summary

Section 11.1: Fundamentals of Annuities: What Do I Need to Know?

- Understanding what an annuity is
- The four different types of annuities
- The difference between annuities and single payments
- The annuity timeline format

Section 11.2: Future Value of Annuities: Will I Have Enough?

- The future value of ordinary annuities
- Variable changes in future value annuity calculations
- The future value of annuities due

Section 11.3: Present Value of Annuities: How Much Do I Need Now?

- The present value of both ordinary annuities and annuities due
- Variable changes in present value annuity calculations
- Applying both future value and present value calculations to loans
 - Determining loan balances
 - Selling loan contracts between companies

Section 11.4: Annuity Payment Amounts: What Commitment Am I Making?

- Calculating the annuity payment amount for both ordinary annuities and annuities due

Section 11.5: Number of Annuity Payments: Exactly How Long Are We Talking About?

- Calculating the number of annuity payments (term) for both ordinary annuities and annuities due
- What to do when N has decimals

Section 11.6: Annuity Interest Rates: How Does the Money Grow?

- Calculating the interest rate for both ordinary annuities and annuities due

The Language of Business Mathematics

annuity A continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation.

annuity payment The dollar amount of the equal periodic payment in an annuity environment.

due Annuity payments that are each made at the beginning of a payment interval.

future value of any annuity The sum of all the future values for all of the annuity payments when they are moved to the end of the last payment interval.

general annuity An annuity in which the payment frequency and compounding frequency are unequal.

general annuity due An annuity where payments are made at the beginning of the payment intervals and the payment and compounding frequencies are unequal. The first payment occurs on the same date as the beginning of the annuity, while the end of the annuity is one payment interval after the last payment.

ordinary Annuity payments that are each made at the end of a payment interval; this is the most common form of an annuity payment.

ordinary general annuity An annuity where payments are made at the end of the payment intervals and the payment and compounding frequencies are unequal. The first payment occurs one interval after the beginning of the annuity, while the last payment is on the same date as the end of the annuity.

ordinary simple annuity An annuity where payments are made at the end of the payment intervals and the payment and compounding frequencies are equal. The first payment occurs one interval after the beginning of the annuity while the last payment is on the same date as the end of the annuity.

payment frequency The number of annuity payments in a complete year.

payment interval The amount of time between each continuous and equal annuity payment.

present value of any annuity The sum of all the present values for all of the annuity payments when they are moved to the beginning of the first payment interval.

simple annuity An annuity in which the payment frequency and compounding frequency are equal.

simple annuity due An annuity where payments are made at the beginning of the payment intervals and the payment and compounding frequencies are equal. The first payment occurs on the same date as the beginning of the annuity, while the end of the annuity is one payment interval after the last payment.

The Formulas You Need to Know

Symbols Used

CY = compounding per year or compounding frequency

FV_{DUE} = future value of annuity due

FV_{ORD} = future value of an ordinary annuity

IY = nominal interest rate

i = periodic interest rate

N = number of annuity payments

PY = payments per year or payment frequency

PMT = annuity payment amount

PV_{DUE} = present value of annuity due

PV_{ORD} = present value of ordinary annuity

Years = the term of the annuity

Formulas Introduced

Formula 11.1 Number of Annuity Payments: $N = PY \times \text{Years}$ (Section 11.1)

Formula 11.2 Ordinary Annuity Future Value:
$$FV_{ORD} = PMT \left[\frac{\left[(1+i)^{\frac{CY}{PY}} - 1 \right]^N}{(1+i)^{\frac{CY}{PY}} - 1} \right] \quad (\text{Section 11.2})$$

Formula 11.3 Annuity Due Future Value:
$$FV_{DUE} = PMT \left[\frac{\left[(1+i)^{\frac{CY}{PY}} - 1 \right]^N}{(1+i)^{\frac{CY}{PY}} - 1} \right] \times (1+i)^{\frac{CY}{PY}} \quad (\text{Section 11.2})$$

Formula 11.4 Ordinary Annuity Present Value:
$$PV_{ORD} = PMT \left[\frac{1 - \left[\frac{1}{(1+i)^{\frac{CY}{PY}}} \right]^N}{(1+i)^{\frac{CY}{PY}} - 1} \right] \quad (\text{Section 11.3})$$

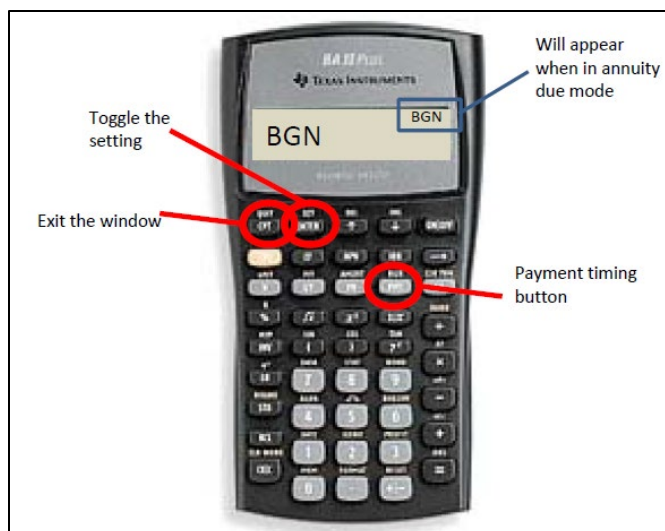
Formula 11.5 Annuity Due Present Value:
$$PV_{DUE} = PMT \left[\frac{1 - \left[\frac{1}{(1+i)^{\frac{CY}{PY}}} \right]^N}{(1+i)^{\frac{CY}{PY}} - 1} \right] \times (1+i)^{\frac{CY}{PY}} \quad (\text{Section 11.3})$$

Technology

Calculator

Annuity Type Settings

- The calculator default is for END mode, which is the ordinary annuity.
- The annuity type (payment timing) setting can be found on the second shelf above the PMT key. This function works as a toggle.
- To toggle the setting, complete the following sequence:
 1. 2nd BGN (the current payment timing of END or BGN is displayed)
 2. 2nd SET (it toggles to the other setting)
 3. 2nd Quit (to get out of the window)
- When the calculator is in annuity due mode, a tiny BGN is displayed in the upper right of your calculator.



Chapter 12: Compound Interest: Special Applications of Annuities

(What Can I Do with This Knowledge of Annuities?)

It has been a hectic day at work. This morning your newly leased computer equipment arrived and your manager reminded you that as soon as it is set up, the debt obligation associated with the computer lease must be recorded in the accounting books. As well, a few of your corporate partners had contributed toward a new scholarship program for your employees and you must determine exactly how much the scholarship can afford to pay out from its interest earnings without touching the principal. If that were not enough, your company's pension plan is under review. Your pension earners' monthly income payments need to be increased based on the rate of inflation. On a related note, some of your employees want to participate in a special component of your RRSP benefits program in which they invest single payments today and receive future benefit payments down the road. You must help them figure out how much money they must deposit today to participate in this program.

What does this example illustrate? A majority of businesses lease computers and equipment for tax benefit purposes along with the ability to regularly upgrade equipment to meet rapid technological changes. Whether you are the business selling the leased products or acquiring them, you must understand how leases work. Company and government pension plans such as the Canada Pension Plan and Old Age Security regularly increase their payments to match inflation. Many companies, organizations, and even not-for-profit enterprises beyond just the educational sector set up scholarship programs for employees and their families. These scholarships are designed to provide benefits today and endlessly into the future.

On the personal side, you must arrange for the retirement income that you will eventually withdraw from your RRSP savings to increase regularly to match inflation so that your purchasing power remains the same throughout all stages of retirement. On a similar note, most contributions to RRSPs are regularly increased as annual income rises throughout a career. In addition, as a Canadian you have access to Canada Pension Plan and Old Age Security benefits, which are constantly increasing annuity payments available to you in your retirement years. You also save for a lot of events and big purchases in your lives; many consumers defer the purchase of these items until cash is available.

In this chapter, you will explore four specific situations relating to annuities: deferred annuities, constant growth annuities, perpetuities, and leases. Then you will practise with some personally relevant applications in the form of procedures for purchasing your vehicle at the lowest cost and for basic RRSP planning.

Outline of Chapter Topics

- 12.1: Deferred Annuities (Can I Receive Those Payments Later?)
- 12.2: Constant Growth Annuities (Always Wanting More)
- 12.3: Perpetuities (Are You Going to Live Forever?)
- 12.4: Leases (I Will Just Borrow It for a While)
- 12.5: Application: How to Purchase a Vehicle (Get Your Motor Running)
- 12.6: Application: Planning Your RRSP (When Can I Retire?)

12.1: Deferred Annuities

(Can I Receive Those Payments Later?)

The power of compound interest, as you have already seen, is amazing. Your investment realizes exponential growth, and over long periods of time the results are spectacular.

Taking advantage of this principle, many parents (and grandparents too) invest large amounts of money when their children are young to have enough money to pay for their college or university education. A simple investment of \$10,000 at birth with no further contributions could sustain approximately four years' worth of \$1,200 monthly payments to the child starting at age 18. That is over \$55,000 toward the child's education!

Many students now stay at home longer with their parents while they pursue postsecondary education. Some accumulate sums of wealth at an early age through part-time earnings when they have little or no expenditure. Assume a 21-year old student had accumulated \$50,000 and instead of using that money on a new car or backpacking across Europe, she invested it into her RRSP. By age 65 and without contributing another dime, she could have a retirement fund that sustains 20 years of \$15,000 monthly income, or approximately \$3.6 million!

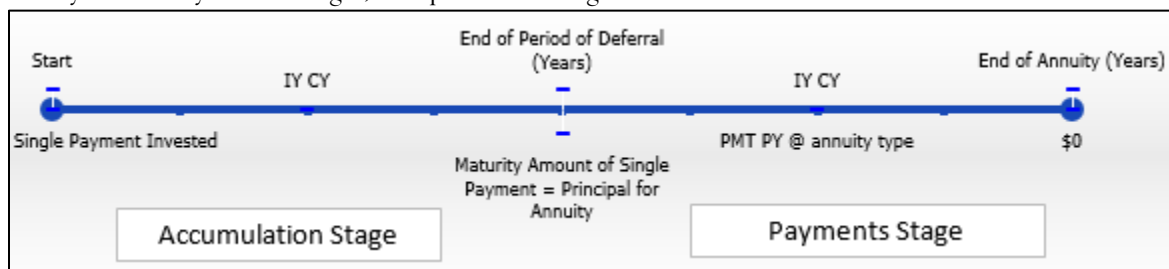
Compare this to someone who makes \$300 monthly payments every month from age 21 to age 65, therefore contributing 528 payments totalling \$158,400 out-of-pocket over the years. Under equal conditions, this person will receive about \$12,000 monthly for 20 years, or approximately \$2.9 million. See the difference? This person not only put \$108,400 more into the RRSP but receives \$0.7 million less in income during retirement.

Although investing the \$50,000 into an RRSP instead of buying a new car is not quite as sexy, can you imagine never having to contribute to your RRSP again and being secure in your retirement before you even start your career? The financial freedom you will experience for the rest of your life would be enviable.

This section explores the concept of investing single payments today with the goal of using the maturity value to sustain an annuity afterwards. This is more commonly referred to as a deferred annuity.

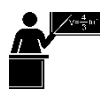
What Is a Deferred Annuity?

A **deferred annuity** is a financial transaction where annuity payments are delayed until a certain period of time has elapsed. Usually the annuity has two stages, as depicted in this figure.



1. **Accumulation Stage.** A single payment is allowed to earn interest for a specified duration. There are no annuity payments during this period of time, which is commonly referred to as the **period of deferral**.
2. **Payments Stage.** The annuity takes the form of any of the four annuity types and starts at the beginning of this stage as per the financial contract. Note that the maturity value of the accumulation stage is the same as the principal for the payments stage.

The interest rate on deferred annuities can be either variable or fixed. However, since deferred annuities are commonly used to meet a specific need, fixed interest rates are more prevalent since they allow for certainty in the calculations.



The Formula

For a deferred annuity, you apply a combination of formulas that you have already used throughout this book. The accumulation stage is not an annuity, so it uses the various single payment compound interest formulas from Chapter 9. The payments stage is an annuity, so it uses the various annuity formulas from Chapter 11.



How It Works

For deferred annuities, the most common unknown variables are either the present value, the length of the period of deferral, the annuity payment amount, or the number of annuity payments that are sustainable for a fixed income payment. Follow this sequence of steps for each of these variables:

Solving for the Present Value	Solving for the Period of Deferral	Solving for the Annuity Payment Amount	Solving for the Number of Annuity Payments
Step 1: Draw a timeline and identify the variables that you know, along with the annuity type.			
Step 2: Starting at the end of your timeline, calculate the present value of the annuity using the steps from Section 11.3 (Formulas 11.4 or 11.5). Round your answer to two decimals.		Step 2: Starting at the beginning of the timeline, calculate the future value of the single payment using the steps from Section 9.2 (Formula 9.3). Round your answer to two decimals.	
Step 3: Take the principal of the annuity, and using the steps from Section 9.3 (Formula 9.3) calculate the present value for the single amount.	Step 3: Solve for the number of compounding periods using the applicable steps from Section 9.7 (Formula 9.3). The single payment investment is the present value, and the principal of the annuity is the future value.	Step 3: Calculate the annuity payment amount using steps from Section 11.4 (Formula 11.4 or 11.5).	Step 3: Calculate the number of annuity payments using steps from Section 11.5 (Formula 11.4 or 11.5).



Important Notes

Rounding. The maturity value of the single payment or the present value of the annuity is always rounded to two decimals. Since an accumulation fund is different from a payment annuity, logistically the money is transferred between different bank accounts, which means that only two decimals are carried either forwards or backwards through this step of the required calculations.



Things To Watch Out For

Avoid these three common sources of error when you work with deferred annuities:

1. **Combining the Deferral Period and the Annuity Term.** It is an error to treat the period of deferral and the term of the annuity as simultaneous time periods. For example, if a deferred annuity has a three-year period of deferral and a 10-year annuity term, this is sometimes interpreted, mistakenly, as an annuity ending 10 years from today. These time segments are separate and consecutive on the timeline! The correct interpretation is that the annuity term ends 13 years from today, since the 10-year term does not start until the three-year deferral terminates.
2. **Incorrect Timing between Stages.** A common mistake is to determine incorrectly when the period of deferral ends and the annuity starts. This error usually results from forgetting that the payments on ordinary annuities start one payment interval after the annuity starts, whereas annuity due payments start immediately. Thus, if the first quarterly payment on an ordinary annuity is to be paid $6\frac{3}{4}$ years from today, then the period of deferral is $6\frac{1}{2}$ years. If the deferral is for an annuity due, then the period of deferral is $6\frac{3}{4}$ years.
3. **Confusing N.** A deferred annuity requires different calculations of N using either Formula 9.2 or Formula 11.1. In the accumulation stage, recall that N must represent the number of compound periods calculated by Formula 9.2. In the payments stage, the N must represent the number of annuity payments calculated by Formula 11.1.



Give It Some Thought

1. If a deferred annuity has a four-year period of deferral and a seven-year annuity term, how many years from today will the term of the annuity end?
2. If an ordinary deferred annuity makes its first monthly payment 25 months from now, how long is the period of deferral? What if it were a deferred annuity due?

2. 24 months; 25 months

1. 11 years
Solutions:

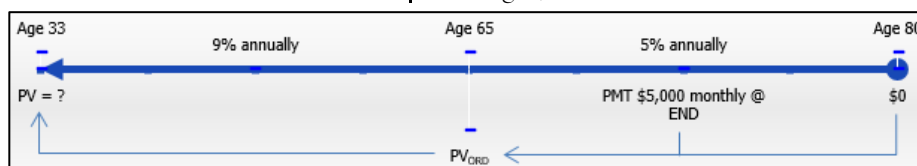
Example 12.1A: Investing an Inheritance for your Retirement

Frasier is 33 years old and just received an inheritance from his parents' estate. He wants to invest an amount of money today such that he can receive \$5,000 at the end of every month for 15 years when he retires at age 65. If he can earn 9% compounded annually until age 65 and then 5% compounded annually when the fund is paying out, how much money must he invest today?

Plan Calculate the single payment that must be invested today. This is the present value (PV) of the deferred annuity.

Understand **What You Already Know**
Step 1: The deferred annuity has monthly payments at the end with an annual interest rate. Therefore, this is an ordinary general annuity. The timeline for the deferred annuity appears below.
Ordinary General Annuity: FV = \$0, IY = 5%, CY = 1, PMT = \$5,000, PY = 12, Years = 15
Period of Deferral: FV = PV_{ORD}, IY = 9%, CY = 1, Years = 32

How You Will Get There
Step 2: Calculate the periodic interest rate (i , Formula 9.1), number of annuity payments (N , Formula 11.1), and present value of the ordinary general annuity (PV_{ORD} , Formula 11.4).
Step 3: Discount the principal of the annuity back to today.
Calculate the periodic interest rate (i , Formula 9.1), number of single payment compound periods (N , Formula 9.2), and present value of a single payment (PV , Formula 9.3 rearranged).



Perform Step 2: $i = 5\%/1 = 5\%$; $N = 12 \times 15 = 180$ payments $PV_{ORD} = \$5,000 \left[\frac{1 - \frac{1}{(1+0.05)^{180}}}{(1+0.05)^{12} - 1} \right] = \$636,925.79$

Step 3: $i = 9\%/1 = 9\%$; $N = 1 \times 32 = 32$ compounds
 $\$636,925.79 = PV(1 + 0.09)^{32}$ $PV = \$636,925.79 \div (1.09)^{32} = \$40,405.54$

Calculator Instructions

Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Payments	END	180	5	Answer: -636,925.79	5000	0	12	1
Accumulation	✓	32	9	Answer: 40,405.53861	0	636925.79	1	1

Present If Frasier invests \$40,405.54 today, he will have enough money to sustain 180 withdrawals of \$5,000 in retirement.

Example 12.1B: Planning the Deferral Period

Bashir wants an annuity earning 4.3% compounded semi-annually to pay him \$2,500 at the beginning of every month for 10 years. To achieve his goal, how far in advance of the start of the annuity does Bashir need to invest \$50,000 at 8.25% compounded quarterly? Assume 91 days in a quarter.

Plan

Calculate the amount of time between today and the start of the annuity. This is the period of deferral, or N.

Understand**What You Already Know**

Step 1: The deferred annuity has monthly payments at the beginning with a semi-annual interest rate.

Therefore, this is a general annuity due. The timeline for the deferred annuity appears below.

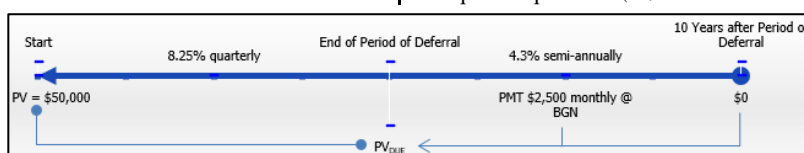
General Annuity Due: FV = \$0, IY = 4.3%, CY = 2, PMT = \$2,500, PY = 12, Years = 10

Period of Deferral: PV = \$50,000, FV = PV_{DUE}, IY = 8.25%, CY = 4

How You Will Get There

Step 2: Calculate the periodic interest rate (*i*, Formula 9.1), number of annuity payments (*N*, Formula 11.1), and present value of the ordinary general annuity (PV_{ORD}, Formula 11.4).

Step 3: Determine the number of compounds during the accumulation stage. Calculate the periodic interest rate (*i*, Formula 9.1) followed by the number of single payment compound periods (*N*, Formula 9.3 rearranged).

**Perform**

Step 2: $i = 4.3\% / 2 = 2.15\%$; $N = 12 \times 10 = 120$ payments

$$PV_{DUE} = \$2,500 \left[\frac{1 - \frac{1}{(1 + 0.0215)^{120}}}{(1 + 0.0215)^{\frac{2}{12}} - 1} \right] \times (1 + 0.0215)^{\frac{2}{12}} = \$244,780.93$$

Step 3: $i = 8.25\% / 4 = 2.0625\%$

$$\$244,780.93 = \$50,000(1 + 0.020625)^N$$

$$4.895618 = 1.020625^N$$

$$\ln(4.895618) = N \times \ln(1.020625)$$

$$N = \frac{1.588340}{0.020415} = 77.801923 \text{ quarterly compounds}$$

$$\text{Years} = 77 \div 4 = 19.25 = 19 \text{ years, 3 months}$$

$$0.801923 \times 91 \text{ days} = 72.975105 = 73 \text{ days}$$

Calculator Instructions

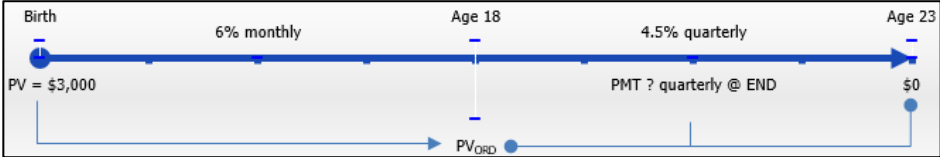
Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Payments	BGN	120	4.3	Answer: -244,780.9336	2500	0	12	2
Accumulation	END	Answer: 77.801924	8.25	-50000	0	244780.93	4	4

Present

To achieve his goal, Bashir needs to invest the \$50,000 19 years, 3 months and 73 days before the annuity starts.

Example 12.1C: How Much Income Will It Provide?

On the day of their granddaughter's birth, Henri and Frances deposited \$3,000 into a trust fund for her future education. The fund earns 6% compounded monthly. When she turns 18, they then want it to make payments at the end of every quarter for five years. If the income annuity can earn 4.5% compounded quarterly, what is the amount of each annuity payment to the granddaughter?

Plan	Calculate the amount of the annuity payment (PMT) during the income payments stage of the deferred annuity.																											
Understand	What You Already Know Step 1: The deferred annuity has quarterly payments at the end with a quarterly interest rate. Therefore, this is an ordinary simple annuity. The timeline for the deferred annuity appears below. Period of Deferral: PV = \$3,000, IY = 6%, CY = 12, Years = 18 Ordinary Simple Annuity: PV_{ORD} = FV after deferral, FV = \$0, IY = 4.5%, CY = 4, PY = 4, Years = 5	How You Will Get There Step 2: Calculate the future value of the single payment investment. Calculate the periodic interest rate (<i>i</i>, Formula 9.1), number of single payment compound periods (<i>N</i>, Formula 9.2), and future value of a single payment amount (FV, Formula 9.3). Step 3: Work with the ordinary simple annuity. First, calculate the periodic interest rate (<i>i</i>, Formula 9.1), number of annuity payments (<i>N</i>, Formula 11.1), and finally the annuity payment amount (PMT, Formula 11.4).																										
																												
Perform	Step 2: $i = 6\%/12 = 0.5\%$; $N = 12 \times 18 = 216$ compounds $FV = \\$3,000(1 + 0.005)^{216} = \\$8,810.30$ Step 3: $i = 4.5\%/4 = 1.125\%$; $N = 4 \times 5 = 20$ payments $\\$8,810.30 = PMT \left[\frac{1 - \left[\frac{1}{(1+0.01125)^4} \right]^N}{(1+0.01125)^4 - 1} \right]$$PMT = \frac{\\$8,810.30}{\frac{1 - \left[\frac{1}{(1+0.01125)^4} \right]^N}{(1+0.01125)^4 - 1}} = \frac{\\$8,810.30}{\frac{0.200480}{0.01125}} = \\$494.39$																											
	Calculator Instructions <table><tr><th>Stage</th><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Accumulation</td><td>END</td><td>216</td><td>6</td><td>−3000</td><td>0</td><td>Answer: 8,810.297916</td><td>12</td><td>12</td></tr><tr><td>Payments</td><td>✓</td><td>20</td><td>4.5</td><td>−8810.30</td><td>Answer: 494.392721</td><td>0</td><td>4</td><td>4</td></tr></table>		Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	Accumulation	END	216	6	−3000	0	Answer: 8,810.297916	12	12	Payments	✓	20	4.5	−8810.30	Answer: 494.392721	0	4
Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y																				
Accumulation	END	216	6	−3000	0	Answer: 8,810.297916	12	12																				
Payments	✓	20	4.5	−8810.30	Answer: 494.392721	0	4	4																				
Present	The granddaughter will receive \$494.39 at the end of every quarter for five years starting when she turns 18. Since the payment is rounded, the very last payment is a slightly different amount, which could be determined exactly using techniques discussed in Chapter 13.																											

Example 12.1D: How Long Can the Annuity Be Sustained?

Emile received a \$25,000 one-time bonus from his employer today, and he immediately invested it at 8% compounded annually. Fourteen years from now, he plans to withdraw \$2,300 at the beginning of every month to use as his retirement income. If the income annuity can earn 3.25% compounded semi-annually, what is the term of the annuity before it is depleted (including the smaller final payment)?

Plan	Figure out how long the income annuity is able to sustain the income payments. This requires you to calculate the number of annuity payments, or <i>N</i> .

Understand	What You Already Know Step 1: The deferred annuity has monthly payments at the beginning with a semi-annual interest rate. Therefore, this is a general annuity due. The timeline for the deferred annuity appears below. Period of Deferral: $PV = \\$25,000$, $IY = 8\%$, $CY = 1$, $Years = 14$ General Annuity Due: $PV_{DUE} = FV$ after deferral, $FV = \\$0$, $IY = 3.25\%$, $CY = 2$, $PMT = \\$2,300$, $PY = 12$	How You Will Get There Step 2: Calculate the future value of the single deposit. Calculate the periodic interest rate (i, Formula 9.1), number of single payment compound periods (N, Formula 9.2), and future value of a single payment amount (FV, Formula 9.3). Step 3: Work with the general annuity due. Calculate the periodic interest rate (i, Formula 9.1) and the number of annuity payments (N, Formula 11.5 rearranged for N). Finally substitute into the annuity payments Formula 11.1 to solve for Years.																										
Perform	Step 2: $i = 8\%/1 = 8\%$; $N = 1 \times 14 = 14$ compounds $FV = \\$25,000(1 + 0.08)^{14} = \\$73,429.84$ Step 3: $i = 3.25\%/2 = 1.625\%$ $\$73,429.84 = \$2,300 \left[\frac{1 - \left[\frac{1}{(1 + 0.01625)^{\frac{2}{12}}} \right]^N}{(1 + 0.01625)^{\frac{2}{12}} - 1} \right] \times (1 + 0.01625)^{\frac{2}{12}}$ $31.840361 = \frac{1 - 0.997317^N}{0.002690}$ $0.085656 = 1 - 0.997317^N$ $\ln(0.997317) = N \times \ln(0.914343)$ $N = 33.332019 = 34 \text{ payments}$ $34 = 12 \times \text{Years} \qquad \text{Years} = 2.8\bar{3} = 2 \text{ years, 10 months}$																											
	Calculator Instructions <table><tr><th>Stage</th><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Accumulation</td><td>END</td><td>14</td><td>8</td><td>-25000</td><td>0</td><td>Answer: 73,429.84061</td><td>1</td><td>1</td></tr><tr><td>Payments</td><td>BGN</td><td>Answer: 33.332019</td><td>3.25</td><td>-73429.84</td><td>2300</td><td>0</td><td>12</td><td>2</td></tr></table>	Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	Accumulation	END	14	8	-25000	0	Answer: 73,429.84061	1	1	Payments	BGN	Answer: 33.332019	3.25	-73429.84	2300	0	12	2
Stage	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y																				
Accumulation	END	14	8	-25000	0	Answer: 73,429.84061	1	1																				
Payments	BGN	Answer: 33.332019	3.25	-73429.84	2300	0	12	2																				
Present	The annuity will last 2 years and 10 months before being depleted.																											



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12.2: Constant Growth Annuities

(Always Wanting More)

Many experts recommend that you should save around 10% of your annual income toward your RRSP contributions. Like most people, when you graduate college or university, your employer will offer you a starting salary that is usually at the lower end of their pay scale. In most companies, you are then eligible to receive annual raises in accordance with a predetermined pay structure or through performance reviews. This represents an annual income that always rises. As a result, your RRSP contributions should always rise annually too.

All annuity calculations so far have permitted fixed contributions only. More realistically, in many financial situations, such as your RRSP, the annuity payments should constantly increase on a regular basis. For this situation you need to study constant growth annuities.

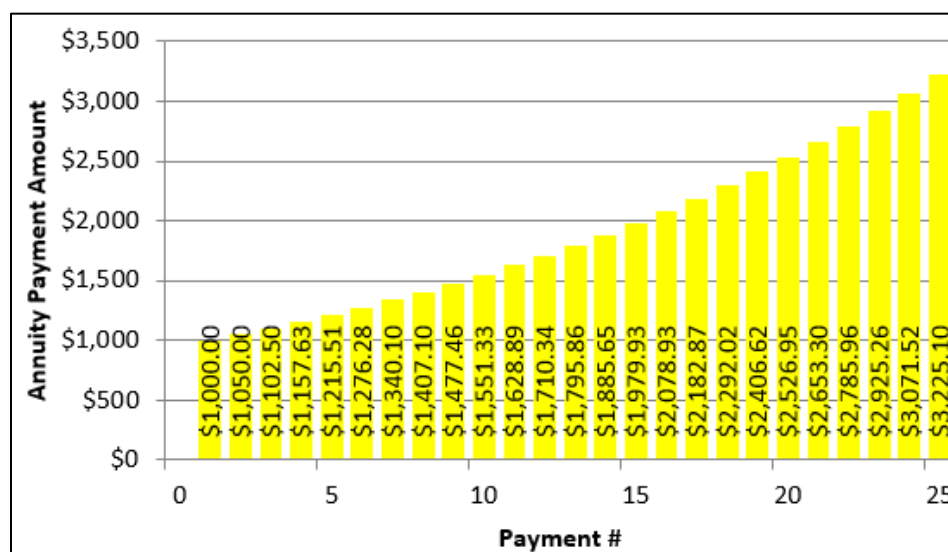
The Concept of Constant Growth

A **constant growth annuity** is an annuity in which each annuity payment is increased by a fixed percentage. The figure here illustrates a \$1,000 initial payment growing by 5% with each subsequent payment.

In addition to your RRSP contributions, annuity payments are regularly increased in many situations:

- When you draw from your retirement savings, you must increase your payments each year to keep up with the cost of living. This increasing demand on your retirement fund must be factored into a proper RRSP savings plan.
- Payments by the Canada Pension Plan and Old Age Security, along with most private company pension plans, increase annually to match the rate of inflation.

The change in each payment is a fixed percentage and therefore represents a percent change in the annuity payment. This can be represented by the percent change symbol, $\Delta\%$, used in Section 3.1. This variable represents the periodic percent change, where the period is consistent with the payment interval. Thus, if the payments are monthly then the percent change variable needs to be expressed as the percent change per month.



The Impact of Constant Growth on the Annuity Payment

To understand how constant growth affects the annuity payment variable, take a look at how the first four payments in any constant growth annuity are represented. Recall in annuities that N represents the number of payments, therefore:

$N = 1$: First payment = $PMT = PMT(1 + \Delta\%)^{N-1}$ (since $N = 1$, this produces a zero exponent on the percent change)

$N = 2$: Second payment = $PMT(1 + \Delta\%) = PMT(1 + \Delta\%)^{N-1}$

$N = 3$: Third payment = $PMT(1 + \Delta\%)(1 + \Delta\%) = PMT(1 + \Delta\%)^2 = PMT(1 + \Delta\%)^{N-1}$

$N = 4$: Fourth payment = $PMT(1 + \Delta\%)(1 + \Delta\%)(1 + \Delta\%) = PMT(1 + \Delta\%)^3 = PMT(1 + \Delta\%)^{N-1}$

Notice that every annuity payment ultimately is represented by $PMT(1 + \Delta\%)^{N-1}$ regardless of the number of annuity payments made.

The Impact of Constant Growth on the Annuity Interest Rate

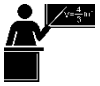
You must adjust the annuity periodic interest rate to isolate the growth in the annuity from interest, since the growth in the annuity payments is already reflected in the $PMT(1 + \Delta\%)^{N-1}$ above. Over the course of any single period, interest compounds at a rate calculated as $1 + i$. On the other hand, contributions over the same time period grow by $1 + \Delta\%$. Therefore, the interest in the account grows more than the growth in the contributions by a rate that is calculated by:

$$\frac{(1 + i)}{(1 + \Delta\%)} - 1$$

For illustrative purposes, assume an annuity with a periodic interest rate of 10% and a periodic growth rate of 5%. Apply the above calculation:

$$\frac{1 + 0.1}{1 + 0.05} - 1 = \frac{1.1}{1.05} - 1 = 0.047619$$

Therefore, each period the net growth attributable solely to interest represents a periodic compounding of 4.7619% and not 10%. This interest rate, sometimes called the **net rate**, must replace the periodic interest rate you use in all annuity formulas.



The Formula

The four formulas for the future value and present value of both ordinary and annuity dues are shown below, incorporating the concept of constant growth. Notice in every formula that the periodic interest rate is changed to net rate.

The percent change formula in the denominator does not require the $\frac{CY}{PY}$ exponent since the variable is already expressed in the same periodic terms as the payments.

1. **Future Value Formulas.** The annuity payment is modified to incorporate the growth in the payments from $PMT(1 + \Delta\%)^{N-1}$ as previously illustrated. The first payment has zero growth, which results in an exponent having one period of growth less than the number of payments made.
2. **Present Value Formulas.** When bringing the annuity back to its beginning, this represents the 0th payment since a first payment has not occurred yet. Applying the $PMT(1 + \Delta\%)^{N-1}$ formula results in $PMT(1 + \Delta\%)^{0-1} = PMT(1 + \Delta\%)^{-1} = \frac{PMT}{1 + \Delta\%}$, which then substitutes for PMT in the formula.

$FV_{ORD} = PMT(1 + \Delta\%)^{N-1} \left[\frac{\left[\frac{(1 + i)^{\frac{CY}{PY}}}{1 + \Delta\%} \right]^N - 1}{\frac{(1 + i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right]$ Formula 12.1: Future Value of a Constant Growth Ordinary Annuity	$PV_{ORD} = \frac{PMT}{1 + \Delta\%} \left[\frac{1 - \left[\frac{1 + \Delta\%}{(1 + i)^{\frac{CY}{PY}}} \right]^N}{\frac{(1 + i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right]$ Formula 12.3: Present Value of a Constant Growth Ordinary Annuity
$FV_{DUE} = PMT(1 + \Delta\%)^{N-1} \left[\frac{\left[\frac{(1 + i)^{\frac{CY}{PY}}}{1 + \Delta\%} \right]^N - 1}{\frac{(1 + i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right] \times (1 + i)^{\frac{CY}{PY}}$ Formula 12.2: Future Value of a Constant Growth Annuity Due	$PV_{DUE} = \frac{PMT}{1 + \Delta\%} \left[\frac{1 - \left[\frac{1 + \Delta\%}{(1 + i)^{\frac{CY}{PY}}} \right]^N}{\frac{(1 + i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right] \times (1 + i)^{\frac{CY}{PY}}$ Formula 12.4: Present Value of a Constant Growth Annuity Due

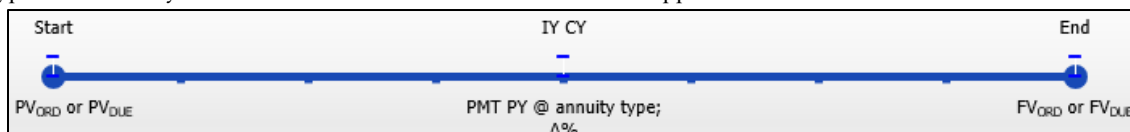
Make the following notes about the variables:

1. **PMT.** Every payment is constantly increasing in a constant growth annuity. Therefore, the first payment is the clearly identifiable value. To ascertain the value of any other payment, use the formula $PMT(1 + \Delta\%)^{N-1}$, as illustrated previously. For example, if a \$1,000 payment is growing at 5% and the value of the 10th payment needs to be known, it is calculated as $\$1,000(1 + 0.05)^{10-1} = \$1,551.33$.
2. **i.** The rate of interest resulting from Formula 9.1, converted to match the payment interval if necessary. The periodic interest rate is adjusted to reflect the net rate by dividing it by $1 + \Delta\%$.
3. **$\Delta\%$.** The periodic percent change matching the payment interval between each successive payment in the annuity. If payments are made quarterly, then the percent change per quarter is required. The value of $\Delta\%$ in these formulas is restricted to a value less than the equivalent periodic interest rate i (where i is expressed as the interest rate per payment interval). Situations where $\Delta\% \geq (1 + i)^{\frac{CY}{PY}} - 1$ are beyond the scope of this textbook. If $\Delta\%$ is assigned a value of 0% then these formulas revert back to Formulas 11.2 to 11.5, respectively. Hence, the formulas from Chapter 11 are sometimes referred to as **zero growth annuity formulas** and represent simplified versions of these complete annuity formulas.



How It Works

Follow the same steps discussed for future value in Section 11.2 and the steps for present value in Section 11.3. The only notable difference is that you must identify the periodic growth rate for the annuity payment and, of course, use the new Formulas 12.1 to 12.4. Figure 12.12 illustrates a typical timeline when constant growth is involved. Note the $\Delta\%$ after the annuity type and that only one of FV_{ORD} , FV_{DUE} , PV_{ORD} , or PV_{DUE} ever appears on the timeline.



The most common applications involving constant growth annuities require you to calculate the future value, present value, or the first annuity payment. In the first two cases, you use Formulas 12.1 to 12.4 to solve for FV and PV . If the first annuity payment is the unknown variable, you can rearrange any of Formulas 12.1 to 12.4 algebraically to isolate PMT as needed.



Important Notes

Most financial calculators, including the Texas Instruments BAII Plus, are not pre-programmed with constant growth annuities. They are designed to handle fixed payment annuities only.

That said, on the web or in your readings you may come across methods that show you how to adapt your calculator input to "trick" the calculator. It is important to note that these methods are generally complex, requiring you to memorize a lot of adaptations and conditional modifications. Outputs are not necessarily correct unless further adapted. Ultimately, these tricks are not recommended. This textbook solves constant growth annuities only using formulas.



Paths To Success

Although all of the discussion has been about growth, you can also use these formulas in situations involving constant reduction since the requirement of $\Delta\% < (1 + i)^{\frac{CY}{PY}} - 1$ remains true. Use a negative value for the growth rate in all calculations.

Additionally, if you wish to know the total value of the annuity payments in a constant growth annuity, you can use Formula 11.2 with a few adaptations:

- Treat the first payment as the annuity payment amount or PMT.
- Substitute the periodic growth rate for the equivalent periodic interest rate while setting CY and PY both to 1. This eliminates the $\frac{CY}{PY}$ exponent.
- Retain using N as the number of payments.

Thus, substituting and simplifying produces the following calculation:

$$\text{Sum of constant growth payments} = \text{PMT} \left[\frac{[(1 + \Delta\%)]^N - 1}{\Delta\%} \right]$$



Give It Some Thought

If an annuity payment is increased by \$5 each time from \$100 to \$105 to \$110 to \$115, does this represent a situation of constant growth?

Solution: No. A constant growth is represented by a fixed percentage growth, not a fixed dollar amount growth.

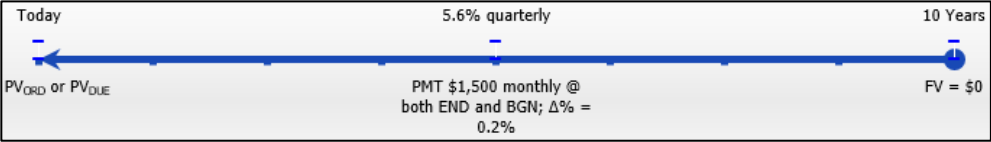
Example 12.2A: Constant Growth RRSP Contributions

Bradford is 27 years old and starts investing \$3,000 per year into his RRSP earning 9% compounded annually. He decides that each year he will increase his contributions by 2.5%. Calculate the maturity value at age 65 for both an ordinary annuity and an annuity due.

Plan	If every payment is increasing by a fixed percentage, then this is a constant growth annuity. Calculate two maturity values, one for the ordinary annuity, or FV_{ORD} , and one for the annuity due, or FV_{DUE} .	
Understand	What You Already Know Step 1: There are annual contributions with an annual compound interest rate. The beginning payment creates a simple constant growth annuity due, while the end payments create an ordinary simple constant growth annuity. The timeline is below. Step 2: Both annuities: $PV = \\$0$, $IY = 9\%$, $CY = 1$, $PMT = \\$3,000$, $PY = 1$, Years = 38, $\Delta\% = 2.5\%$	How You Will Get There Step 3: Apply Formula 9.1. Step 4: With $PV = \\$0$, skip this step. Step 5: Apply Formula 11.1 and Formulas 12.1 and 12.2.
Perform	Step 3: $i = 9\% / 1 = 9\%$ Step 5: $N = 1 \times 38 = 38$ payments $FV_{ORD} = \$3,000(1 + 0.025)^{37} \left[\frac{\left[\frac{(1 + 0.09)^{1/1}}{(1 + 0.025)} \right]^{38} - 1}{\frac{(1 + 0.09)^{1/1}}{(1 + 0.025)} - 1} \right] = \$1,102,199.91$ $FV_{DUE} = \$1,102,199.91 \times (1 + 0.09)^{1/1} = \$1,201,397.90$	
Present	The 38 contributions amount to a total principal contribution of \$186,681.89. The maturity value of the ordinary annuity is \$1,102,199.91 and the annuity due is one compound higher at \$1,201,397.90.	

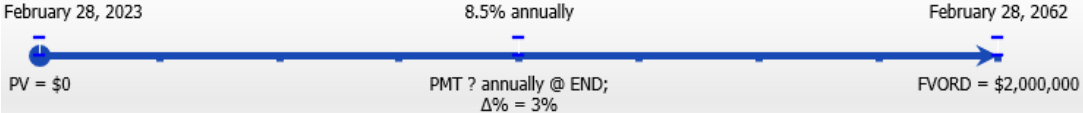
Example 12.2B: Investment Necessary to Support a Constant Growth Annuity

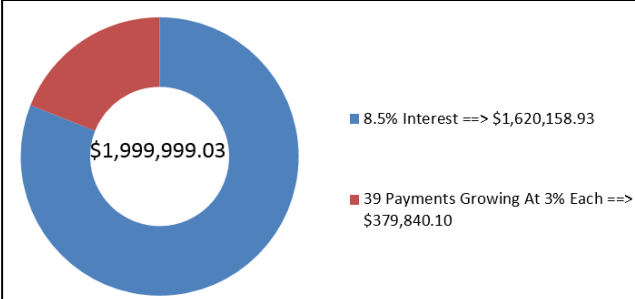
How much money is required today to fund a 10-year annuity earning 5.6% compounded quarterly where the first monthly payment will be \$2,000 and each payment will grow by 0.2%? Calculate as both an ordinary annuity and annuity due.

Plan	If every payment is increasing by a fixed percentage, then this is a constant growth annuity. Calculate two principal amounts, one for the ordinary annuity, or PV_{ORD} , and one for the annuity due, or PV_{DUE} .	
Understand	What You Already Know Step 1: There are monthly payments with a quarterly compound interest rate. The beginning payments create a general constant growth annuity due, while the end payments create an ordinary general constant growth annuity. The timeline appears below. Step 2: Both annuities: $FV = \\$0$, $IY = 5.6\%$, $CY = 4$, $PMT = \\$2,000$, $PY = 12$, $Years = 10$, $\Delta\% = 0.2\%$	How You Will Get There Step 3: Apply Formula 9.1. Step 4: With $FV = \\$0$, skip this step. Step 5: Apply Formula 11.1 and Formulas 12.3 and 12.4.
		
Perform	Step 3: $i = 5.6\% / 4 = 1.4\%$ Step 5: $N = 12 \times 10 = 120$ payments $PV_{ORD} = \frac{\$2,000}{1 + 0.002} \left[\frac{1 - \left[\frac{(1 + 0.002)^{120}}{(1 + 0.014)^{12}} \right]}{\frac{(1 + 0.014)^{\frac{4}{12}}}{(1 + 0.002)} - 1} \right] = \$205,061.74$ $PV_{DUE} = \$205,061.7409 \times (1 + 0.014)^{\frac{4}{12}} = \$205,061.7409 \times 1.004645 = \$206,014.26$	
Present	The 120 payments each increased by 0.2% amounts to a total payout of \$270,944.49. To fund these payouts, an ordinary constant growth annuity requires \$205,061.74 today, whereas a constant growth annuity due requires a little more, equalling \$206,014.26 because the first payment is withdrawn immediately.	

Example 12.2C: Where to Start?

After a discussion with her financial adviser on February 28, 2023, Jennifer has determined that she needs \$2 million in her RRSP when she retires on February 28, 2062. She has decided to start making annual contributions to her RRSP starting February 28, 2024, and grow those contributions by 3% every year. If her RRSP can earn 8.5% compounded annually, in what amount should she make her first contribution?

Plan	If every payment is increasing by a fixed percentage, then this is a constant growth annuity. Calculate the amount of her first annuity payment (PMT).	
Understand	What You Already Know Step 1: There are annual contributions at the end of the interval with an annual compound interest rate. Therefore, this is an ordinary simple constant growth annuity. The timeline appears below. Step 2: $PV = \\$0$, $FV_{ORD} = \\$2,000,000$, $IY = 8.5\%$, $CY = 1$, $PY = 1$, $Years = 39$, $\Delta\% = 3\%$	How You Will Get There Step 3: Apply Formula 9.1. Step 4: With $PV = \\$0$, skip this step. Step 5: Apply Formula 11.1 and Formula 12.1. Once you have substituted and simplified this formula, rearrange it for PMT.
		

Perform	Step 3: $i = 8.5\%/1 = 8.5\%$ Step 5: $N = 1 \times 39 = 39$ payments $\$2,000,000 = PMT(1 + 0.03)^{38} \left[\frac{\left[\frac{(1+0.085)^1}{(1+0.03)} \right]^{39} - 1}{\frac{(1+0.085)^1}{(1+0.03)} - 1} \right]$ $\$2,000,000 = PMT(380.340030)$ $\$5,258.45 = PMT$ $\$2,000,000 = PMT(3.074783) \left[\frac{6.605154}{0.053398} \right]$
Present	Jennifer's first payment is \$5,258.45 and then increased by 3% each year for her 39 payments to provide enough principal to achieve her \$2,000,000 financial goal. Note that because of the rounding of the first annuity payment, she in fact ends up marginally short of her goal in the amount of \$0.97. 



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12.3: Perpetuities

(Are You Going to Live Forever?)

Some of the scholarships offered to students at your college were created decades ago, yet these scholarships continue to pay out money to students each and every year. Where does all of the money come from? Did somebody bestow a tremendously large endowment fund all those many years ago to sustain the scholarship over decades? Or does an individual or supporting organization simply pay it out of pocket each year? How is it possible for these scholarships to pay out through all the past years and all of the foreseeable years?

What would you do if you win a large lottery prize such as Lotto 6/49 or Lotto Max? Today's headlines continually showcase lottery winners who win "the big one," becoming overnight millionaires. With their new-found earnings, they rush out to buy million dollar homes and luxury vehicles and indulge in the pleasures of life. Sometime shortly thereafter, stories are told about how these millionaires file for bankruptcy. What if someone had told these people that instead of spending the money all at once, they could invest it and use the interest to live on forever; whole generations of their family could benefit from the winnings. Just imagine: if you invest \$5 million at 5% interest, you would earn \$250,000 of interest each and every year that you could withdraw endlessly.

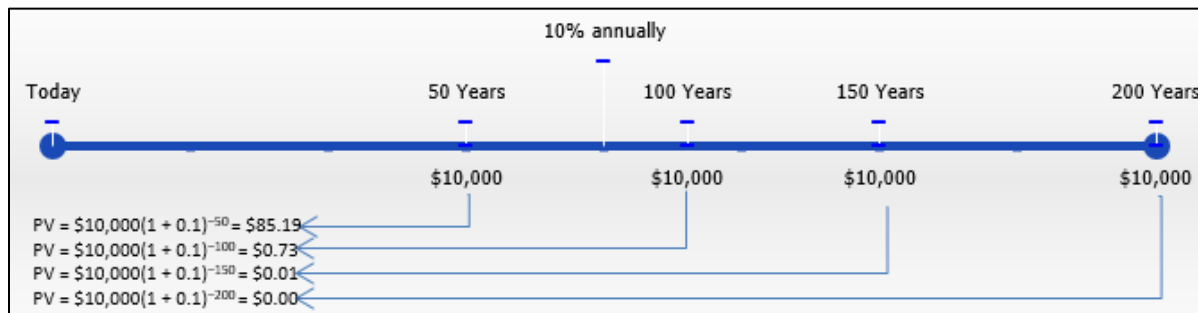
These scenarios highlight the importance of perpetuities, which are annuities that have an infinite term. At your individual level, any sum of money you invest as a perpetuity can be used to generate income forever. At the professional level, many companies and not-for-profit organizations such as sports clubs establish scholarship programs and bursaries for their employees, members, or clients. In some contracts, payments such as royalties continue forever. Some dividend stocks have their price determined by indefinite future dividend amounts.

This section explores the concept of perpetuities. You will calculate the investment required to sustain a perpetuity along with the payment amount.

What Are Perpetuities?

A **perpetuity** is a special type of annuity that has fixed, regular payments continuing indefinitely. If the principal of the investment is never withdrawn, then the interest earned each period can be withdrawn without affecting the future interest earnings of the investment. Therefore, the annuity continues to earn the same amount of interest each and every future interval and can pay out interest forever.

How can a single deposit of principal sustain an infinite amount of payments? Mathematically, money that occurs in a far distant future becomes worthless when it is brought back to today, where it has no value.

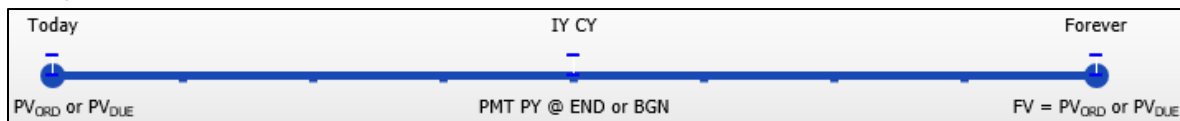


To illustrate the point of money being worthless today, the figure here calculates some present values of \$10,000 payments that occur far into the future. Through the various calculations, notice that when the payments occur farther into the future the present value diminishes to practically nothing. In fact, after about 160 years the \$10,000 becomes worth almost \$0 today. Any payments made after this point in time result in such minuscule additions to the present value that in essence they really have no further impact. Under this premise, it is then possible to determine a value today that is equivalent to the infinite future annuity payment stream.

Ordinary Perpetuities and Perpetuities Due

A perpetuity is a special type of annuity. It comes in both ordinary and annuity due types. As well, the payment frequency and compounding frequency create either a simple or general annuity structure. Perpetuities can even occur after a deferral period, and hence deferred perpetuities are possible.

A typical timeline for a perpetuity appears here. It is identical to any other annuity timeline except for two distinguishing characteristics:



1. **Future Time Period.** Instead of a specific time period being indicated on the right-hand side of the timelines, the word “Forever” appears, to represent the infinite nature of the perpetuity.
2. **Equivalency of FV and PV.** The future value of the perpetuity is the same as the present value since only the interest is ever paid out and the principal is never touched.



The Formula

Like the annuity formulas, the perpetuity formulas are designed to accommodate both simple and general annuities through the $\frac{CY}{PY}$ exponent, which ensures that the compounding interval matches the payment interval. These new formulas represent simplified versions of Formulas 11.4 and 11.5. To illustrate, recall Formula 11.4:

$$PV_{ORD} = PMT \left[\frac{1 - \left[\frac{1}{(1+i)^{\frac{CY}{PY}}} \right]^N}{(1+i)^{\frac{CY}{PY}} - 1} \right]$$

Looking at the numerator, as N becomes increasingly larger the $\left[\frac{1}{(1+i)^{\frac{CY}{PY}}} \right]^N$ approaches a value of zero, which effectively removes it from the equation. This leaves a numerator of 1 over the denominator.

PMT is Perpetuity Payment Amount: The perpetuity payment is the money that is available for withdrawal each payment interval. It is equal to the exact amount of interest that the principal earns during a single payment interval.

+1 is Perpetuity Due Adjustment: The perpetuity due is larger than the ordinary perpetuity by one perpetuity payment since that payment is immediately taken from the principal, reducing the balance to an amount equal to the ordinary perpetuity.

$$PV_{ORD} = \frac{PMT}{(1+i)^{\frac{CY}{PY}} - 1}$$

Formula 12.5: Ordinary Perpetuity Present Value

$$PV_{DUE} = PMT \left(\frac{1}{(1+i)^{\frac{CY}{PY}} - 1} + 1 \right)$$

Formula 12.6: Perpetuity Due Present Value

i is Periodic Interest Rate: This is the rate of interest that is used in converting the interest to principal. It results from Formula 9.1. Continuing with the same requirement as any annuity, the interest rate period must always match the payment interval.



How It Works

The steps required to solve for the present value of a perpetuity remain essentially unchanged from solving for the present value of any annuity. These steps, introduced in Section 11.3, are reviewed for the case of perpetuities:

Step 1: Identify the perpetuity type. Draw a timeline to visualize the question.

Step 2: Identify the variables that may be known, including IY , CY , PMT , PY , and either PV_{ORD} or PV_{DUE} . If a PV_{ORD} or PV_{DUE} is known, then set the FV to the same amount. If no present value is known, then set FV to \$0, since money in the distant future is worthless. Unlike in regular annuities, you do not need to identify Years because of the absence of N from all perpetuity calculations.

Step 3: Use Formula 9.1 to calculate i .

Step 4: You never require this step since the FV is located so far into the future that you can automatically ascertain that its present value is equal to zero.

Step 5: To calculate the present value, apply either Formula 12.5 or Formula 12.6, depending on the type of perpetuity.

As with regular annuities, if you are dealing with a deferred perpetuity then you must modify these procedures as you would to calculate deferred annuities (refer to Section 12.1). Most perpetuity applications require you to calculate either the investment needed (PV_{ORD} or PV_{DUE}) or the payment (PMT). Use algebraic substitution and rearrangement when other variables are required.



Important Notes

The BAII Plus calculator is set up for fixed term annuities only. Therefore, it has no specific built-in function or manner in which to enter a perpetuity. However, since a perpetuity is a specialized version of a regular annuity, a few minor adaptations to the annuity inputs allow you to calculate perpetuities. Input all variables as usual except for the changes shown in the table below.

Variable	Solving for PV_{ORD} or PV_{DUE}	Solving for PMT
FV	Assign $FV = 0$ since money in the distant future is worthless today.	Assign FV equal to either PV_{ORD} or PV_{DUE} , whichever is known. Recall that the principal is never touched.
N	Use a large value such as 10,000.*	Use a large value such as 2,500.*

* This will place the FV far into the distant future. A larger value could be used, but this must be done with caution. Depending on the quantities appearing in the perpetuity calculation, the power resulting from the exponent of a very large N may exceed the computational abilities of the technology and produce an error. If you experience an error, try lowering the value of N by a little bit.



Paths To Success

Formulas 12.5 and 12.6 represent simplified versions of Formulas 11.4 and 11.5. If you find these new formulas confusing, or if you just want to remember a single formula for present values, you can solve any perpetuity question with the Chapter 11 formulas. If you use these formulas, you must substitute an extremely large value of N into the formula using the technology substitutions discussed in the table above.



Give It Some Thought

1. If a single payment is placed today into an ordinary perpetuity, will its future value be higher, lower, or the same?
2. If a lump sum is placed today into a perpetuity due, will its future value be higher, lower, or the same?

Solutions:
 1. It is the same after every withdrawal of interest and higher between withdrawals.
 2. It is the same immediately before any withdrawal of interest and always lower between withdrawals. Recall that the first payment is withdrawn immediately, resulting in a principal that remains smaller than the original amount.

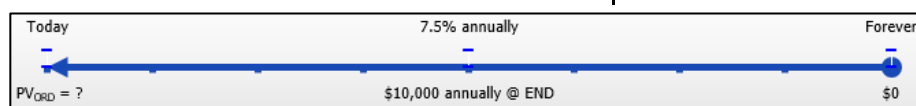
Example 12.3A: Implementing a New Employee Benefit Program

Kendra is the human resources director and wants to offer a new employee perk through a corporate scholarship program. Each year, 10 \$1,000 scholarships are to be made available to the children of the company's employees. If the first scholarships are to be offered annually starting one year from now, what amount must be invested today at 7.5% compounded annually to fund the scholarship program in perpetuity?

Plan Step 1: Calculate an amount today that must be invested such that it can pay out the needed funds at the end of every year in perpetuity. Payment and compounding intervals are the same, therefore this is an ordinary simple perpetuity. Determine the present value (PV_{ORD}).

Understand **What You Already Know**
 Step 1 (continued): The timeline for the perpetuity appears below.
 Step 2: $IY = 7.5\%$, $CY = 1$, $PMT = 10 \times \$1,000 = \$10,000$, $PY = 1$

How You Will Get There
 Step 3: Apply Formula 9.1.
 Step 4: Not needed for perpetuities.
 Step 5: Apply Formula 12.5.



Perform Step 3: $i = 7.5\%/1 = 7.5\%$

Step 5:

$$PV_{ORD} = \frac{\$10,000}{(1 + 0.075)^1 - 1} = \$133,333.33$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
END	10000	7.5	Answer: -133,333.3333	10000	0	1	1

Present If Kendra has her company place \$133,333.33 into the perpetuity account, it will earn \$10,000 interest at 7.5% compounded annually each year, which will fund the scholarship program in perpetuity.

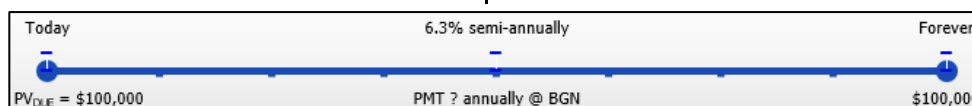
Example 12.3B: How Much Can Be Paid Out?

In memory of his late father, who was a highly successful marketing manager, Brian would like to set up a marketing scholarship program with his university's school of business. He is able to donate \$100,000 from his father's estate to set up the fund. If the perpetuity fund can earn 6.3% compounded semi-annually and the first scholarship is to be provided immediately, what annual scholarship amount can the fund offer?

Plan Step 1: Calculate the amount of the perpetuity payment that the contribution can sustain. Payments and compounding intervals are different, while payments start immediately. Therefore, this is a general perpetuity due. Determine the perpetuity payment amount (PMT).

Understand **What You Already Know**
 Step 1 (continued): The timeline for the scholarship appears below.
 Step 2: $IY = 6.3\%$, $CY = 2$, $PV_{DUE} = \$100,000$, $PY = 1$

How You Will Get There
 Step 3: Apply Formula 9.1.
 Step 4: Not needed in perpetuities.
 Step 5: Substitute into Formula 12.6, rearranging for PMT.



Perform Step 3: $i = 6.3\%/2 = 3.15\%$

Step 5:

$$\$100,000 = PMT \left(\frac{1}{(1 + 0.0315)^2 - 1} + 1 \right)$$

$$\$100,000 = PMT(16.626892)$$

$$\$6,014.35 = PMT$$

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	2500	6.3	-100000	Answer: 6,014.353018	100000	1	2

Present

Brian's donation will allow an immediate scholarship amount of \$6,014.35, which will leave $\$100,000 - \$6,014.35 = \$93,985.65$ principal in the account. This principal amount will be able to earn \$6,014.35 in perpetuity each year.

Example 12.3C: Valuation of Common Shares

One of the methods for valuing common shares is to determine the present value of its future dividends. If IBM Corporation is forecast to have year-end dividends of \$1.50 per share for the next three years followed by \$1.80 per share in perpetuity, how much should an investor be willing to pay if he requires a 12% compounded quarterly rate of return?

Plan

Step 1: There are different compounding and payments, and payments are made at the end of the interval. Therefore, this is a combination of an ordinary general annuity for the first three years followed by an ordinary general perpetuity. Calculate the present value of the common shares (PV_{ORD}).

Understand

What You Already Know

Step 1 (continued): The timeline for the common shares appears below.
Step 2: Ordinary General Perpetuity: $IY = 12\%$, $CY = 4$, $PMT = \$1.80$, $PY = 1$

How You Will Get There

Starting from the right side of the timeline, calculate the present value of the perpetuity first.

Step 3: Apply Formula 9.1.

Step 4: Not needed in perpetuities.

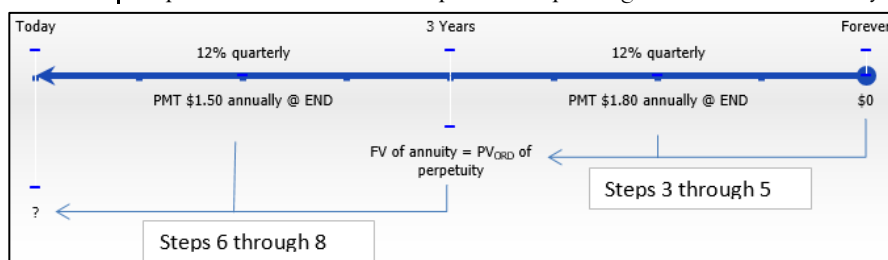
Step 5: Apply Formula 12.5.

Then calculate the present value of the three-year annuity.

Step 6: Apply Formulas 9.2 and 9.5 (rearranging for PV) to find the future value single payment (which is the PV_{ORD} of the perpetuity).

Step 7: Apply Formulas 11.1 and 11.4 to the annuity.

Step 8: Add the results of step 6 and step 7 to get the share value today.



Perform

Step 3: $i = 12\%/4 = 3\%$

$$\text{Step 5: } PV_{ORD} = \frac{\$1.80}{(1 + 0.03)^{\frac{1}{4}} - 1} = \$14.341622$$

Step 6: $N = 4 \times 3 = 12$ compounds

$$\$14.341622 = PV(1 + 0.03)^{12}$$

$$PV = \$14.341622 \div (1 + 0.03)^{12} = \$10.058925$$

Step 7: $N = 1 \times 3 = 3$ payments

$$PV_{ORD} = \$1.50 \left[\frac{1 - \left[\frac{1}{(1 + 0.03)^{\frac{1}{4}}} \right]^3}{(1 + 0.03)^{\frac{1}{4}} - 1} \right] = \$3.568914$$

Step 8: $\$10.058925 + \$3.568914 = \$13.63$

Calculator Instructions

Steps	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
3-5	END	10000	12	Answer: -14.341622	1.8	0	1	4
6-8	✓	3	✓	Answer: -13.627839	1.5	14.341622	✓	✓

Present

If an investor desires a 12% rate of return on his investment, he would be willing to pay \$13.63 per share to receive the dividends in perpetuity.



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12.4: Leases

(I Will Just Borrow It for a While)

Average vehicle costs in North America rose from \$10,668 in 1982 to \$25,683 in 2009,¹ a whopping 141% increase over 27 years. With vehicle purchase prices running well ahead of inflation, leasing now represents an attractive option for many Canadians with many added conveniences. On the upside, lease payments are substantially lower than purchase payments. As a bonus you are able to return your vehicle after a certain number of years and upgrade to the most recent models available. And while you are in possession of the vehicle, it usually remains under the manufacturer warranty, which minimizes your maintenance and repair expenses. On the downside, you never actually own the vehicle and always make car payments, while various research has also calculated that leasing is actually much more expensive than vehicle ownership in the long run.

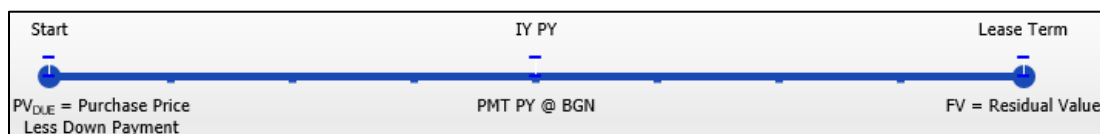
Businesses take advantage of leasing not only for vehicles, but for real estate and office space along with production equipment and office equipment. Businesses constantly need to change to avoid obsolescence, particularly in their fleet of computers. As a business grows, it must expand or even change its office space. Leasing allows organizations to do all of this. Leasing also offers significant tax benefits to most companies as well.

This section explains the basic principles of a lease and then illustrates leasing calculations such as purchase prices, residual values, lease payments, and leasing interest rates.

How Leases Work

A **lease** is a contract by which the owner of an asset gives another party an exclusive right to possess and use the asset under specified conditions for a specific period of time in return for agreed-upon payments. In other words, you are able to borrow something for a period of time and make payments on it. At the end of the lease, the asset is usually returned to the owner, or it can be purchased outright for a depreciated price. The owner of the asset is known as the **lessor**, while the borrowing party is known as the **lessee**.

A typical leasing timeline is illustrated below. Important lease components include the present value, annuity timing, term, interest rate, and future value.



1. **Present Value of the Lease.** The purchase price of an asset is known by many names, such as its selling price, list price, or even the manufacturer's suggested retail price (MSRP). A **down payment**, which is a portion of the purchase price required up front, may or may not be required depending on the asset being leased. If it is required, it is deducted from the purchase price to determine the amount of money being financed.
2. **Annuity Timing.** All leases take the form of annuities due since you must first pay for the item before you are allowed to borrow it. Car dealerships will not let you drive that new car home until you demonstrate the financial capability to pay for it!
3. **Lease Term.** A lease term is no different from an annuity term. It is the specified period of time for which the lessor leases the asset.
4. **Lease Interest Rate.** The nominal interest rate and compounding frequency are determined by the owner of the asset. Most vehicle leases default to rates that are compounded monthly; however, advertisements typically present effective rates.
5. **Future Value of the Lease.** The future value is more commonly known as the **residual value**, which is the projected value of an asset at the end of its lease term. A residual value does not necessarily occur in every lease calculation, as some assets, like computer equipment, may only have an insubstantial value upon lease expiration. Vehicle leases always have a

¹ Statistics Canada, *New Motor Vehicle Sales, June 2011* (Catalogue no. 63-007-X, p. 15), http://publications.gc.ca/collections/collection_2011/statcan/63-007-X/63-007-x2011006-eng.pdf (accessed August 18, 2013).

residual value. Thus, the lessee has a choice to give the car dealership either the keys to the car or a cheque in the amount of the residual value, thereby purchasing the car.

In accounting, under certain circumstances, when an officer of the company signs a lease for the acquisition of a capital good (i.e., a lease of an asset, not a rental of property like a building), this creates a debt. Most leases are closed and cannot be cancelled without severe financial penalties; therefore, a liability is recorded on the balance sheet. The amount recorded is the present value of the remaining lease payments, and it is calculated using a discount rate equivalent to the interest rate the business would have had to pay if the business had purchased the asset instead. The residual value, if any, is included in the amount of this debt, which is known as a **capitalized lease liability**.²

For example, assume a company signed a two-year lease with quarterly payments of \$1,000 and no residual value. Alternatively, it could have purchased the asset and financed it through a bank loan for 8% quarterly. Therefore, applying Formula 11.5, the present value of the eight \$1,000 payments discounted at 8% quarterly is \$7,471.99. This is the amount that appears on the balance sheet.



The Formula

A lease is an annuity due involving some special characteristics. To solve a leasing situation you can use any of the Chapter 11 annuity due formulas.



How It Works

The typical unknown variable in a leasing situation is the present value, future value, lease payment amount, or lease interest rate. To solve a lease for those variables you follow the exact same steps as found in Sections 11.2, 11.3, 11.4, and 11.6.



Things To Watch Out For

Here are some of the most common mistakes made when working with leases:

1. **Cash Flow Violation.** This mistake occurs when the cash flow sign convention on the calculator is violated. From the perspective of the lessee, the present value of the lease is a positive number since the lessee is receiving the asset. Payments on the lease are therefore negative. The future value or residual value of the lease (if other than zero) is also a negative number, since it represents the amount still owing on the asset if the lessee wishes to purchase it.
2. **Confusing the Present Value and the Purchase Price.** When calculating the present value of the lease, be careful to answer the question that is asked. If you need to know how much money was financed for the lease, this is your present value, whereas if you need to know the purchase price of the asset, you must add any down payment to the present value to arrive at the right answer.



Give It Some Thought

Assume two cars are financed at the same amount at the same interest rate for the same period of time, but one is leased and the other is purchased. Are the lease payments larger than, smaller than, or equal to the purchase payments?

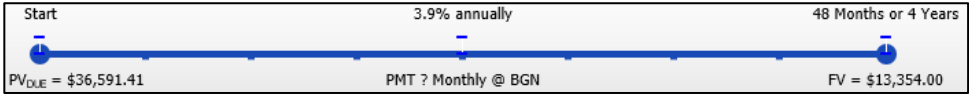
Solution: The lease has a residual value at the end, meaning the lease payments do not have to pay as much principal as would be the case under the purchase payments.

² For illustrative purposes this textbook makes some simplifying assumptions:

1. Accounting rules require the asset to be recorded at the present value of the minimum lease obligations or its fair market value, whichever is less. The examples in this book assume that the present value is the lesser number.
2. Accounting rules require at least one of four specific conditions to be met for a lease to qualify as a capital lease. The examples assume that this criterion is met.
3. Issues involving capital leases such as depreciation or executory costs are ignored.

Example 12.4A: What Will the Lease Payments Be?

A new sports car is being advertised with an MSRP inclusive of all taxes, deliveries, and fees for \$39,091.41. It offers a 48-month lease with a lease rate of 3.9% effectively. The residual value of the vehicle is \$13,354.00. Determine the monthly lease payments if a \$2,500 down payment is made.

Plan	Step 1: This is a leasing question, so payments are made at the beginning of each period. Monthly payments with annual compounding create a general annuity due. Calculate the annuity payment amount (PMT).																		
Understand	What You Already Know Step 1 (continued): The timeline for the vehicle lease appears below. Step 2: $PV_{DUE} = \$39,091.41 - \$2,500.00 = \$36,591.41$, $IY = 3.9\%$, $CY = 1$, $PY = 12$, $Years = 4$, $FV = \$13,354.00$	How You Will Get There Step 3: Apply Formula 9.1. Step 4: The future value needs to be moved to the start, the same date as PV_{DUE}. Apply Formulas 9.2 and 9.3 (rearranged for PV). The calculated PV is money the lease is not required to pay off, so it is subtracted from PV_{DUE} to arrive at the amount required by the annuity. Step 5: Apply Formulas 11.1 and 11.5 (rearranged for PMT) using the calculated step 4 PV_{DUE} amount.																	
																			
Perform	Step 3: $i = 3.9\% / 1 = 3.9\%$ Step 4: $N = 1 \times 4 = 4$ compounds, $\\$13,354 = PV(1 + 0.039)^4$ $PV = \\$13,354 \div (1 + 0.039)^4 = \\$11,459.06497$ new $PV_{DUE} = \\$36,591.41 - \\$11,459.06497 = \\$25,132.34503$ Step 5: $N = 12 \times 4 = 48$ payments $\$25,132.34503 = PMT \frac{1 - \left[\frac{1}{(1 + 0.039)^{\frac{1}{12}}} \right]^{48}}{(1 + 0.039)^{\frac{1}{12}} - 1} \times (1 + 0.039)^{\frac{1}{12}}$ $\$25,132.34503 = PMT \frac{[0.141900]}{[0.003193]} \times 1.003193$ $PMT = \frac{\$25,132.34503}{44.578554} = \$563.78$	Calculator Instructions <table border="1"> <thead> <tr> <th>Mode</th><th>N</th><th>I/Y</th><th>PV</th></tr> </thead> <tbody> <tr> <td>BGN</td><td>48</td><td>3.9</td><td>36591.41</td></tr> </tbody> </table> <table border="1"> <thead> <tr> <th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr> </thead> <tbody> <tr> <td>Answer: -563.776579</td><td>-13354</td><td>12</td><td>1</td></tr> </tbody> </table>		Mode	N	I/Y	PV	BGN	48	3.9	36591.41	PMT	FV	P/Y	C/Y	Answer: -563.776579	-13354	12	1
Mode	N	I/Y	PV																
BGN	48	3.9	36591.41																
PMT	FV	P/Y	C/Y																
Answer: -563.776579	-13354	12	1																
Present	With \$2,500 down on this vehicle, a four-year lease has beginning-of-month payments of \$563.78.																		

Example 12.4B: Recording the Lease Liability

Bradford Enterprises just updated its computer mainframe systems and leased its new equipment under a two-year contract requiring fixed quarterly payments of \$17,300. Alternatively, it could have borrowed the money through a financial institution at 6.5% compounded semi-annually. What liability amount (the capitalized lease liability) should be recorded on the company's balance sheet

- on the day the contract was signed?
- after five payments have been made?

Plan	Step 1: This is a leasing question, so payments are made at the beginning of the period. Quarterly payments with semi-annual compounding create a general annuity due. Calculate the present value (PV_{DUE}) at two time points—today and after five payments.

Understand	What You Already Know Step 1 (continued): The timeline for the computer equipment lease appears below. Step 2: $FV = \\$0$, $IY = 6.5\%$, $CY = 2$, $PMT = \\$17,300$, $PY = 4$, Years = 2	How You Will Get There Step 3: Apply Formula 9.1. Step 4: Since $FV = \\$0$, skip this step. Step 5: Apply Formulas 11.1 and 11.5 twice (for both today and after five payments).																										
Perform	Step 3: $i = 6.5\%/2 = 3.25\%$ Step 5: a. For today, $N = 4 \times 2 = 8$ payments $PV_{DUE} = \$17,300 \left[\frac{1 - \left[\frac{1}{(1 + 0.0325)^{\frac{2}{4}}} \right]^8}{(1 + 0.0325)^{\frac{2}{4}} - 1} \right] \times (1 + 0.0325)^{\frac{2}{4}} = \$130,954.37$ b. After five payments, $N = 4 \times \frac{3}{4} = 3$ payments left $PV_{DUE} = \$17,300 \left[\frac{1 - \left[\frac{1}{(1 + 0.0325)^{\frac{2}{4}}} \right]^3}{(1 + 0.0325)^{\frac{2}{4}} - 1} \right] \times (1 + 0.0325)^{\frac{2}{4}} = \$51,080.99$																											
	Calculator Instructions <table><tr><th>Calculation</th><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Today</td><td>BGN</td><td>8</td><td>6.5</td><td>Answer: 130,954.37</td><td>-17300</td><td>0</td><td>4</td><td>2</td></tr><tr><td>After 5 payments</td><td>✓</td><td>3</td><td>✓</td><td>Answer: 51,080.99</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr></table>		Calculation	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	Today	BGN	8	6.5	Answer: 130,954.37	-17300	0	4	2	After 5 payments	✓	3	✓	Answer: 51,080.99	✓	✓	✓
Calculation	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y																				
Today	BGN	8	6.5	Answer: 130,954.37	-17300	0	4	2																				
After 5 payments	✓	3	✓	Answer: 51,080.99	✓	✓	✓	✓																				
Present	On the day that the computer equipment is leased, the accountants will record a capital lease liability of \$130,954.37. This amount will reduce with each payment, such that after five payments the liability would be \$51,080.99.																											

Example 12.4C: What Is It Worth at the End?

A vehicle advertised in the local paper offers monthly payments of \$728.70 for 48 months at 3.83% compounded monthly. The lease requires a \$5,000 down payment, and the vehicle has an MSRP (including all fees and taxes) of \$53,614.83. What residual value is used in the calculation?

Plan	Step 1: This is a leasing question, so payments are made at the beginning of each period. Monthly payments with monthly compounding create a simple annuity due. Calculate its residual value, which is the same thing as the future value (FV_{DUE}).	
Understand	What You Already Know Step 1 (continued): The timeline for the vehicle lease appears below. Step 2: $PV = \\$53,614.83 - \\$5,000 = \\$48,614.83$, $IY = 3.83\%$, $CY = 12$, $PMT = \\$728.70$, $PY = 12$, Years = 4	How You Will Get There Step 3: Apply Formula 9.1. Step 4: For the PV, apply Formulas 9.2 and 9.3. Step 5: Apply Formulas 11.1 and 11.2. The residual value is the difference between the answers to step 4 (what the lessee owes) and step 5 (what the lessee paid).

Perform	Step 3: $i = 3.83\%/12 = 0.3191\overline{6}\%$ Step 4: $N = 12 \times 4 = 48$ compounds; $FV = \$48,614.83(1 + 0.003191)^{48}$ $= \$56,649.58522$ Step 5: $N = 12 \times 4 = 48$ payments $FV_{DUE} = \$728.70 \left[\frac{\left[(1 + 0.003191)^{\frac{12}{12}} - 1 \right]^{48}}{(1 + 0.003191)^{\frac{12}{12}} - 1} \right] \times (1 + 0.003191)^{\frac{12}{12}}$ $= \$37,854.63271$ Residual Value $= \$56,649.58522 - \$37,854.63271 = \$18,794.95$	Calculator Instructions <table><tr><td>Mode</td><td>N</td><td>I/Y</td><td>PV</td></tr><tr><td>BGN</td><td>48</td><td>3.83</td><td>48614.83</td></tr></table> <table><tr><td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>-728.7</td><td>Answer: -18,794.95252</td><td>12</td><td>12</td></tr></table>	Mode	N	I/Y	PV	BGN	48	3.83	48614.83	PMT	FV	P/Y	C/Y	-728.7	Answer: -18,794.95252	12	12
	Mode	N	I/Y	PV														
BGN	48	3.83	48614.83															
PMT	FV	P/Y	C/Y															
-728.7	Answer: -18,794.95252	12	12															
Present	In determining the lease payments, it is expected to have a residual value of \$18,794.95 after the four-year lease term expires. The owner will either have to pay this amount or give the keys back.																	

Example 12.4D: What Rate Are They Using?

An advertisement on television tells you that you can lease a family sedan for \$161.00 biweekly for five years. The MSRP of the vehicle is \$26,480, and a \$1,500 down payment is required. If the residual value of the vehicle is \$8,146.16, what annually compounded interest rate does the calculation use?

Plan	Step 1: This is a leasing question, so payments are made at the beginning of each period. Monthly payments with monthly compounding create a simple annuity due. Calculate the annually compounded interest rate (IY) with CY = 1.																	
Understand	What You Already Know Step 1 (continued): The timeline for the vehicle lease appears below. Step 2: $PV_{DUE} = \\$26,480 - \\$1,500 = \\$24,980$, CY = 1, PMT = \$161, PY = 26, Years = 5, FV = \$8,146.16	How You Will Get There Step 3: Apply Formulas 11.1 and 11.5. Enter this information into the calculator to solve for IY (which automatically applies Formula 9.1 to convert the i to an IY).																
Perform	Step 3: $N = 26 \times 5 = 130$ payments $\$24,980 = \$161 \left[\frac{1 - \left[\frac{1}{(1 + i)^{\frac{1}{26}}} \right]^{130}}{(1 + i)^{\frac{1}{26}} - 1} \right] \times (1 + i)^{\frac{1}{26}}$ See calculator instructions for the solution. IY = 4.99%	Calculator Instructions <table><tr><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>BGN</td><td>130</td><td>Answer: 4.989999</td><td>24980</td><td>-161</td><td>-8146.16</td><td>26</td><td>1</td></tr></table>	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	BGN	130	Answer: 4.989999	24980	-161	-8146.16	26	1
	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y										
BGN	130	Answer: 4.989999	24980	-161	-8146.16	26	1											
Present	To arrive at payments of \$161 biweekly on the family sedan with \$1,500 down and a residual value of \$8,146.16, a 4.99% compounded annually interest rate is being used.																	



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12.5: Application: How to Purchase a Vehicle

(Get Your Motor Running)

What is the cheapest way to acquire a vehicle? With 20 million cars on the road for 12 million Canadian households, that is an average of 1.7 cars per household in Canada. In fact, on average Canadians purchase about 10 to 12 cars in their lifetime. With low-end average prices in the \$25,000 to \$30,000 range per vehicle, and factoring in rising inflation over your lifetime, you will spend approximately \$550,000 on acquiring vehicles! Yet despite this huge amount of money being spent, shockingly few consumers take the time to analyze their financing options critically. Would you spend \$550,000 without doing some research? I thought not.

This section guides you through the practical task of finding the cheapest of the alternatives for obtaining a vehicle. It lays out the four primary means of vehicle ownership and develops a procedure through which you can make a smart purchase decision.

What Are the Choices?

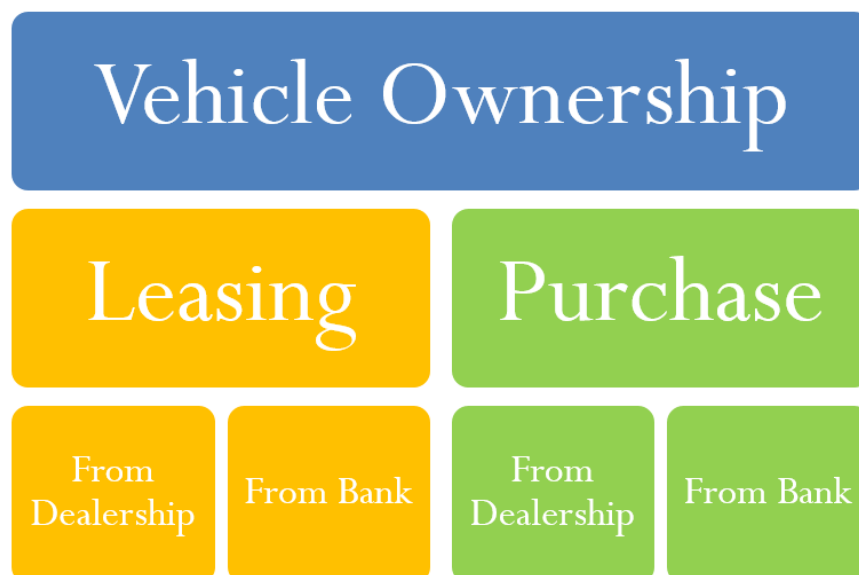
A few matters need to be cleared up from the outset. This section sticks to examining vehicle leases aimed at eventual ownership. Studies of vehicle leases have revealed that leasing a vehicle without intent to own is much more costly than ownership, even after factoring in maintenance and repairs. Give it some thought: Leasing a vehicle with no intent ever to own it means that you will be making monthly car payments for the rest of your driving life. That is called the "forever-in-debt" club. That does not sound like much fun, does it?

Smart vehicle ownership means exploring the various financial paths to vehicle ownership and ultimately choosing the lowest cost option that results both in 100% ownership of the vehicle for as long as you decide to retain the vehicle and in eventual termination of your car payments. That definitely sounds much more financially appealing.

One of the ways to purchase the vehicle is to pay cash for it. While this is technically possible, very few Canadians have \$30,000 lying around that they can just write a cheque against. Therefore, this option is not pursued as a normal alternative. If you cannot pay cash, then you must finance the purchase by borrowing the money from an organization.

The figure to the right illustrates the four paths available for a typical vehicle acquisition. The two basic options are to lease the vehicle or purchase it. You can borrow the money from either the dealership or a financial institution of your choice (from here on just called a *bank* for simplicity), such as RBC or a credit union. Note that if you lease the vehicle, you intend to pay the residual value upon the termination of the lease term.

Through the Internet and publications such as newspapers and flyers, you can usually obtain a lot of information about each of these four options without having to visit the dealership or a financial institution. Sometimes you may have to look in the fine print. The goal is to explore your monthly payment under equal conditions for each of the four options and then select the lowest payment.





The Formula

Borrowing funds for a car requires either a loan or a lease. Both options represent annuities and have previously been discussed in this textbook. Therefore, to figure out your lowest cost alternative to vehicle ownership you do not need any new formulas, just a new process.

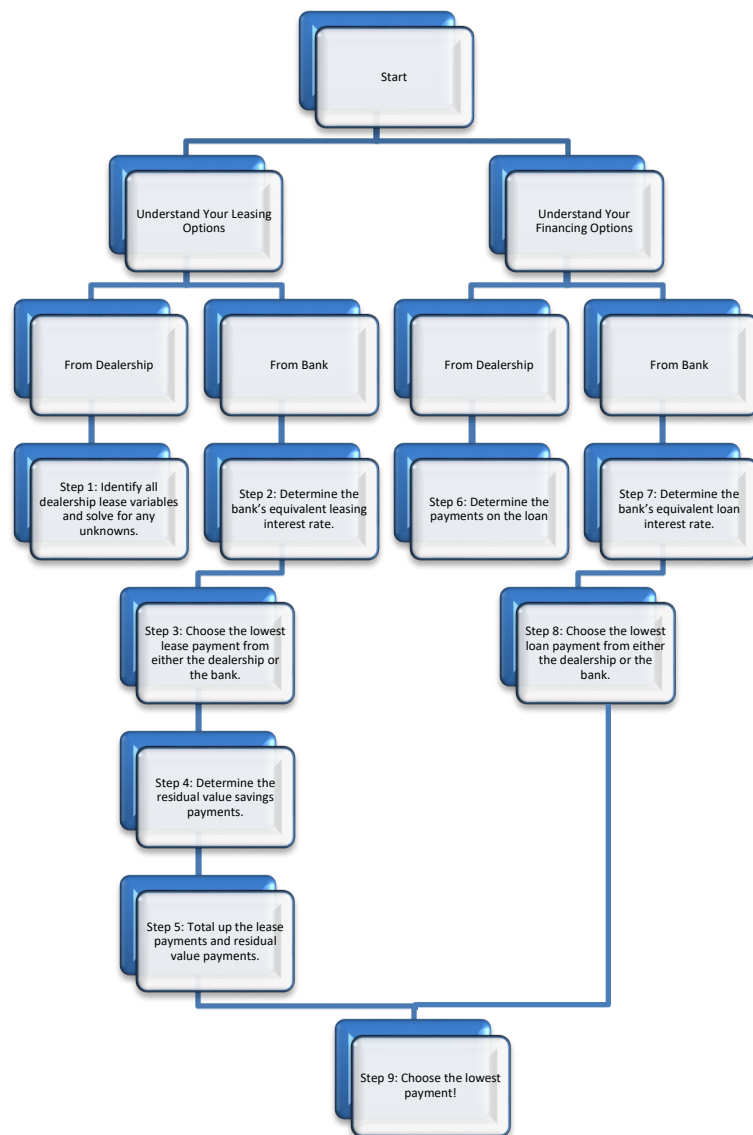


How It Works

The secret to choosing well is that most vehicles actually have two prices at the dealership. One is the financing price and the other is a cash price, which results from a cash rebate that lowers the price. To be eligible to receive this cash rebate, the dealership must receive payment in full for the vehicle on the date of purchase. If you elect to have a bank be your source of funds, then the bank will provide a cheque to the dealership for the vehicle on your behalf, and therefore you will have paid cash! Thus, you need to borrow less money from the bank to acquire the vehicle since the cash price is lower. If you elect for the dealership to be your source of funds, then the dealership is not paid in full right away, and you are not eligible for the cash rebate. Thus, you pay the financing price, which is higher.

Of course, the dealerships want you to borrow from them since you pay the higher price and they receive an additional revenue stream from the interest they charge you. Dealerships put very tempting offers in front of consumers. Because most consumers do not understand the time value of money, they fail to compare these offers with other alternatives. Do you pay more for the car but at a lower interest rate from the dealership, or do you pay less for the car but at a higher interest rate from the bank? Only a time value of money calculation can determine that. The consumer's best decision is based on which alternative has the lowest payment under equal conditions.

The steps for figuring out the lowest cost automotive option are illustrated in the figure to the right. To understand the process illustrated, it is much easier to obtain information on vehicles from dealership advertisements or websites. Most commonly, the price of the vehicle is the same regardless of which dealership you might visit. Therefore, acquiring the vehicle from a dealership commonly represents a singular path with known information. On the other hand, approximately 75 banks and 485 credit unions are available to consumers in Canada. This represents up to 560 different choices, from which it would be extremely time consuming



to gather all required information. The process establishes a threshold against which the banks are compared. You need contact only those financial institutions that pass the initial screening.

Step 1: Understand your leasing terms from the dealership. You must identify your leasing variables, including the lease interest rate, lease term, leasing payment structure, any down payment required, the purchase price, and the residual value. If any of these variables is unknown, identify its value by applying the appropriate annuity due formula.

Step 2: Determine the equivalent leasing interest rate from your bank. Banks advertise interest rates, not payments. Thus, you calculate the interest rate a bank must charge to be equivalent to the dealership using the identical leasing structure using the same lease term, lease payments, compounding, and residual value. The present value used in this calculation is the dealer's purchase price from step 1 less the cash rebate. Solve for the bank's equivalent nominal interest rate (IY).

Step 3: Make your leasing decision. After conducting some research, if you find a bank with a lower interest rate than the calculated IY from step 2, then choose the bank lease (you should contact the bank to confirm the rate). Recalculate the leasing payments at the bank's interest rate. If you cannot find a lower interest rate from a bank, then select the dealership lease payments from step 1.

Step 4: A lease requires you to save up for the residual value. In choosing a leasing option, you also must save enough funds to pay off the residual value at the lease end. Set up an annuity due savings account using the same term and payment frequency. Use an appropriate interest rate from your savings institution. Calculate the annuity payment required to meet your residual value savings goal.

Step 5: Total the lease payment. Your lease payment from step 3 plus the residual value annuity payment from step 4 is the total out-of-pocket commitment required per payment interval to own the vehicle through leasing.

That completes the analysis of the leasing alternative. The remainder of the steps now involve examining your financing options.

Step 6: Figure out the loan payment from the dealership. Using the same term and payment frequency as the lease, calculate the payments to purchase the vehicle. Remember that a vehicle loan is an ordinary annuity. Use the financing price of the vehicle and the financing interest rate (not the lease rate) offered by the dealership.

Step 7: Figure out the equivalent loan interest rate from your bank. Using the same process as in step 2, aim to determine the equivalent interest rate from a bank loan that matches the payments with the dealership. With an identical structure and using the cash price, calculate the bank's equivalent nominal interest rate (IY).

Step 8: Make the loan decision. If a bank offers an interest rate lower than the calculated IY from step 7, then select the bank loan (and you should probably contact the bank to confirm the rate). Recalculate the loan payments at the bank's interest rate. If you are unable to find a bank with a lower interest rate, then select the dealership loan payments from step 6.

Step 9: The final decision. In step 5, you calculated the total lease payment. From step 8, you have the lowest loan payment. The obvious choice is the lowest payment.



Important Notes

1. At any given point in time, numerous vehicle rebates that adjust the purchase price of the vehicle are available (subject to various conditions). If such rebates apply to your purchase situation, modify the vehicle's purchase price accordingly in your calculations.
2. While leases are generally non-negotiable, the purchase price of a vehicle may be negotiable depending on the dealership and automotive manufacturer's policies. If haggling is allowed, estimate the amount by which you could reduce the purchase price and use the reduced amount for the calculations in steps 6 and 7.
3. If you have a trade-in for your vehicle acquisition, the value of the trade-in is treated in the same manner as a down payment toward the vehicle. It reduces the purchase price, so you modify your calculations accordingly.
4. The fundamental concept of time value of money says that to decide between the different leasing and financing timelines, all money must be moved to the same focal date. Note that in the process explained above you are making the final decision by comparing the payment amounts on the annuity due against the payment amounts on the ordinary annuity. A clear discrepancy in the timing of the payments is evident. In technically correct financial theory, to obey the fundamental

concept of time value of money these payment amounts should then be present valued to the start of the transaction (the focal date) and the lower present value should be chosen. However, the reality is that most consumers base their purchase decision on the payment amount and what they can afford rather than on the total amount. Therefore, it is more logical to base the car acquisition decision strictly on the physical out-of-pocket annuity payment amounts.

Example 12.5A: How Should You Acquire This Vehicle?

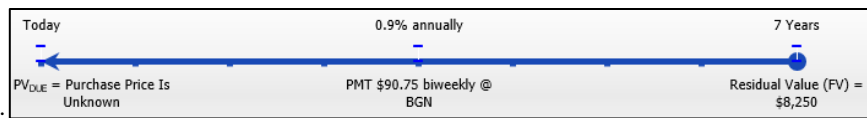
Haber Suzuki advertises a Suzuki Hatchback for sale. The following information is known:

- Leasing information: Seven-year term, 0.9% compounded annually lease rate, \$90.75 biweekly payments, \$8,250 residual value, no down payment required.
- Purchase information: Cash rebate of \$3,500 available, purchase financing at 2.99% compounded annually.
- Bank options: The lowest vehicle lease rate from a bank is 4.4% compounded annually, while the lowest loan rate is 8.25% compounded annually.
- Savings accounts: The best rate available is 2% compounded annually.

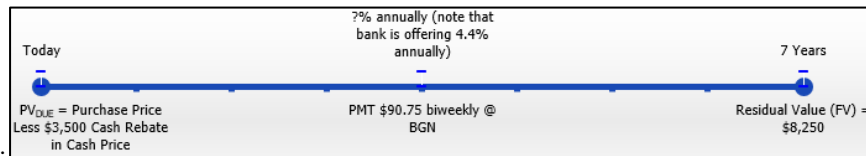
What is the lowest cost method to purchase this vehicle?

Plan	Examine the four paths to vehicle ownership and choose the path with the lowest biweekly payment.	
Understand	What You Already Know There are five timelines, one for each of the options (two leases, one savings account, two purchases): Dealership Lease: $IY = 0.9\%$, $CY = 1$, $PMT = \\$90.75$, $PY = 26$, $FV = \\$8,250$, $Years = 7$ Bank Lease: $PV_{DUE} = \text{Purchase Price} - \\$3,500$, $CY = 1$, $PMT = \\$90.75$, $PY = 26$, $FV = \\$8,250$, $Years = 7$ Residual Savings Plan: $PV = \\$0$, $IY = 2\%$, $CY = 1$, $PY = 26$, $FV = \\$8,250$, $Years = 7$ Dealership Loan: $PV_{ORD} = \text{Purchase Price}$, $IY = 2.99\%$, $CY = 1$, $PY = 26$, $FV = \\$0$, $Years = 7$ Bank Loan: $PV_{ORD} = \text{Purchase Price} - \\$3,500$, $CY = 1$, $PMT = \text{Dealer loan payment}$, $PY = 26$, $FV = \\$0$, $Years = 7$	How You Will Get There Step 1: In the dealership lease, the purchase price of the vehicle for the lease is missing. Find the present value of the residual value by applying Formulas 9.1, 9.2, and 9.3 (rearranged for PV). Then apply Formulas 11.1 and 11.5 to find the present value of the lease payments. Add the PV and PV_{DUE} to arrive at the purchase price. Step 2: In the bank lease, calculate the equivalent bank leasing interest rate. Adjust the purchase price to reflect the cash price and use your calculator to solve Formula 11.5 for IY. Step 3: If the bank rate is higher, choose the payment from step 1. If the bank rate is lower, then recalculate the lease payment using Formulas 9.1, 9.2, and 9.3 (rearranged for PV) to adjust the PV_{DUE} and apply Formula 11.5 (rearranged for PMT). Step 4: In the residual savings plan, save to pay off the residual value using the savings account interest rate. Apply Formulas 9.1 and 11.3 (rearranged for PMT). Step 5: Add the payments from steps 3 and 4. This is the total lease payment. Step 6: In the dealership loan, calculate the loan payments from the dealership by applying Formulas 9.1 and 11.4 (rearranged for PMT). Step 7: In the bank loan, calculate the equivalent bank loan interest rate. Adjust the purchase price and use your calculator to solve Formula 11.4 for IY. Step 8: If the bank rate is higher, choose the payment from step 6. If the bank rate is lower, recalculate the loan payment, first using Formulas 9.1 and 9.3 (rearranged for PV) to adjust the PV_{ORD} and then using Formula 11.4 (rearranged for PMT). Step 9: Choose the lowest payment from step 5 or step 8.

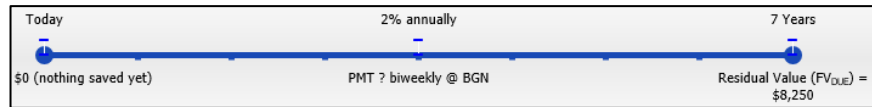
Dealership Lease:



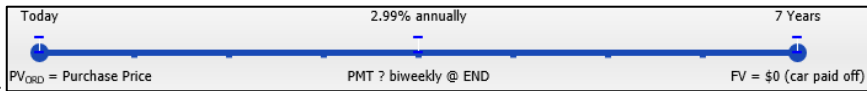
Bank Lease:



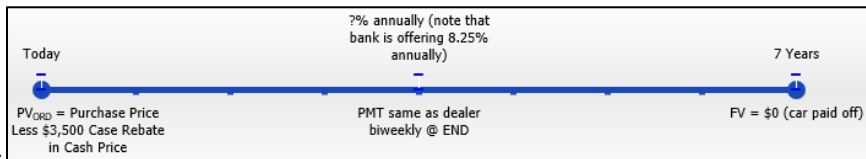
Residual Savings:



Dealership Loan:



Bank Loan:

**Step 1:** Residual Value: $i = 0.9\%/1 = 0.9\%$; $N = 1 \times 7 = 7$ compounds;

$$\$8,250 = PV(1 + 0.009)^7$$

$$PV = \$8,250 \div (1 + 0.009)^7 = \$7,748.466949$$

Lease Payments: $N = 26 \times 7 = 182$ payments

$$PV_{DUE} = \$90.75 \left[\frac{1 - \left[\frac{1}{(1 + 0.009)^{\frac{1}{26}}} \right]^{182}}{(1 + 0.009)^{\frac{1}{26}} - 1} \right] \times (1 + 0.009)^{\frac{1}{26}} = \$16,011.97657$$

Total PV = $\$7,748.466949 + \$16,011.97657 = \$23,760.44$ = purchase price of the vehicle**Step 2:** Cash price of vehicle = $\$23,760.44 - \$3,500.00 = \$20,260.44$. Solve Formula 11.5 (adjusting for residual value):

$$\$20,260.44 - \frac{\$8,250}{\left((1 + i)^{\frac{1}{26}} \right)^{182}} = \$90.75 \left[\frac{1 - \left[\frac{1}{(1 + i)^{\frac{1}{26}}} \right]^{182}}{(1 + i)^{\frac{1}{26}} - 1} \right] \times (1 + i)^{\frac{1}{26}}$$

From calculator, $IY = 4.5331\%$ compounded annually.**Step 3:** The bank lease rate of 4.4% is lower than the equivalent lease rate of 4.5331%. Choose the lease from the bank and recalculate the payment. $i = 4.4\%/1 = 4.4\%$; $N = 1 \times 7 = 7$ compounds;

$$\$8,250 = PV(1 + 0.044)^7$$

$$PV = \$8,250 \div (1 + 0.044)^7 = \$6,103.099605$$

$$\text{new } PV_{DUE} = \$20,260.44 - \$6,103.099605 = \$14,157.34039$$

 $N = 182$ payments (same as step 1)

$$\$14,157.34039 = PMT \left[\frac{1 - \left[\frac{1}{(1 + 0.044)^{\frac{1}{26}}} \right]^{182}}{(1 + 0.044)^{\frac{1}{26}} - 1} \right] \times (1 + 0.044)^{\frac{1}{26}}$$

$$\$14,157.34039 = PMT \left[\frac{0.260230}{0.001657} \right] \times 1.001657$$

Perform

$$PMT = \frac{\$14,157.34039}{157.261348} = \$90.02$$

Step 4: $i = 2\%/1 = 2\%$; $N = 182$ payments (same as step 1)

$$\$8,250 = PMT \left[\frac{\left[(1 + 0.02)^{\frac{1}{26}} - 1 \right]^{182}}{(1 + 0.02)^{\frac{1}{26}} - 1} \right] \times (1 + 0.02)^{\frac{1}{26}}$$

$$\$8,250 = PMT \left[\frac{0.148685}{0.000761} \right] \times 1.000761$$

$$PMT = \frac{\$8,250}{195.292254} = \$42.24$$

Step 5: The total lease payment is $\$90.02 + \$42.24 = \$132.26$ biweekly.

Step 6: Purchase price = $\$23,760.44$ (from step 1); $i = 2.99\%/1 = 2.99\%$; $N = 182$ payments (same as in step 1)

$$\$23,760.44 = PMT \left[\frac{1 - \left[\frac{1}{(1 + 0.0299)^{\frac{1}{26}}} \right]^{182}}{(1 + 0.0299)^{\frac{1}{26}} - 1} \right]$$

$$\$23,760.44 = PMT \left[\frac{0.186355}{0.001133} \right]$$

$$PMT = \frac{\$23,760.44}{164.365992} = \$144.56$$

Step 7: Cash price of vehicle = $\$20,260.44$ (from step 2).

$$\$20,260.44 = \$144.56 \left[\frac{1 - \left[\frac{1}{(1 + i)^{\frac{1}{26}}} \right]^{182}}{(1 + i)^{\frac{1}{26}} - 1} \right]$$

From calculator, $IY = 8.0841\%$ compounded annually.

Step 8: Since the bank loan rate of 8.25% is higher than the equivalent loan rate of 8.0841% , remain with the dealership loan and make payments of $\$144.56$.

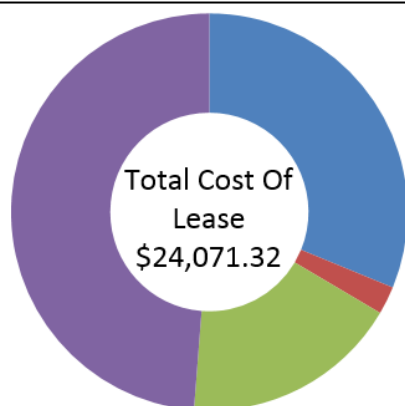
Step 9: The lowest payment is from step 5, which is $\$132.26$.

Calculator Instructions

Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	BGN	182	0.9	Answer: 23,760.44352	-90.75	-8250	26	1
2	✓	✓	Answer: 4.533113	20260.44	✓	✓	✓	✓
3	✓	✓	4.4	✓	Answer: -90.024284	✓	✓	✓
4	✓	✓	2	0	Answer: -42.244378	8250	✓	✓
6	END	✓	2.99	23760.44	Answer: -144.558127	0	✓	✓
7	✓	✓	Answer: 8.084126	20260.44	-144.56	✓	✓	✓

Present

The best option to acquire this vehicle is to lease it from the bank. This requires biweekly lease payments of $\$90.02$ and residual savings payments of $\$42.24$ for a total of $\$132.26$ per period for seven years. At the end of the lease, the accumulated savings is used to pay the $\$8,250$ residual value.



- Contributions to Residual Value Savings Plan ==> \$7,687.68
- 2% Interest Earned On Residual Value Savings Plan ==> \$562.32
- 4.4% Interest Paid Through Lease Payments ==> \$4,373.20
- Principal Paid Through Lease Payments ==> \$12,010.44



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12.6: Application: Planning Your RRSP

(When Can I Retire?)

How and when should you start planning your RRSP? When you are in your late teens or early twenties, you are probably not thinking much about your retirement income. You are at a stage in life when, besides paying for education or paying off student loans, you move out on your own and need to acquire many possessions ranging from appliances to furniture to transportation. Your income is probably at its lowest point in your career. You find yourself on a limited budget with high demands.

Ironically, though, this is the best time to start your RRSP. Throughout the last few chapters you have already witnessed the incredible power of compound interest. Through discussion, examples, and even some of the exercises, it should be clear that the earlier you invest your money, the less you have to pay out of your pocket to reach your retirement income goals.

But why should you go to the trouble and expense of contributing to an RRSP? For those counting on the government to provide retirement income, it was pointed out that the typical Canadian in retirement earns \$689.17 per month from the Canada Pension Plan (CPP) and \$592.29 from Old Age Security (OAS). Both of these amounts are pre-tax, which means your net income from these sources will be even less. This is not much to live off of.

Are companies not supposed to provide pension plans for their employees? While some companies do offer pension plans, these public and private-sector companies are few and far between. The Pension Investment Association of Canada (PIAC) manages approximately 130 pension funds representing approximately 80% of Canada's pension industry by asset size. In a study conducted by the PIAC, it found that approximately 20.5% of Canadian membership companies are either closing or considering closing their defined benefit pension funds. In Ontario, the percentage is much higher at 36.2%. These closures are in the private sector. Additionally approximately 20% of Canadian organizations are reducing future benefits.³

The bottom line is that you should not count on somebody else to create your retirement income for you. Also, do not bank on being lucky enough to work for a company that has a good pension plan. You need to take care of yourself.

This section introduces a simplified model of RRSP planning that reasonably approximates the financial commitment that will provide a satisfactory retirement income. Ultimately, you should always consult a financial adviser as you plan your RRSP.

Income Planning

Income planning requires you to project future earnings, determine income needs in retirement, and then develop a savings plan toward that goal. Look at these details:

1. **Projected Retirement Income.** The first step is to determine your annual retirement income at the age of retirement. The general consensus is that a retiree needs approximately 70% of pre-retirement gross income to live comfortably and maintain the same lifestyle. It is difficult to know what that amount will be 30 or 40 years from now. However, today's typical retirement income is known. The average Canadian needs about \$40,000 of gross income. You can adjust this number higher if necessary. You can then project today's amount forward to the age of retirement using an estimated rate of inflation based on historical data.
2. **Principal Required at Age of Retirement.** The second step requires you to determine the principal needed to fund the annual income requirement at the age of retirement. You must estimate the number of years for which withdrawals are to be made from the RRSP balance. In essence, how long are you going to live? While no one knows the answer, the average Canadian male lives to age 78 while the average Canadian female lives to age 82. These are good starting points. It would not make sense to lower these numbers since you do not want to be caught short, but you could raise these numbers if necessary. A look at your parents, grandparents, and other ancestors combined with your own lifestyle choices might indicate the typical life span you may expect. While you are not guaranteed to live that long, at least you can base a reasonable financial decision upon it. You must also factor in inflation during the retirement years. Each year, your

³ Pension Investment Association of Canada, *Pension Plan Funding Challenges: 2007 PIAC Survey*, December 2007, www.piacweb.org/files/Pension%20Plan%20Funding%20Challenges%202007%20PIAC%20Survey%20Dec%202007.pdf (accessed September 26, 2010).

income must rise to keep up with the cost of living. Finally, you should use a conservative, low-risk interest rate in these calculations because you certainly cannot afford to lose your savings to another "Black Monday" (which refers to October 19, 1987, when stock markets around the world crashed).

3. **Annuity Savings Payments.** The final step is to determine the payments necessary to reach the principal required for retirement. You should consider that your income generally rises over your lifetime, which means as you get older you could afford to contribute a larger amount to your RRSP. Thus, contributions start small when you are young and increase as you grow older. The interest rate you use must reflect the market and your risk tolerance. Since 1980, the Toronto Stock Exchange (TSX) has averaged approximately 6% annual growth, which is a good starting point for determining what interest rate to use in your calculations.



The Formula

RRSP planning does not involve any new formulas. Instead, you must combine previously studied single payment concepts from Chapter 9 and annuity concepts from both Chapters 11 and 12.



How It Works

Your end goal is to calculate the regular contributions to your RRSP necessary to achieve a balance in your RRSP from which you can regularly withdraw in retirement to form your income. The calculations presented ignore other sources of retirement income such as the CPP or OAS, as if you were solely responsible for your own financial well-being. You can then treat any other income as bonus income. Follow the steps :

Step 1: Calculate the annual retirement income you will need. Choose a value of annual income in today's dollars along with an annual rate of inflation to use. Then, by applying Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compound Periods for Single Payments), and 9.3 (Compound Interest for Single Payments) you can move that income to your required age of retirement.

Step 2: Calculate the present value of the retirement income. Most people receive their retirement income monthly, so you divide the result from step 1 by 12. Retirement income usually starts one month after retirement, thus forming an ordinary annuity. Using a reasonable annual rate of inflation, you also divide the inflation rate by 12 to approximate the growth rate to be used in calculating a constant growth annuity required during retirement. Estimate the number of years the retirement fund must sustain and select a low-risk conservative interest rate. Then apply Formulas 9.1 (Periodic Interest Rate), 11.1 (Number of Annuity Payments), and 12.3 (Present Value of a Constant Growth Ordinary Annuity) to arrive at the principal required at the age of retirement.

Step 3: Calculate the annuity payment required to achieve your goal. If a single payment is already invested today, deduct its future value at the age of retirement (using Formulas 9.1, 9.2, and 9.3) from the amount of money determined in step two. The principal at the age of retirement (from step 2) now becomes the future value for the ordinary constant growth annuity. Use an interest rate that reflects market rates and your risk level, along with an appropriate growth rate you select for the annuity payments you will make. This growth rate may or may not match the growth rate determined in step 2. Thus, arriving at the annuity payment amount requires Formulas 9.1 (Periodic Interest Rate), 11.1 (Number of Annuity Payments), and Formula 12.1 (Future Value of a Constant Growth Ordinary Annuity) rearranged for PMT.



Important Notes

This procedure approximates the required regular RRSP contribution. However, recognize that the procedure is somewhat simplified and certainly does not factor in all variables that may apply in an individual situation. It is always best to consult a certified financial planner to ensure that your RRSP plan works under current market conditions and under any applicable restrictions.

Also, the steps listed above also assume an ordinary annuity structure. In the event of an annuity due, substitute the appropriate annuity due formula as required.

**TOP
SECRET**

Paths To Success

One of the hardest tasks in planning an RRSP is to guess interest rates and rates of inflation. If you are unsure of what numbers to use, some safe values are as follows:

1. A rate of inflation or growth rate of 3% compounded annually, equalling 0.25% per month
2. During retirement, an interest rate of 4% compounded annually
3. While you contribute toward your RRSP, an interest rate of 6% compounded annually, based on the approximate historical average for the TSX.



Give It Some Thought

Assume that two people—Person A and Person B—are saving for the same retirement amount under equal conditions except as noted. In each of the following cases, determine whose RRSP contributions are higher:

- a. Person A starts contributing at age 18 while Person B starts contributing at age 25.
- b. Person A earns 7% compounded annually while Person B earns 6% compounded annually.
- c. Person A contributes monthly while Person B contributes semi-monthly.
- d. Person A contributes through an ordinary annuity while Person B uses an annuity due.
- e. Person A earns 4.5% interest during retirement while Person B earns 4% interest during retirement.
- f. Person A has quarterly compounded earnings while Person B has monthly compounded earnings.

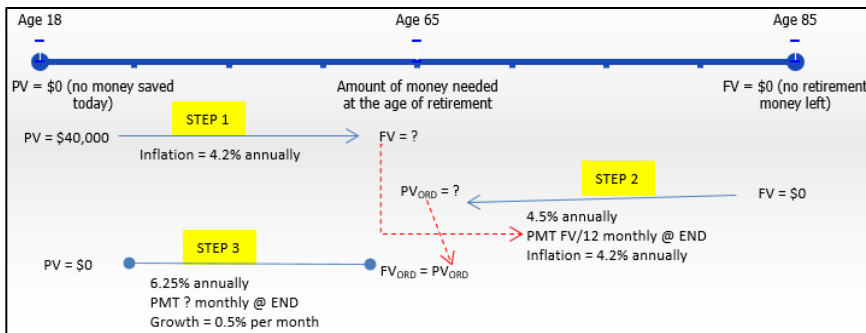
Solutions:
 a. Person B, because he has a shorter period to accumulate the same amount as Person A.
 b. Person B, because a lower interest rate requires more principal, hence larger contributions.
 c. Person A, because the principal does not increase as often as Person B's does.
 d. Person A, because there is one fewer compounds than for Person B.
 e. Person B earns less interest, so a larger principal is required at retirement than Person A requires.
 f. Person A, because the interest is not converted to principal as often as it is for Person B.

Example 12.6A: An 18-Year Old with No Savings Planning for Retirement

Jesse just turned 18 and plans on retiring when he turns 65. In today's funds, he wants to earn \$40,000 annually in retirement. Based on the past 50 years, he estimates inflation at 4.2% compounded annually. He would like to receive 20 years of monthly payments from his retirement money, which is forecasted to earn 4.5% compounded annually. He believes his RRSP can earn 6.25% compounded annually during his contributions, he has no money saved to date, and he would like to increase each payment by 0.5%. One month from now, what is the amount of his first monthly RRSP contribution?

Plan

Jesse wants to make regularly increasing contributions to his RRSP at the end of the monthly payment interval with annual compounding. Therefore, this is a constant growth general ordinary annuity. Calculate his first payment (PMT).

Understand	What You Already Know Jesse's RRSP plan appears in the timeline. Step 1: $PV = \\$40,000$, $IY = 4.2\%$, $CY = 1$, Years = 47 Step 2: $FV = \\$0$, $IY = 4.5\%$, $CY = 1$, $PY = 12$, Years = 20, $\Delta\% = 4.2\%/12 = 0.35\%$ per payment Step 3: $PV = \\$0$, $FV_{ORD} = \text{Step 2 answer}$, $IY = 6.25\%$, $CY = 1$, $PY = 12$, Years = 47, $\Delta\% = 0.5\%$	How You Will Get There Step 1: Move Jesse's desired annual income from today to the age of retirement by applying Formulas 9.1, 9.2, and 9.3. Step 2: Calculate the amount of money needed at the age of retirement by applying Formulas 9.1, 11.1, and 12.3. Step 3: Calculate Jesse's regular monthly payment by applying Formulas 9.1, 11.1, and 12.1.
		

Perform	Step 1: $i = 4.2\%/1 = 4.2\%$; $N = 1 \times 47 = 47$ compounds; $FV = \\$40,000(1 + 0.042)^{47} = \\$276,594.02$ Step 2: $i = 4.5\%/1 = 4.5\%$; $N = 12 \times 20 = 240$ payments Note that the monthly payment is $\\$276,594.02 \div 12 = \\$23,049.50$ $PV_{ORD} = \frac{\$23,049.50}{1 + 0.0035} \left[\frac{1 - \left[\frac{1 + 0.0035}{(1 + 0.045)^{\frac{1}{12}}} \right]^{240}}{\frac{(1 + 0.045)^{\frac{1}{12}}}{1 + 0.0035} - 1} \right] = \$5,398,479.88$ Step 3: $i = 6.25\%/1 = 6.25\%$; $N = 12 \times 47 = 564$ payments $\$5,398,479.88 = PMT(1 + 0.005)^{564-1} \left[\frac{\left[\frac{(1 + 0.0625)^{\frac{1}{12}}}{1 + 0.005} \right]^{564} - 1}{\frac{(1 + 0.0625)^{\frac{1}{12}}}{1 + 0.005} - 1} \right]$ $\$5,398,479.88 = PMT(16.576497)[574.367284]$ $\$5,398,479.88 = PMT(9,520.997774)$ $\$567.01 = PMT$																		
	Calculator Instructions <table><tr><th>Step</th><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>1</td><td>END</td><td>47</td><td>4.2</td><td>-40000</td><td>0</td><td>Answer: 276,594.0176</td><td>1</td><td>1</td></tr></table>	Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	1	END	47	4.2	-40000	0	Answer: 276,594.0176	1	1
	Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y										
1	END	47	4.2	-40000	0	Answer: 276,594.0176	1	1											

Present	On the basis that the rate of inflation is relatively accurate, Jesse's first RRSP contribution one month from now is \$567.01. Each subsequent payment increases by 0.5% resulting in a balance of \$5,398,479.88 at retirement, from which he can make his first withdrawal of \$23,049.50 increasing at 0.35% per month for 20 years.

Example 12.6B: A 30-Year Old with Savings Planning for Retirement

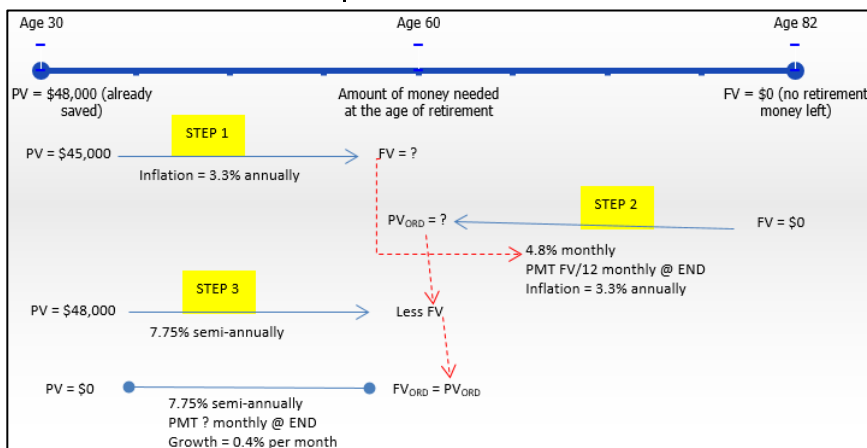
Marilyn just turned 30 and plans on retiring when she turns 60. In today's funds, she wants to earn \$45,000 in retirement. Based on the past 30 years, she estimates inflation at 3.3% compounded annually. She would like to receive 22 years of month-end payments from her retirement fund, which is forecasted to earn 4.8% interest compounded monthly. Based on the results to date, she believes her RRSP can earn 7.75% compounded semi-annually during her contributions. She has already saved \$48,000, and she will increase each contribution by 0.4%. One month from now, what is the amount of her first monthly RRSP contribution?

Plan She will make regularly increasing contributions to her RRSP at the end of the monthly payment interval with semi-annual compounding. Therefore, this is a constant growth general ordinary annuity. Calculate her first payment (PMT).

What You Already Know
Marilyn's RRSP plan appears in the timeline.
Step 1: $PV = \$45,000$, $IY = 3.3\%$, $CY = 1$, $Years = 30$
Step 2: $FV = \$0$, $IY = 4.8\%$, $CY = 12$, $PY = 12$, $Years = 22$,
 $\Delta\% = 3.3\%/12 = 0.275\%$ per payment
Step 3: $PV = \$48,000$, $FV_{ORD} = \text{Step 2 answer}$, $IY = 7.75\%$, $CY = 2$, $PY = 12$, $Years = 30$, $\Delta\% = 0.4\%$

How You Will Get There

Step 1: Move her desired annual income from today to the age of retirement by applying Formulas 9.1, 9.2, and 9.3.
Step 2: Calculate the amount of money needed at the age of retirement by applying Formulas 9.1, 11.1, and 12.3.
Step 3: Deduct her current savings from her required future contributions by calculating the future value of her present savings and deducting it from the step 2 answer by applying Formulas 9.1, 9.2, and 9.3. Calculate her regular monthly payment by applying Formulas 9.1, 11.1, and 12.1:

Understand

Perform	Step 1: $i = 3.3\%/1 = 3.3\%$; $N = 1 \times 30 = 30$ compounds; $FV = \$45,000(1 + 0.033)^{30} = \$119,185.15$ Step 2: $i = 4.8\%/12 = 0.4\%$; $N = 12 \times 22 = 264$ payments Note that the monthly payment is $\$119,185.15/12 = \$9,932.10$ $PV_{ORD} = \frac{\$9,932.10}{1 + 0.00275} \left[\frac{1 - \left[\frac{1 + 0.00275}{(1 + 0.004)^{\frac{12}{12}}} \right]^{264}}{\frac{(1 + 0.004)^{\frac{12}{12}}}{1 + 0.00275} - 1} \right] = \$2,226,998.08$																										
	Step 3: Current Savings: $i = 7.75\%/2 = 3.875\%$; $N = 2 \times 30 = 60$ compounds; $FV = \$48,000(1 + 0.03875)^{60} = \$469,789.65$ new $FV_{ORD} = \$2,226,998.08 - \$469,789.65 = \$1,757,208.43$ RRSP Payments: $i = 7.75\%/2 = 3.875\%$; $N = 12 \times 30 = 360$ payments $\$1,757,208.43 = PMT(1 + 0.004)^{360-1} \left[\frac{\left[\frac{(1 + 0.03875)^{\frac{2}{12}}}{1 + 0.004} \right]^{360} - 1}{\frac{(1 + 0.03875)^{\frac{2}{12}}}{1 + 0.004} - 1} \right]$ $\$1,757,208.43 = PMT(4.191822)[564.766990]$ $\$1,757,208.43 = PMT(2,367.403054)$ $\$742.25 = PMT$																										
	Calculator Instructions																										
	<table><tr><th>Step</th><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>1</td><td>END</td><td>30</td><td>3.3</td><td>-45000</td><td>0</td><td>Answer: 119,185.1533</td><td>1</td><td>1</td></tr><tr><td>3 (Savings)</td><td>✓</td><td>60</td><td>7.75</td><td>-48000</td><td>✓</td><td>Answer: 469,789.6504</td><td>2</td><td>2</td></tr></table>	Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	1	END	30	3.3	-45000	0	Answer: 119,185.1533	1	1	3 (Savings)	✓	60	7.75	-48000	✓	Answer: 469,789.6504	2
Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y																			
1	END	30	3.3	-45000	0	Answer: 119,185.1533	1	1																			
3 (Savings)	✓	60	7.75	-48000	✓	Answer: 469,789.6504	2	2																			
Present	On the basis that the rate of inflation is relatively accurate, Marilyn’s first RRSP contribution one month from now is \$742.25. Each subsequent payment increases by 0.4% resulting in a balance of \$2,226,998.08 (including her current savings) at retirement, from which she makes her first withdrawal of \$9,932.10 increasing at 0.275% per month for 22 years.																										



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Chapter 12 Summary

Key Concepts Summary

Section 12.1: Deferred Annuities (Can I Receive Those Payments Later?)

- The stages of deferred annuities
- The four common unknown variables and how to solve for them

Section 12.2: Constant Growth Annuities (Always Wanting More)

- The concept of constant growth and the modifications required to annuity formulas
- The four new annuity formulas
- How to solve constant growth scenarios

Section 12.3: Perpetuities (Are You Going to Live Forever?)

- An explanation of perpetuities
- Ordinary and due types of perpetuities
- The two perpetuity formulas
- How to solve perpetuity scenarios

Section 12.4: Leases (I Will Just Borrow It for a While)

- An explanation of lease characteristics and how leases operate
- Capitalized lease liabilities explained
- How to solve leasing scenarios

Section 12.5: Application: How to Purchase a Vehicle (Get Your Motor Running)

- Four financial choices available to make payments on a vehicle
- The basis and procedure upon which the selection of the vehicle ownership choice is made
- Key considerations to keep in mind when purchasing a vehicle

Section 12.6: Application: Planning Your RRSP (When Can I Retire?)

- The three components of retirement income planning
- The procedure for retirement income planning

The Language of Business Mathematics

capitalized lease liability The present value of the remaining lease payments and residual value on a capital good using a discount rate equivalent to the interest rate the business would have had to pay if it had purchased the asset instead.

constant growth annuity An annuity in which each annuity payment is increased by a fixed percentage.

deferred annuity A financial transaction where annuity payments are delayed until a certain period of time has elapsed.

down payment A portion of the purchase price required up front.

lease A contract by which the owner of an asset gives another party an exclusive right to possess and use the asset under specified conditions for a specific period of time in return for agreed-upon payments.

lessee The borrower of a leased asset.

lessor The owner of a leased asset.

net rate The growth in a constant growth annuity that is attributable solely to the interest rate and not to the growth in the annuity payment.

period of deferral The time segment of a deferred annuity where the single payment earns interest and no contributions are made to the investment.

perpetuity A special type of annuity that has fixed, regular payments continuing indefinitely.

residual value The projected value of an asset at the end of its lease term.

zero growth annuity formula Any annuity formula where the growth rate is 0%. All formulas presented in Chapter 11 incorporate zero growth.

The Formulas You Need to Know

Symbols Used

$\Delta\%$ = the constant growth rate per payment interval (percent change)

CY = compounding frequency

FV_{DUE} = future value of an annuity due

FV_{ORD} = future value of an ordinary annuity

i = periodic interest rate

N = number of annuity payments

PMT = annuity payment amount

PV_{DUE} = present value of an annuity due

PV_{ORD} = present value of an ordinary annuity

PY = payment frequency

Formulas Introduced

Formula 12.1 Future Value of a Constant Growth Ordinary Annuity:

$$FV_{ORD} = PMT(1 + \Delta\%)^{N-1} \left[\frac{\left[\frac{(1+i)^{\frac{CY}{PY}}}{1 + \Delta\%} \right]^N - 1}{\frac{(1+i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right] \quad (\text{Section 12.2})$$

Formula 12.2 Future Value of a Constant Growth Annuity Due:

$$FV_{DUE} = PMT(1 + \Delta\%)^{N-1} \left[\frac{\left[\frac{(1+i)^{\frac{CY}{PY}}}{1 + \Delta\%} \right]^N - 1}{\frac{(1+i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right] \times (1 + i)^{\frac{CY}{PY}} \quad (\text{Section 12.2})$$

Formula 12.3 Present Value of a Constant Growth Ordinary Annuity:

$$PV_{ORD} = \frac{PMT}{1 + \Delta\%} \left[\frac{1 - \left[\frac{1 + \Delta\%}{(1+i)^{\frac{CY}{PY}}} \right]^N}{\frac{(1+i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right] \quad (\text{Section 12.2})$$

Formula 12.4 Present Value of a Constant Growth Annuity Due:

$$PV_{DUE} = \frac{PMT}{1 + \Delta\%} \left[\frac{1 - \left[\frac{1 + \Delta\%}{(1+i)^{\frac{CY}{PY}}} \right]^N}{\frac{(1+i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right] \times (1 + i)^{\frac{CY}{PY}} \quad (\text{Section 12.2})$$

Formula 12.5 Ordinary Perpetuity Present Value: $PV_{ORD} = \frac{PMT}{(1+i)^{\frac{CY}{PY}} - 1}$ (Section 12.3)

Formula 12.6 Perpetuity Due Present Value: $PV_{\text{DUE}} = PMT \left(\frac{1}{\frac{CY}{(1+i)^{PY}-1}} + 1 \right)$ (Section 12.3)



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Chapter 13: Understanding Amortization and Its Applications

(Is That Where My Money Went?)

It would be nice to pay cash for everything that you acquire; however, goods such as housing, automobiles, furniture, and electronics require more money than most Canadians have lying around. If you tried saving up and paying cash for these large purchases, it would take a long time to accumulate enough funds. Take the example of a new home. If you saved \$300 per month at 5% compounded annually, it would take approximately 30 years to save up \$250,000 in cash, and in the meantime, of course, you would still need to pay for somewhere to live.

So it is not necessarily bad to borrow money at some point. However, to borrow wisely you had better understand where your money is going. Would it shock you to learn that in a typical mortgage arrangement most homeowners pay approximately double for their homes? That is, \$250,000 for the home and \$250,000 in interest!

Businesses borrow money for many of the same reasons as consumers. Loans and mortgages are commonplace. Certain business activities need to be financed upfront. For example, new products must be researched and developed before a single unit can be sold. This requires investments that will not be reimbursed until the products turn a profit. All company debts must be accurately recorded onto balance sheets to reflect the balances owing. Income statements must appropriately track interest expenses or earnings. Because businesses can deduct their interest expenses against corporate taxable income and lower their taxes, it is important to see what proportion of their loan payments is going to interest.

When you take out a mortgage for yourself or your business, where does your money go? You need a chart of your loan payments showing how much interest the bank charges and how much is applied against your principal.

This chapter takes you through calculating the principal and interest components of any single payment or series of payments for both loans and investment annuities. Recall from Chapter 11 that you estimated the final payment on a loan; you will now see how to calculate that payment precisely. Finally, you will be guided through the largest transactions you personally are likely to make: basic mortgages and renewals.

Outline of Chapter Topics

- 13.1: Calculating Interest and Principal Components (Where Did My Money Go?)
- 13.2: Calculating the Final Payment (The Bank Wants Every Penny)
- 13.3: Amortization Schedules (How Much Is Mine, How Much Is Theirs?)
- 13.4: Special Application: Mortgages (Your Biggest Purchase)

13.1: Calculating Interest and Principal Components

(Where Did My Money Go?)

How much of the principal do you pay off when you make a loan payment? One year ago you purchased your \$250,000 dream home on a 25-year mortgage at a fixed 5% compounded semi-annually interest rate. With monthly contributions of \$1,454.01, or \$17,448.12 in total for the past year, you figure you must have put a serious dent in the balance owing.

You get a rude shock when you inspect your mortgage statement, as you realize the balance owing is not what you expected. Your remaining balance is \$244,806.89, reflecting a principal reduction of only \$5,193.11! The other 70% of your hard-earned money, amounting to \$12,255.01, went solely toward the bank's interest charges.

Many people do not fully understand how their loan payments are portioned out. Over the full course of the 25-year mortgage you will pay \$186,204.46 in interest charges at 5% compounded semi-annually, or approximately 74.5% of the home's price tag. That is a total of \$436,204.46 paid on a \$250,000 home. What if interest rates rise? At a more typical 7% semi-annual rate, you would owe \$275,311.51 in total interest, or 110% of the value of your home.

Knowing these numbers, what can you do about them? Term, interest rates, payment amounts, and payment frequency all affect the amount of interest you pay. What if you made one extra mortgage payment per year? Did you know that it would only take approximately 21 years instead of 25 years to own your home? Instead of paying \$186,204.46 in interest you would pay only \$156,789.33, a savings of almost \$30,000!

These calculations should make it clear that both businesses and consumers need to understand the interest and principal components of annuity payments. This section shows you how to calculate principal and interest components both for single payments and for a series of payments.

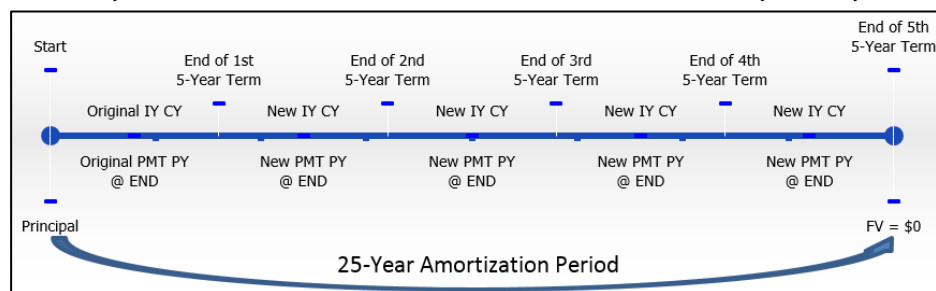
What Is Amortization?

Amortization is a process by which the principal of a loan is extinguished over the course of an agreed-upon time period through a series of regular payments that go toward both the accruing interest and principal reduction. Two components make up the agreed-upon time component:

1. **Amortization Term.** The **amortization term** is the length of time for which the interest rate and payment agreement between the borrower and the lender will remain unchanged. Thus, if the agreement is for monthly payments at a 5% fixed rate over five years, it is binding for the entire five years. Or if the agreement is for quarterly payments at a variable rate of prime plus 2% for three years, then interest is calculated on this basis throughout the three years.
2. **Amortization Period.** The **amortization period** is the length of time it will take for the principal to be reduced to zero. For example, if you agree to pay back your car loan over six years, then after six years you reduce your principal to zero and your amortization period is six years.

In most relatively small purchases, the amortization term and amortization period are identical. For example, a vehicle loan has an agreed-upon interest rate and payments for a fixed term. At the end of the term, the loan is fully repaid. However, larger purchases such as real estate transactions typically involve too much money to be repaid under short time frames. Financial institutions hesitate to agree to amortization terms of much more than five to seven years because of the volatility and fluctuations of interest rates. As a result, a term of five years may be established with an amortization period of

25 years. When the five years elapse, a new term is established as agreed upon between the borrower and lender. The conditions of the new term reflect prevailing interest rates and a payment plan that continues to extinguish the debt within the original amortization period. The figure shows the timeline for a 25-year mortgage in



which the borrower establishes five sequential five-year terms throughout the 25-year amortization period to extinguish the debt.

With regard to amortization, all loans take the structure of either simple ordinary annuities or general ordinary annuities unless otherwise stated. Therefore, this textbook will primarily focus on ordinary annuities.

Calculating Interest and Principal Components for a Single Payment

At any point during amortization you can precisely calculate how much any single payment contributes toward principal and interest. Businesses must separate the principal and interest components for two reasons:

1. **Interest Expense.** Any interest paid on a debt is an accounting expense that must be reported in financial statements. In addition, interest expenses have tax deduction implications for a business.
2. **Interest Income.** Any interest that a company receives is a source of income. This must be reported as revenue in its financial statements and is subject to taxation rules.



The Formula

To calculate the interest and principal components of any annuity payment, follow this sequence of two formulas.

1. Calculate the interest portion of the payment (Formula 13.1).
2. Calculate the principal portion of the payment (Formula 13.2).

INT is Interest Portion: In calculating the interest portion of any annuity payment, this formula accommodates both simple and general ordinary annuities.

CY is Compound Frequency: The number of compounding periods in a single year.

Formula 13.1 - Interest Portion of an Ordinary Single Payment:

$$\text{INT} = \text{BAL} \times \left((1 + i)^{\frac{\text{CY}}{\text{PY}}} - 1 \right)$$

BAL is Previous Balance Owning: This is the principal balance owing immediately prior to the current annuity payment.

i is Periodic Interest Rate: This is the rate of interest that is used in converting the interest to principal resulting from Formula 9.1. Note the continued usage of the exponent $\frac{\text{CY}}{\text{PY}}$ to ensure that the payment and compounding intervals match.

PY is Payment Frequency: The number of payment intervals in a single year.

PRN is Principal Portion: The amount of the annuity payment that is deducted from the principal. Once the interest portion is known from Formula 13.1, the amount left over must be the principal portion of the payment.

Formula 13.2 - Principal Portion of a Single Payment: $\text{PRN} = \text{PMT} - \text{INT}$

PMT is Annuity Payment Amount: The annuity payment amount.

INT is Interest Portion: The interest portion of the single payment calculated in Formula 13.1.



How It Works

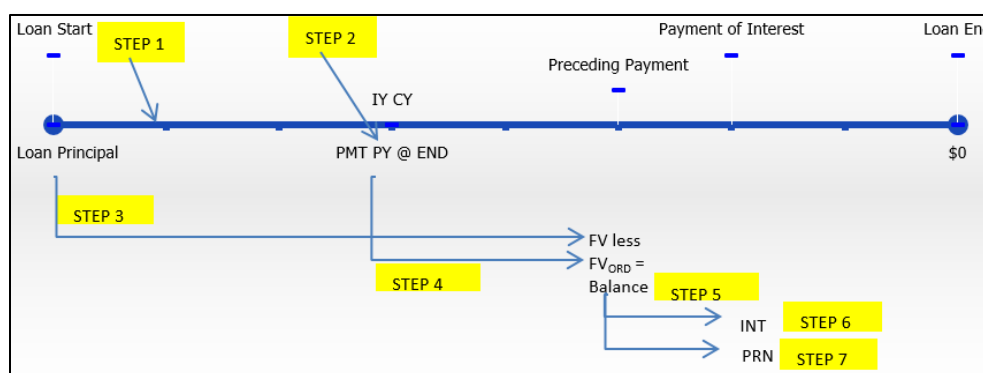
Follow these steps to calculate the interest and principal components for a single annuity payment:

Step 1: Draw a timeline (seen below). Identify the known time value of money variables, including IY, CY, PY, Years, and one of PV_{ORD} or FV_{ORD} . The annuity payment amount may or may not be known.

Step 2: If the annuity payment amount is known, proceed to step 3. If it is unknown, solve for it using Formulas 9.1 (Periodic Interest Rate) and 11.1 (Number of Annuity Payments) and by rearranging Formula 11.4 (Ordinary Annuity Present Value). Round the payment to two decimals.

Step 3: Calculate the future value of the original principal immediately prior to the payment being made. Use Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compounding Periods for Single Payments), and 9.3 (Compound Interest for Single Payments). For example, when you calculate the interest and principal portions for the 22nd payment, you need to know the balance immediately after the 21st payment.

Step 4: Calculate the future value of all annuity payments already made. Use Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value). For example, if you need to calculate the interest and principal portions for the 22nd payment, you need to know the future value of the first 21 payments.



Step 5: Calculate the balance (BAL) prior to the payment by subtracting step 4 (the future value of the payments) from step 3 (the future value of the original principal). The fundamental concept of time value of money allows you to combine these two numbers on the same focal date.

Step 6: Calculate the interest portion of the current annuity payment using Formula 13.1.

Step 7: Calculate the principal portion of the current annuity payment using Formula 13.2.



Important Notes

Investment Annuities. The formulas and techniques being discussed in this section also apply to any type of investment annuity from which annuity payments are received. For example, most people receive annuity payments from their accumulated RRSP savings when in retirement. In these cases, view the investment as a loan to the financial institution at an agreed-upon interest rate. The financial institution then makes annuity payments to the retiree to extinguish its debt at some future point; these payments consist of the principal and interest being earned.

Your BAII Plus Calculator. The function that calculates the interest and principal components of any single payment on your BAII Plus calculator is called AMORT. It is located on the 2nd shelf above the PV button.

AMORT
Function



The Amortization window has five variables (use ↓ or ↑ to scroll through them). The first two, P1 and P2, are data entry variables. The last three, BAL, PRN, and INT, are output variables.

- P1 is the starting payment number. The calculator works with a single payment or a series of payments.
- P2 is the ending payment number. This number is the same as P1 when you work with a single payment. When you work with a series of payments later in this section, you set it to a number higher than P1.
- BAL is the principal balance remaining after the P2 payment number. The cash flow sign is correct as indicated on the calculator display.
- PRN is the principal portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.
- INT is the interest portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.

To use the Amortization function, the commands are as follows:

1. You must enter all seven time value of money variables accurately (N, I/Y, PV, PMT, FV, P/Y, and C/Y). If PMT was computed, you must re-enter it with only two decimals and the correct cash flow sign.
2. Press 2nd AMORT.
3. Enter a value for P1 and press Enter followed by ↓.
4. Enter a value for P2 and press Enter followed by ↓. Note that the higher the numbers entered in P1 or P2, the longer it takes the calculator to compute the outputs. It is possible that your calculator will go blank for a few moments before displaying the outputs.
5. Using the ↓ and ↑, scroll through BAL, PRN, and INT to read the output.



Things To Watch Out For

A common misunderstanding when using the AMORT function on the calculator occurs when inputting the values for P1 and P2. Many people think that when they solve for a single payment they need to set these values one apart. For example, if they are looking for the 22nd payment they imagine that $P1 = 21$ and $P2 = 22$. This is incorrect, as the calculator would then compute the total values for both payments 21 and 22.

If you are interested in a single payment, you must set P1 and P2 to the exact same value. In the example, if you want the 22nd payment then both $P1 = 22$ and $P2 = 22$.



Give It Some Thought

1. For any loan, if you calculated the interest portions of the second payment and the tenth payment, which payment has a smaller interest portion?
2. For any loan, if you calculated the principal portions of the fifth payment and the twelfth payment, which payment has a smaller principal portion?
3. Holding all other variables constant, if Loan A had an interest rate of 4% while Loan B had an interest rate of 6%, which loan has the higher interest portion on any payment?

Solutions:

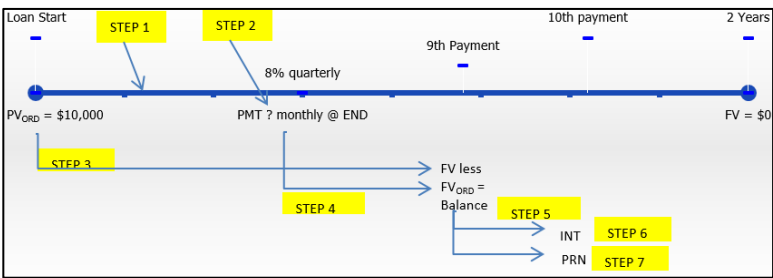
1. The tenth payment, since the principal is much smaller by then so less interest is charged.
 2. The fifth payment, since the balance is higher so more interest is charged.
 3. Loan B, since the interest rate is higher.

Example 13.1A: Interest and Principal of a Loan Payment

The accountant at the accounting firm of Nichols and Burnt needs to separate the interest and principal on the tenth loan payment. The company borrowed \$10,000 at 8% compounded quarterly with month-end payments for two years.

Plan

Note that this is an ordinary general annuity. Calculate the principal portion (PRN) and the interest portion (INT) of the tenth payment on the two-year loan.

Understand	What You Already Know Step 1: The information about the accounting firm's loan are in the timeline. PV_{ORD} = \$10,000, IY = 8%, CY = 4, PY = 12, Years = 2, FV = \$0	How You Will Get There Step 2: PMT is unknown. Apply Formulas 9.1, 11.1, and 11.4. Step 3: Calculate the future value of the loan principal using Formulas 9.2 and 9.3. Step 4: Calculate the future value of the first nine payments using Formulas 11.1 and 11.2. Step 5: Calculate the principal balance after nine payments through BAL = FV – FV_{ORD}. Step 6: Calculate the interest portion by using Formula 13.1. Step 7: Calculate the principal portion by using Formula 13.2.																								
																										
Perform	Step 2: $i = 8\%/4 = 2\%$; $N = 12 \times 2 = 24$ payments $\\$10,000 = PMT \left[\frac{1 - \left[\frac{1}{(1 + 0.02)^{\frac{4}{12}}} \right]^{24}}{(1 + 0.02)^{\frac{4}{12}} - 1} \right]$$PMT = \frac{\\$10,000}{\left[\frac{1 - \left[\frac{1}{(1 + 0.02)^{\frac{4}{12}}} \right]^{24}}{(1 + 0.02)^{\frac{4}{12}} - 1} \right]} = \frac{\\$10,000}{\left[\frac{0.146509}{0.006622} \right]} = \\$452.03$																									
	Step 3: $N = 4 \times \frac{9}{12} = 3$ compounds; $FV = \\$10,000(1 + 0.02)^3 = \\$10,612.08$ Step 4: $N = 12 \times \frac{9}{12} = 9$ payments $FV_{ORD} = \\$452.03 \left[\frac{\left[(1 + 0.02)^{\frac{4}{12}} \right]^9 - 1}{(1 + 0.02)^{\frac{4}{12}} - 1} \right] = \\$4,177.723934$ Step 5: $BAL = \\$10,612.08 - \\$4,177.723934 = \\$6,434.356066$ Step 6: $INT = \\$6,434.356066 \times \left((1 + 0.02)^{\frac{4}{12}} - 1 \right) = \\$42.612871$ Step 7: $PRN = \\$452.03 - \\$42.612871 = \\$409.42$																									
	Calculator Instructions <table><tr><td>N</td><td>I/Y</td><td>PV</td><td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>24</td><td>8</td><td>10000</td><td>Answer: -452.032375 Rekeyed as: -452.03</td><td>0</td><td>12</td><td>4</td></tr></table> <table><tr><td>P1</td><td>P2</td><td>BAL (output)</td><td>PRN (output)</td><td>INT (output)</td></tr><tr><td>10</td><td>10</td><td>6,024.938937</td><td>409.417128</td><td>42.612871</td></tr></table>		N	I/Y	PV	PMT	FV	P/Y	C/Y	24	8	10000	Answer: -452.032375 Rekeyed as: -452.03	0	12	4	P1	P2	BAL (output)	PRN (output)	INT (output)	10	10	6,024.938937	409.417128	42.612871
	N	I/Y	PV	PMT	FV	P/Y	C/Y																			
24	8	10000	Answer: -452.032375 Rekeyed as: -452.03	0	12	4																				
P1	P2	BAL (output)	PRN (output)	INT (output)																						
10	10	6,024.938937	409.417128	42.612871																						
Present	The accountant for Nichols and Burnt records a principal reduction of \$409.42 and an interest expense of \$42.61 for the tenth payment.																									

Example 13.1B: Interest and Principal of an Investment Annuity Payment

Baxter has \$50,000 invested into a five-year annuity that earns 5% compounded quarterly and makes regular end-of-quarter payments to him. For his fifth payment, he needs to know how much of his payment came from his principal and how much interest was earned on the investment.

Plan Note that this is an ordinary simple annuity. Calculate the principal portion (PRN) and the interest portion (INT) of the fifth payment on the five-year investment annuity.

What You Already Know
 Step 1: You know the following about the investment annuity, as illustrated in the timeline.
 $PV_{ORD} = \$50,000$,
 $IY = 5\%$, $CY = 4$,
 $PY = 4$, Years = 5, $FV = \$0$

How You Will Get There

Step 2: PMT is unknown. Apply Formulas 9.1, 11.1, and 11.4.

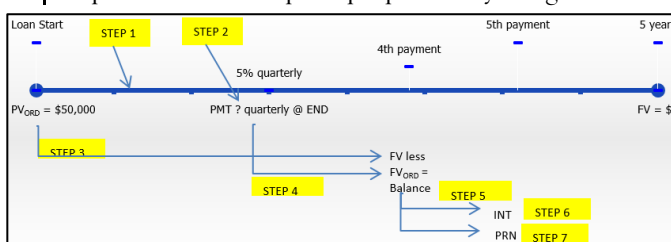
Step 3: Calculate the future value of the loan principal using Formulas 9.2 and 9.3.

Step 4: Calculate the future value of the first four payments using Formulas 11.1 and 11.2.

Step 5: Calculate the principal balance after four payments through $BAL = FV - FV_{ORD}$.

Step 6: Calculate the interest portion by using Formula 13.1.

Step 7: Calculate the principal portion by using Formula 13.2.



Step 2: $i = 5\%/4 = 1.25\%$; $N = 4 \times 5 = 20$ payments

$$\$50,000 = PMT \left[\frac{1 - \left[\frac{1}{(1 + 0.0125)^4} \right]^{20}}{(1 + 0.0125)^4 - 1} \right]$$

$$PMT = \frac{\$50,000}{\left[\frac{1 - \left[\frac{1}{(1 + 0.0125)^4} \right]^{20}}{(1 + 0.0125)^4 - 1} \right]} = \frac{\$50,000}{\left[\frac{0.219991}{0.0125} \right]} = \$2,841.02$$

Step 3: $N = 4 \times 1 = 4$ compounds; $FV = \$50,000(1 + 0.0125)^4 = \$52,547.26685$

Step 4: $N = 4 \times 1 = 4$ payments

$$FV_{ORD} = \$2,841.02 \left[\frac{\left[(1 + 0.0125)^4 \right]^4 - 1}{(1 + 0.0125)^4 - 1} \right] = \$11,578.93769$$

Step 5: $BAL = \$52,547.26685 - \$11,578.93769 = \$40,968.32916$

Step 6: $INT = \$40,968.32916 \times \left((1 + 0.0125)^4 - 1 \right) = \$512.104114$

Step 7: $PRN = \$2,841.02 - \$512.104114 = \$2,328.92$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
20	5	-50000	?Answer: 2,841.019482 Re-keyed as: 2,841.02	0	4	4

P1	P2	BAL (output)	PRN (output)	INT (output)
5	5	-38,639.41327	2,328.915886	512.104114

Present On Baxter's fifth payment of \$2,841.02, he has \$2,328.92 deducted from his principal and the remaining \$512.10 comes from the interest earned on his investment.

Calculating Interest and Principal Components for a Series of Payments

Many times in business, you need to know the principal and interest portions for a series of annuity payments. For example, when completing tax forms a company needs the total loan interest paid annually. If the loan payments are monthly, using Formulas 13.1 and 13.2 requires you to perform the calculations 12 times (once for each payment) to arrive at the total interest paid. Clearly, that is time consuming and tedious. In this section, you learn new formulas and a process for calculating



the principal and interest portions involving a series of payments.

The Formula

Formulas 13.3 and 13.4 are used to determine the interest and principal components for a series of annuity payments.

PRN is Principal Portion: By taking the starting principal less the ending principal, you obtain the difference, which must be the total principal portion of the series of payments made.

Formula 13.3 - Principal Portion for a Series of Payments: $PRN = BAL_{P1} - BAL_{P2}$

BAL_{P1} is Principal Balance Immediately Prior to First Payment: This is the principal balance owing immediately prior to the first payment in the series. It is found by calculating the future value of the annuity one payment earlier than the starting payment.

BAL_{P2} is Principal Balance Immediately After Last Payment: This is the principal balance owing after the last payment in the series. It is found by calculating the future value of the annuity just after the last payment is made.

INT is Interest Portion: The interest portion of the series of payments.

N is Number of Payments: The number of payments involved in the time segment inclusive. For example, if you are calculating interest on payments four through seven, there are four payments.

Formula 13.4 - Interest Portion for a Series of Payments: $INT = N \times PMT - PRN$

PMT is Annuity Payment Amount: The amount of each annuity payment.

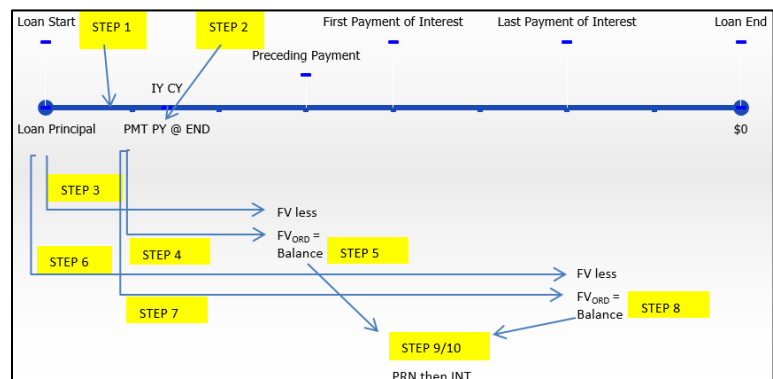
PRN is Principal Portion: This is the total principal portion of all the annuity payments calculated by Formula 13.3.



How It Works

Follow these steps to calculate the interest and principal components for a series of annuity payments:

Step 1: Draw a timeline. Identify the known time value of money variables, including IY, CY, PY, Years, and one of PV_{ORD} or FV_{ORD} . The annuity payment amount may or may not be known.



Step 2: If the annuity payment amount is known, proceed to step 3. If it is unknown, solve for it using Formulas 9.1 (Periodic Interest Rate) and 11.1 (Number of Annuity Payments) and by rearranging Formula 11.4 (Ordinary Annuity Present Value). Round the payment to two decimals.

Step 3: Calculate the future value of the original principal immediately prior to the series of payments being made. Use Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compounding Periods for Single Payments), and 9.3 (Compound Interest for Single Payments). For example, when calculating the interest and principal portions for the 22nd through 25th payments, you need the balance immediately after the 21st payment.

Step 4: Calculate the future value of all annuity payments already made prior to the first payment in the series. Use Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value). For example, when calculating the interest and principal portions for the 22nd through 25th payments, you need the future value of the first 21 payments.

Step 5: Calculate the balance (BAL) prior to the series of payments by subtracting step 4 (the future value of the payments) from step 3 (the future value of the original principal). The fundamental concept of time value of money allows you to combine these two numbers on the same focal date. Do not round this number.

Steps 6 to 8: Repeat steps 3 to 5 to calculate the future value of the original principal immediately after the last payment in the series is made. For example, when calculating the interest and principal portions for the 22nd through 25th payments, you need the balance immediately after the 25th payment.

Step 9: Calculate the principal portion of the series of payments using Formula 13.3.

Step 10: Calculate the interest portion of the series of payments using Formula 13.4.



Important Notes

Working with a series of payments on the BAII Plus calculator requires you to enter the first payment number into the P1 and the last payment number into the P2. Thus, if you are looking to calculate the interest and principal portions of payments four through seven, set $P1 = 4$ and $P2 = 7$. In the outputs, the BAL window displays the balance remaining after the last payment entered ($P2 = 7$), and the PRN and INT windows display the total principal interest portions for the series of payments.



Things To Watch Out For

A common mistake occurs in translating years into payment numbers. For example, assume payments are monthly and you want to know the total interest paid in the fourth year. In error, you might calculate that the fourth year begins with payment 36 and ends with payment 48, thus looking for payments 36 to 48. The mistake is to fail to realize that the 36th payment is actually the last payment of the third year. The starting payment in the fourth year is the 37th payment. Hence, if you are concerned only with the fourth year, then you must look for the 37th to 48th payments.

There are two methods to calculate the correct payment numbers:

1. Calculate the payment at the end of the year in question, then subtract the payment frequency less one ($PY - 1$) to arrive at the first payment of the year. In the example, the last payment of the fourth year is 48. With monthly payments, or $PY = 12$, then $48 - (12 - 1) = 37$, which is the first payment of the fourth year.
2. You could determine the last payment of the year prior to the year of interest and add one payment to it. Thus, the end of the third year is payment #36, so the first payment of the fourth year is $36 + 1 = 37$. The last payment of the fourth year remains at payment 48.

Example 13.1C: Interest and Principal of a Series of Loan Payments

Revisit Example 13.1A. The accountant at the accounting firm of Nichols and Burnt is completing the tax returns for the company and needs to know the total interest expense paid during the tax year that encompassed payments 7 through 18 inclusively. Remember, the company borrowed \$10,000 at 8% compounded quarterly with month-end payments for two years.

Plan	Note that this is an ordinary general annuity. Calculate the total principal portion (PRN) and the total interest portion (INT) of the 7th to the 18th payments on the two-year loan.																								
Understand	What You Already Know Step 1: The following information about the accounting firm's loan is known, as illustrated in the timeline. $PV_{ORD} = \\$10,000$, $IY = 8\%$, $CY = 4$, $PMT = \\$452.03$, $PY = 12$, $Years = 2$, $FV = \\$0$	How You Will Get There Step 2: PMT is known. Skip this step. Step 3: Calculate the future value of the loan principal prior to the first payment in the series using Formulas 9.2 and 9.3. Step 4: Calculate the future value of the first six payments using Formulas 11.1 and 11.2. Step 5: Calculate the principal balance prior to the 7th payment through $BAL_{P1} = FV - FV_{ORD}$. Steps 6 to 8: Repeat steps 3 to 5 for the 18 payment to calculate BAL_{P2}. Step 9: Calculate the principal portion by using Formula 13.3. Step 10: Calculate the interest portion by using Formula 13.4.																							
Perform	Step 3: Recall $i = 2\%$; $N = 4 \times \frac{6}{12} = 2$ compounds; $FV = \\$10,000(1 + 0.02)^2 = \\$10,404.00$ Step 4: $N = 12 \times \frac{6}{12} = 6$ payments $FV_{ORD} = \\$452.03 \left[\frac{[(1+0.02)^{\frac{4}{12}} - 1]}{(1+0.02)^{\frac{4}{12}} - 1} \right] = \\$2,757.483449$ Step 5: $BAL_{P1} = \\$10,404.00 - \\$2,757.483449 = \\$7,646.516551$ Step 6: $N = 4 \times \frac{18}{12} = 6$ compounds; $FV = \\$10,000(1 + 0.02)^6 = \\$11,261.62419$ Step 7: $N = 12 \times \frac{18}{12} = 18$ payments $FV_{ORD} = \\$452.03 \left[\frac{[(1+0.02)^{\frac{4}{12}} - 1]}{(1+0.02)^{\frac{4}{12}} - 1} \right] = \\$8,611.157995$ Step 8: $BAL_{P2} = \\$11,261.62419 - \\$8,611.157995 = \\$2,650.466195$ Step 9: $PRN = \\$7,646.516551 - \\$2,650.466195 = \\$4,996.05$ Step 10: $N = 7\text{th through } 18\text{th payment inclusive} = 12$ payments; $INT = 12 \times \\$452.03 - \\$4,996.06 = \\$5,424.36 - \\$4,996.06 = \\$428.30$																								
	Calculator Instructions <table><tr><td>N</td><td>I/Y</td><td>PV</td><td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>24</td><td>8</td><td>10000</td><td>-452.03</td><td>0</td><td>12</td><td>4</td></tr></table> <table><tr><td>P1</td><td>P2</td><td>BAL (output)</td><td>PRN (output)</td><td>INT (output)</td></tr><tr><td>7</td><td>18</td><td>2,650.466197</td><td>4,996.050354</td><td>428.309646</td></tr></table>		N	I/Y	PV	PMT	FV	P/Y	C/Y	24	8	10000	-452.03	0	12	4	P1	P2	BAL (output)	PRN (output)	INT (output)	7	18	2,650.466197	4,996.050354
N	I/Y	PV	PMT	FV	P/Y	C/Y																			
24	8	10000	-452.03	0	12	4																			
P1	P2	BAL (output)	PRN (output)	INT (output)																					
7	18	2,650.466197	4,996.050354	428.309646																					
Present	For the tax year covering payments 7 through 18, total payments of \$5,424.36 are made, of which \$4,996.05 was deducted from principal while \$428.31 went to the interest charged.																								

Example 13.1D: Interest and Principal of a Series of Investment Annuity Payments

Revisit Example 13.1B, in which Baxter has \$50,000 invested into a five-year annuity that earns 5% compounded quarterly and makes regular end-of-quarter payments to him. For his third year, he needs to know how much of his payments came from his principal and how much was interest earned on the investment.

Plan Note that this is an ordinary simple annuity. Calculate the principal portion (PRN) and the interest portion (INT) of the third-year payments for the five-year investment annuity. This is the 9th through the 12th payments inclusive.

What You Already Know
 Step 1: The following information about the investment annuity is known, as illustrated in the timeline.
 $PV_{ORD} = \$50,000$,
 $IY = 5\%$, $CY = 4$,
 $PMT = \$2,841.02$,
 $PY = 4$, $Years = 5$,
 $FV = \$0$

How You Will Get There
 Step 2: PMT is known. Skip this step.
 Step 3: Calculate the future value of the loan principal prior to the first payment in the series using Formulas 9.2 and 9.3.
 Step 4: Calculate the future value of the first eight payments using Formulas 11.1 and 11.2.
 Step 5: Calculate the principal balance prior to the ninth payment through $BAL_{P1} = FV - FV_{ORD}$.
 Steps 6 to 8: Repeat steps 3 to 5 for the 12th payment to calculate BAL_{P2} .
 Step 9: Calculate the principal portion by using Formula 13.3.
 Step 10: Calculate the interest portion by using Formula 13.4.

Perform

Step 3: Recall $i = 1.25\%$; $N = 4 \times 2 = 8$ compounds; $FV = \$50,000(1 + 0.0125)^8 = \$55,224.30506$

Step 4: $N = 4 \times 2 = 8$ payments $FV_{ORD} = \$2,841.02 \left[\frac{[(1 + 0.0125)^4]^8 - 1}{(1 + 0.0125)^4 - 1} \right] = \$23,747.76825$

Step 5: $BAL_{P1} = \$55,224.30506 - \$23,747.76825 = \$31,476.53681$

Step 6: $N = 4 \times 3 = 12$ compounds; $FV = \$50,000(1 + 0.0125)^{12} = \$58,037.72589$

Step 7: $N = 4 \times 3 = 12$ payments $FV_{ORD} = \$2,841.02 \left[\frac{[(1 + 0.0125)^4]^{12} - 1}{(1 + 0.0125)^4 - 1} \right] = \$36,536.544$

Step 8: $BAL_{P2} = \$58,037.72589 - \$36,536.544 = \$21,501.18189$

Step 9: $PRN = \$31,476.53681 - \$21,501.18189 = \$9,975.35$

Step 10: $N = 9\text{th through } 12\text{th payment inclusive} = 4$ payments;
 $INT = 4 \times \$2,841.02 - \$9,975.35 = \$11,364.08 - \$9,975.35 = \$1,388.73$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
20	5	-50000	2,841.02	0	4	4

P1	P2	BAL (output)	PRN (output)	INT (output)
9	12	-21,501.18189	9,975.354914	1,388.725086

Present In the third year, Baxter receives a total of \$11,364.08 in payments, of which \$9,975.35 is deducted from the principal and \$1,388.73 represents the interest earned on the investment.



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13.2: Calculating the Final Payment

(The Bank Wants Every Penny)

If you have ever paid off a loan you may have noticed that your last payment was a slightly different amount than your other payments. Whether you are making monthly insurance premium payments, paying municipal property tax instalments, financing your vehicle, paying your mortgage, receiving monies from an investment annuity, or dealing with any other situation where an annuity is extinguished through equal payments, the last payment typically differs from the rest, by as little as one penny or up to a few dollars. This difference can be much larger if you arbitrarily chose an annuity payment as opposed to determining an accurate payment through time value of money calculations.

Why is it important for this final payment to differ from all of the previous payments? From a consumer perspective, you do not want to pay a cent more toward a debt than you have to. The average Canadian is more than \$100,000 in debt across various financial tools such as car loans, consumer debt, and mortgages. Imagine if you overpaid every one of those debts by a dollar. Over the course of your lifetime those overpayments would add up to hundreds or even thousands of dollars.

On the business side, particularly where companies are collecting debts from customers, two main issues are legality and profitability:

1. **Legality.** A business can legally collect only the exact amount of money its consumers owe and not a penny more. If a business collects more money than it is owed, it is legally obliged to reimburse the consumer or it will face legal repercussions.
2. **Profitability.** RBC has hundreds of billions in outstanding loans. What if it told every customer who borrowed money last year that they did not have to worry about paying off that final nickel on their loan? This decision would forego millions in revenue. Shareholders would be very unhappy. Giving up even the tiniest amounts of revenue can aggregate into meaningful losses.

Section 11.5 discussed an approximation technique for the final annuity. It is now time to be precise in this calculation. In the current section, you will see why the final payment differs and how to calculate its exact amount. You will also calculate the principal and interest components for a series of payments involving the final payment.

Why Is the Final Payment Different?

Section 11.4 introduced the calculations to determine the annuity payment. Observe that you always needed to round a nonterminating annuity payment to two decimals. It is rare for a calculated annuity payment not to require rounding. The rounding up or down of the annuity payment forms the basis for adjusting the final payment. Observe the implication of each rounding procedure as summarized in this table.

How the Payment Was Rounded	Principal Implications	Interest Implications	Final Payment Implications
Upward	Overpayment	Slight decrease	Needs to be reduced in an amount equal to the overpayment and savings on the interest.
Downward	Underpayment	Slight increase	Needs to be increased in an amount equal to the underpayment and extra charges on the interest.

1. **Annuity Payment Rounded Up.** If the calculated annuity payment is exactly $PMT = \$999.995$, this payment is rounded to two decimals and payments of \$1,000 are made. With each annuity payment you are then overpaying the debt by $\$1,000 - \$999.995 = \$0.005$. Nominally, this means that every two payments made result in an overpayment of \$0.01 toward the debt. If you make 20 such payments, then you nominally overpay the debt by $20 \times \$0.005 = \$0.10$. Therefore, when it comes to the final payment you need to compensate for all of the overpayments made, reducing the final payment by \$0.10. And since the principal is slightly smaller at all times as a result of the overpayment, an additional adjustment may be needed because of less interest being calculated.
2. **Annuity Payment Rounded Down.** The same principles apply when the annuity payment is rounded down. If you calculated $PMT = \$1,000.0025$, your payments of \$1,000 are underpaying the debt by $\$1,000.0025 - \$1,000 = \$0.0025$. Over the course of 20 payments you nominally underpay by $20 \times \$0.0025 = \$0.05$, for which you must

increase the final payment by this amount. As well, since the principal is slightly larger at all times as a result of the underpayment, an additional adjustment may be needed because of more interest being calculated.

For example, take a \$200,000 loan for 25 years at 6% compounded semi-annually with monthly payments. The calculated $PMT = \$1,279.613247$ becomes a rounded down payment of $PMT = \$1,279.61$. Each payment is $\$0.003247$ underpaid. As a result, the final 300th payment is $\$2.23$ more, representing $\$0.003247 \times 300 = \$0.97$ of original principal underpayment plus $\$1.26$ of extra interest on the outstanding principal.



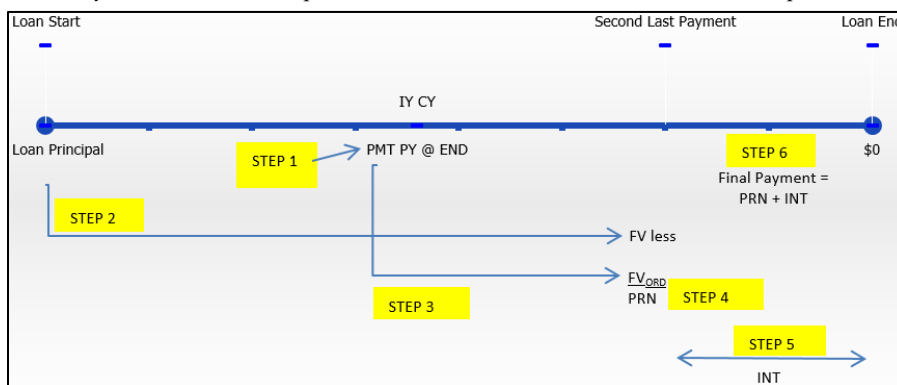
How It Works

The following six steps are needed to calculate the final payment. These steps are designed to integrate with the next section, where the principal and interest components on a series of payments involving the final payment are calculated.

Step 1: Draw a timeline to represent the annuity. A typical timeline format appears in Figure 13.15. Identify all seven time value of money variables. If all are known, proceed to step 2. Most commonly, PMT is unknown. Solve for it using Formulas 9.1 (Periodic Interest Rate), 11.1 (Number of Annuity Payments), and 11.4 (Ordinary Annuity Present Value), rearranging for PMT. Round the PMT to two decimals.

Step 2: Calculate the future value of the original principal at $N - 1$ payments. Obtain the periodic interest rate using Formula 9.1 (Periodic Interest Rate) if you have not already calculated it in step 1. Then use Formulas 9.2 (Number of Compound Periods for Single Payments) and 9.3 (Compound Interest for Single Payments). For example, if your final payment is the 24th payment, you need the balance remaining after the 23rd payment.

Step 3: To calculate the future value of all annuity payments ($N - 1$) already made, apply Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value). Remember that if the final payment is the 24th payment, then only 23 payments have already occurred.



Step 4: Subtract the future value of the payments from the future value of the original principal (step 2 – step 3) to arrive at the principal balance remaining immediately prior to the last payment. This is the principal owing on the account and therefore is the principal portion (PRN) for the final payment. The final payment must reduce the annuity balance to zero!

Step 5: Calculate the interest portion (INT) of the last payment using Formula 13.1 on the remaining principal.

Step 6: Add the principal portion from step 4 to the interest portion from step 5. The sum is the amount of the final payment.



Important Notes

The calculator determines the final payment amount using the AMORT function described in Section 13.1. To calculate the final payment:

1. You must accurately enter all seven time value of money variables (N , I/Y , PV , PMT , FV , P/Y , and C/Y). If PMT was calculated, you must re-enter it with only two decimals while retaining the correct cash flow sign convention.
2. Press 2nd AMORT.
3. Enter the payment number for the final payment into P1 and press Enter followed by ↓.
4. Enter the same payment number for P2 and press Enter followed by ↓.
5. In the BAL window, note the balance remaining in the account after the last payment is made. Watch the cash flow sign to properly interpret what to do with it! The sign matches the sign of your PV . The next table summarizes how to handle this balance.

Type of Transaction	Positive BAL	Negative BAL
Loan	Increase final payment	Decrease final payment
Investment Annuity	Decrease final payment	Increase final payment

- **Loans.** For a loan for which PV is entered as a positive cash flow and hence PMT is a negative cash flow, a positive balance means you are borrowing it. Thus, you need to increase the final payment by this amount to pay off the loan. A negative balance means you overpaid and the bank owes you. Thus, you need to decrease the final payment by this amount.
- **Investment Annuities.** For an investment annuity where PV is entered as a negative cash flow and hence PMT is a positive cash flow, a negative balance means you still have money invested, so you should add it to your final payment to get it back. A positive balance means you have been paid too much, so you need to decrease your final payment by this amount.

A helpful key sequence shortcut to arrive at the final payment is to have BAL on your display and then press

– RCL PMT =

This sequence automatically adjusts the payment accordingly for both loans and investment annuities. Manually round the answer to two decimals when the calculation is complete.

6. If you are interested in the PRN or INT portions of the final payment, the INT output is correct. However, the PRN output is incorrect since the calculator has not adjusted the final payment. You must adjust the PRN output in the same manner and amount as the final payment (by adding or subtracting the BAL remaining).



Things To Watch Out For

In Section 11.5 you calculated the N, and if it turned out to be a decimal number you then approximated the final payment by taking solely the decimal and multiplying it by the annuity payment. Remember that this was only an approximation and not actually the correct answer. For example, revisiting Example 11.5A, it was determined that Samia could sustain 293.660180 payments of \$3,000. The estimated final payment was $0.660180 \times \$3,000 = \$1,980.54$. In applying this chapter's technique, you will find that the precise and correct amount of the final payment is exactly \$1,982.00.

You do not achieve the same answer through the approximation technique, because N does not represent a portion of the payment amount. N represents the portion of the next payment period before the account runs out of money. Recall that in all of the time value formulas N is an exponent, meaning that you must determine the final payment exponentially and not through multiplication.



Paths To Success

An alternative method to adjust the final payment is to calculate the future value of the rounding error on the payment. Recall the \$200,000 loan for 25 years at 6% compounded semi-annually with monthly payments. The calculated $PMT = \$1,279.613247$ was rounded down to $PMT = \$1,279.61$. Each payment is \$0.003247 underpaid. If you move the \$0.003247 underpayment to the 300th final payment this produces a future value of

$$FV_{ORD} = \$0.003247 \left[\frac{\left[(1 + 0.03)^{\frac{2}{12}} \right]^{300} - 1}{(1 + 0.03)^{\frac{2}{12}} - 1} \right] = \$2.23$$

The underpayment is \$2.23, which then means the final payment is increased to $\$1,279.61 + \$2.23 = \$1,281.84$. Note the one disadvantage of this technique is that you are unable to determine the interest and principal components of that final payment. Calculating these components requires you to apply the six-step procedure discussed above.



Give It Some Thought

Consider the following five statements and then answer the two questions that follow.

- a. The final payment is exactly the same as any other annuity payment.
- b. The final payment is smaller, reflecting an overpayment of principal of \$0.04 plus any lower interest charges saved.
- c. The final payment is smaller, reflecting an underpayment of principal of \$0.04 plus any higher interest charges earned.

- d. The final payment is larger, reflecting an overpayment of principal of \$0.04 plus any lower interest charges saved.
- e. The final payment is larger, reflecting an underpayment of principal of \$0.04 plus any higher interest charges earned.
1. If the annuity payment is rounded up by \$0.004 per payment and involves 10 payments, which statement is correct?
2. If the annuity payment is rounded down by \$0.004 per payment and involves 10 payments, which statement is correct?

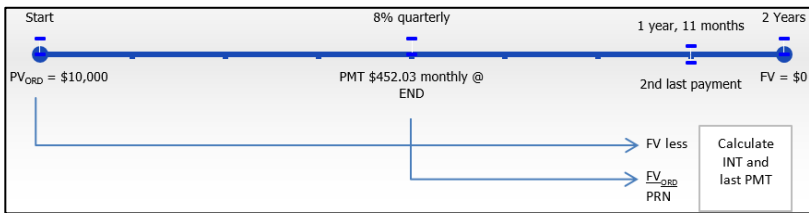
Solutions:

1. b. When rounding up, principal is reduced faster (overpayment) and more interest is saved.

2. e. When rounding down, principal is reduced more slowly (underpayment) and more interest is earned.

Example 13.2A: Final Payment on a Loan

Recall Example 13.1A, in which Nichols and Burnt borrowed \$10,000 at 8% compounded quarterly with month-end payments of \$452.03 for two years. The accountant now needs to record the final payment on the loan with correct portions assigned to principal and interest.

Plan	Calculate the principal portion (PRN) and the interest portion (INT) of the final payment on the two-year loan, along with the amount of the final payment itself (PMT).																													
Understand	What You Already Know Step 1: The following information about the accounting firm's loan is known, as illustrated in the timeline. $PV_{ORD} = \\$10,000$, $IY = 8\%$, $CY = 4$, $PMT = \\$452.03$, $PY = 12$, Years = 2, $N = 12 \times 2 = 24$, $FV = \\$0$	How You Will Get There Step 2: Calculate the future value of the loan principal at the time of the 23rd payment using Formulas 9.1, 9.2, and 9.3. Step 3: Calculate the future value of the first 23 payments using Formulas 11.1 and 11.2. Step 4: Calculate the principal balance remaining after 23 payments through $BAL = FV - FV_{ORD}$. Note that $BAL = PRN$ portion of the final payment. Step 5: Calculate the interest portion by using Formula 13.1. Step 6: Calculate the final payment by totalling steps 4 and 5 above.																												
																														
Perform	Step 2: $i = 8\%/4 = 2\%$; $N = 4 \times 1 \frac{11}{12} = 7.\overline{6}$ compounds; $FV = \\$10,000(1 + 0.02)^{7.\overline{6}} = \\$11,639.50872$ Step 3: $N = 12 \times 1 \frac{11}{12} = 23$ payments $FV_{ORD} = \\$452.03 \left[\frac{[(1+0.02)^{\frac{4}{12}}]^{23} - 1}{(1+0.02)^{\frac{4}{12}} - 1} \right] = \\$11,190.39151$ Step 4: $BAL = \\$11,639.50872 - \\$11,190.39151 = \\$449.117209$ Step 5: $INT = \\$449.117209 \times \left((1 + 0.02)^{\frac{4}{12}} - 1 \right) = \\$2.974372$ Step 6: Final PMT = $\\$449.117209 + \\$2.974372 = \\$452.09$																													
	Calculator Instructions <table><tr><td>N</td><td>I/Y</td><td>PV</td><td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>24</td><td>8</td><td>10000</td><td>-452.03</td><td>0</td><td>12</td><td>4</td></tr></table> <table><tr><td>P1</td><td>P2</td><td>BAL (output)</td><td>PRN (output)</td><td>INT (output)</td></tr><tr><td>24</td><td>24</td><td>0.061582</td><td>449.055627</td><td>2.974372</td></tr><tr><td></td><td></td><td>*added to payment of \$452.03 = \$452.09</td><td>*add BAL = \$449.12</td><td></td></tr></table>		N	I/Y	PV	PMT	FV	P/Y	C/Y	24	8	10000	-452.03	0	12	4	P1	P2	BAL (output)	PRN (output)	INT (output)	24	24	0.061582	449.055627	2.974372			*added to payment of \$452.03 = \$452.09	*add BAL = \$449.12
N	I/Y	PV	PMT	FV	P/Y	C/Y																								
24	8	10000	-452.03	0	12	4																								
P1	P2	BAL (output)	PRN (output)	INT (output)																										
24	24	0.061582	449.055627	2.974372																										
		*added to payment of \$452.03 = \$452.09	*add BAL = \$449.12																											
Present	The accountant for Nichols and Burnt should record a final payment of \$452.09, which consists of a principal portion of \$449.12 and an interest portion of \$2.97.																													

Example 13.2B: Final Payment on an Investment Annuity

Recall Example 13.1B, in which Baxter has \$50,000 invested into a five-year annuity that earns 5% compounded quarterly and makes regular end-of-quarter payments of \$2,841.02 to him. He needs to know the amount of his final payment, along with the principal and interest components.

Plan Calculate the principal portion (PRN) and the interest portion (INT) of the final payment on the five-year investment annuity, along with the amount of the final payment itself (PMT).

Understand

What You Already Know

Step 1: You know the following about the investment annuity, as illustrated in the timeline.

$PV_{ORD} = \$50,000$, $IY = 5\%$,
 $CY = 4$, $PMT = \$2,841.02$,
 $PY = 4$, Years = 5,
 $N = 4 \times 5 = 20$, $FV = \$0$

How You Will Get There

Step 2: Calculate the future value of the investment at the time of the 19th payment using Formulas 9.1, 9.2 and 9.3.

Step 3: Calculate the future value of the first 19 payments using Formulas 11.1 and 11.2.

Step 4: Calculate the principal balance remaining after 19 payments through $BAL = FV - FV_{ORD}$. Note that this is the PRN portion of the final payment.

Step 5: Calculate the interest portion by using Formula 13.1.

Step 6: Calculate the final payment by totalling steps 4 and 5 above.

Perform

Step 2: $i = 5\%/4 = 1.25\%$; $N = 4 \times 4\frac{3}{4} = 19$ compounds; $FV = \$50,000(1 + 0.0125)^{19} = \$63,310.48058$

Step 3: $N = 4 \times 4\frac{3}{4} = 19$ payments $FV_{ORD} = \$2,841.02 \left[\frac{(1+0.0125)^{4 \times 19} - 1}{(1+0.0125)^4 - 1} \right] = \$60,504.54645$

Step 4: $BAL = \$63,310.48058 - \$60,504.54645 = \$2,805.934127$

Step 5: $INT = \$2,805.934127 \times ((1 + 0.0125)^4 - 1) = \$35.074176$

Step 6: Final PMT = $\$2,805.934127 + \$35.074176 = \$2,841.01$

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
20	5	-50000	2841.02	0	4	4

P1	P2	BAL (output)	PRN (output)	INT (output)
20	20	0.011696	2,805.945823	35.074176
		*subtracted from payment of \$2,841.02 = \$2,841.01	*subtract BAL = \$2,805.93	

Present Baxter will receive a final payment of \$2,841.01 consisting of \$2,805.93 in principal plus \$35.08 in interest.

Calculating Principal and Interest Portions for a Series Involving the Final Payment

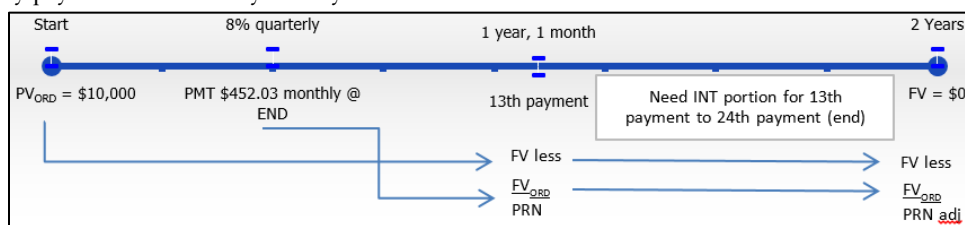
Now that you know how to calculate the last payment along with its interest and principal components, it is time to extend this knowledge to calculating the principal and interest portions for a series of payments that involve the final payment.



How It Works

For a series of payments, you follow essentially the same steps as in Section 13.1; however, you need a few minor modifications and interpretations:

Step 1: Draw a timeline. Identify the known time value of money variables, including IY, CY, PY, Years, and one of PV_{ORD} or FV_{ORD} . The annuity payment amount may or may not be known.



Step 2: If the annuity payment amount is known, proceed to step 3. If it is unknown, then solve for the annuity payment using Formulas 9.1 (Periodic Interest Rate) and 11.1 (Number of Annuity Payments) and by rearranging Formula 11.4 (Ordinary Annuity Present Value). Round this payment to two decimals.

Step 3: Calculate the future value of the original principal immediately prior to the series of payments being made. Use Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compounding Periods for a Single Payment), and 9.3 (Compound Interest for Single Payments).

Step 4: Calculate the future value of all annuity payments already made prior to the first payment in the series. Apply Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value).

Step 5: Calculate the balance (BAL) prior to the series of payments by subtracting step 4 (the future value of the payments) from step 3 (the future value of the original principal). The result of this step determines the amount of principal remaining in the account. This is the PRN for the series of payments, since the remaining payments must reduce the principal to zero.

Step 6: Calculate the future value of the original principal immediately at the end of the timeline. Use Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compounding Periods for Single Payments), and 9.3 (Compound Interest for Single Payments).

Step 7: Calculate the future value of all annuity payments, including the unadjusted final payment. Apply Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value).

Step 8: Calculate the balance (BAL) after the series of payments by subtracting step 7 (the future value of the payments) from Step 6 (the future value of the original principal). The fundamental concept of time value of money allows you to combine these two numbers on the same focal date. Do not round this number. The result of this step determines the amount of overpayment or underpayment, which you must then adjust in the next step.

Step 9: Calculate the interest portion using Formula 13.4, but modify the final amount by the result from step 8. Hence, Formula 13.4 looks like

$$INT = N \times PMT - PRN + (\text{balance from step 8})$$

Example 13.2C: Principal and Interest for a Series Involving the Final Payment

Revisit Example 13.1A. The accountant at the accounting firm of Nichols and Burnt is completing the tax returns for the company and needs to know the total principal portion and interest expense paid during the tax year encompassing payments 13 through 24 inclusively. Recall that the company borrowed \$10,000 at 8% compounded quarterly, with month-end payments of \$452.03 for two years.

Plan

Calculate the total principal portion (PRN) and the total interest portion (INT) of the 13th to 24th payments on the two-year loan. This involves the final payment since the 24th payment is the last payment, requiring usage of the adapted steps discussed in this section.

Understand	What You Already Know Step 1: The following information about the accounting firm's loan is known, as illustrated in the timeline. $PV_{ORD} = \\$10,000$, $IY = 8\%$, $CY = 4$, $PMT = \\$452.03$, $PY = 12$, Years = 2, $N = 2 \times 12 = 24$, $FV = \\$0$	How You Will Get There Step 2: Skip this step, since PMT is known. Step 3: Calculate the future value of the loan principal after the 12th payment using Formulas 9.1, 9.2, and 9.3. Step 4: Calculate the future value of the first 12 payments using Formulas 11.1 and 11.2. Step 5: Calculate the balance remaining after 12 payments through $BAL_{P1} = FV - FV_{ORD}$. Note that for payments 13 through 24, $PRN = BAL_{P1}$. Step 6: Calculate the future value of the loan principal after the 24th payment using Formulas 9.2 and 9.3. Step 7: Calculate the future value of all 24 payments using Formulas 11.1 and 11.2. Step 8: Calculate the balance on the loan after all payments through $BAL_{P2} = FV - FV_{ORD}$. Note that this amount is used to adjust step 9. Step 9: Calculate the interest portion by using the adjusted Formula 13.4.																												
	The diagram shows a timeline from 'Start' to '2 Years'. At 'Start', $PV_{ORD} = \\$10,000$. Quarterly interest is 8%. Monthly payments of $PMT \\$452.03$ are made at the end of each month. At the 12th payment, the balance BAL_{P1} is calculated. At the 13th payment, the interest portion INT is calculated. The final balance at 2 years is $FV = \\$0$.																													
Perform	Step 3: $i = 8\%/4 = 2\%$; $N = 4 \times 1 = 4$ compounds; $FV = \\$10,000(1 + 0.02)^4 = \\$10,824.3216$ Step 4: $N = 12 \times 1 = 12$ payments $FV_{ORD} = \\$452.03 \left[\frac{\left[(1+0.02)^{\frac{4}{12}} \right]^{12} - 1}{(1+0.02)^{\frac{4}{12}} - 1} \right] = \\$5,626.36923$ Step 5: $BAL_{P1} = \\$10,824.3216 - \\$5,626.36923 = \\$5,197.95237$; $PRN = \\$5,197.95$ Step 6: $N = 4 \times 2 = 8$ compounds; $FV = \\$10,000(1 + 0.02)^8 = \\$11,716.59381$ Step 7: $N = 12 \times 2 = 24$ payments $FV_{ORD} = \\$452.03 \left[\frac{\left[(1+0.02)^{\frac{4}{12}} \right]^{24} - 1}{(1+0.02)^{\frac{4}{12}} - 1} \right] = \\$11,716.53223$ Step 8: $BAL_{P2} = \\$11,716.59381 - \\$11,716.53223 = \\$0.061582$ Step 9: $N = 13\text{th through } 24\text{th payment inclusive} = 12$ payments; $INT = 12 \times \\$452.03 - \\$5,197.95237 + \\$0.061582 = \\$5,424.36 - \\$5,197.89 = \\$226.47$																													
	Calculator Instructions <table><tr><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>24</td><td>8</td><td>10000</td><td>-452.03</td><td>0</td><td>12</td><td>4</td></tr></table> <table><tr><th>P1</th><th>P2</th><th>BAL (output)</th><th>PRN (output)</th><th>INT (output)</th></tr><tr><td>13</td><td>24</td><td>0.061582</td><td>5,197.890788</td><td>226.469212</td></tr><tr><td colspan="2"></td><td colspan="3">*add BAL = \$5,197.95</td></tr></table>		N	I/Y	PV	PMT	FV	P/Y	C/Y	24	8	10000	-452.03	0	12	4	P1	P2	BAL (output)	PRN (output)	INT (output)	13	24	0.061582	5,197.890788	226.469212			*add BAL = \$5,197.95	
N	I/Y	PV	PMT	FV	P/Y	C/Y																								
24	8	10000	-452.03	0	12	4																								
P1	P2	BAL (output)	PRN (output)	INT (output)																										
13	24	0.061582	5,197.890788	226.469212																										
		*add BAL = \$5,197.95																												
Present	For the tax year covering payments 13 through 24, total payments of \$5,424.42 are made, of which \$5,197.95 goes toward principal while \$226.47 is the interest charged.																													

**TOP
SECRET**

Paths To Success

Amortization by definition involves the repayment of loans, which are almost always ordinary in nature. What are the implications if you have an investment annuity due, or the rare occurrence of a loan due?

- Step-by-Step Procedures.** Whether you are dealing with ordinary annuities or annuities due, all of the processes and procedures remain unchanged.
- Formulas.** Make the appropriate substitutions from FV_{ORD} to FV_{DUE} .

3. **Time Value of Money Button Calculator Settings.** When the calculator is loaded with the time value of money variables, ensure that BGN mode has been set and that if the PMT has been computed you have re-entered it with two decimals.
 4. **AMORT Function on the BAII Plus Calculator.** A special adaptation is required but is not introduced at this time. This special adaptation will be discussed in Section 13.3 on amortization schedules due.
-



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13.3: Amortization Schedules

(How Much Is Mine, How Much Is Theirs?)

In the previous two sections, you have been working on parts of an entire puzzle. You have calculated the interest and principal portions for either a single payment or a series of payments. Additionally, you calculated the final payment amount along with its principal and interest components. The next task is to put these concepts together into a complete understanding of amortization. This involves developing a complete amortization schedule for an annuity (loan or investment annuity). Additionally, you will create partial amortization schedules that depict specific ranges of payments for a particular annuity.

These concepts primarily apply to accounting. Accountants must record debts properly on balance sheets, make proper journal entries into the accounting books, and track interest expenses clearly. Depreciation of fixed assets uses similar amortization processes and schedules. Bond premiums and discounts (discussed in Chapter 14) also use amortization.

On the personal side, these schedules help you understand any of your loans, mortgages, or investment annuities. Sometimes seeing the true amount of interest you are paying may motivate you to pay the debt off faster. Courts also use these schedules to settle legal matters such as alimony payments.

The Complete Amortization Schedule

An **amortization schedule** shows the payment amount, principal component, interest component, and remaining balance for every payment in the annuity. As the title suggests, it provides a complete understanding of where the money goes.



How It Works

Payment Number	Payment Amount (\$) (PMT)	Interest Portion (\$) (INT)	Principal Portion (\$) (PRN)	Principal Balance Remaining (\$) (BAL)
0—Start	N/A	N/A	N/A	(3)
1	(4)	(5)	(6)	(7)
etc.				
Last payment	(11)	(10)	(9)	\$0.00
Total	*if needed	(13)	*equals original loan balance	N/A

Follow these steps to develop a complete amortization schedule:

Step 1: Identify all of the time value of money variables (N , IY , PV_{ORD} , PMT , FV , PY , CY). If either N or PMT is unknown, solve for it using an appropriate formula. Remember to round PMT to two decimals.

Step 2: Set up an amortization schedule template as per the table above. Cells of the table are marked with the step numbers that follow. The number of payment rows in the table equals the number of payments in the annuity. Note that in the instance of a general annuity, the table does not distinguish between earned interest and accrued interest. The schedule reflects only the total of earned interest and interest accrued at the time of a payment.

Step 3: On the first row, fill in the original principal balance of the annuity, or PV_{ORD} .

Step 4: Fill in the rounded annuity payment (PMT) all the way down the column except for the final payment row.

Step 5: Calculate the interest portion of the current annuity payment using Formula 13.1. Round the number to two decimals for the table but retain the decimals for future calculations.

Step 6: Calculate the principal portion of the current annuity payment using Formula 13.2. The interest component is the unrounded interest number from step 5. Round the result to two decimals for the table but retain the decimals for future calculations.

Step 7: Calculate the new principal balance remaining by taking the previous unrounded balance on the line above and subtracting the unrounded principal portion (PRN) of the current payment. This is Formula 13.3 rearranged such that $BAL_{p2} = BAL_{p1} - PRN$. Round the result to two decimals for the table but retain the decimals for future calculations.

Step 8: Repeat steps 5 through 7 for each annuity payment until you reach the final payment.

Step 9: For the final payment, the principal portion is exactly equal to the previous balance remaining on the line above. Round the result to two decimals for the table but retain the decimals for future calculations. Enter "\$0.00" in the Principal Balance Remaining.

Step 10: Calculate the interest portion of the final annuity payment using Formula 13.1. Round the result to two decimals for the table but retain the decimals for future calculations.

Step 11: Calculate the final payment by adding the unrounded principal and interest portions of the final payment together. Round this number to two decimals.

Step 12: Since all numbers are rounded to two decimals throughout the table, check the table for the "missing penny," as discussed below. Always ensure that for each row the previous balance minus current principal equals the new balance. Then check that the principal portion plus interest portion equals the annuity payment amount. Make any penny adjustments as needed.

Step 13: Sum the interest portion column in the schedule. Note that no total is necessary for the principal portion since it equals the original principal of the annuity!



Important Notes

The "Missing Penny". In the creation of the amortization schedule, you always round the numbers off to two decimals since you are dealing with currency. However, as per the rules of rounding, you do not round any numbers in your calculations until you reach the end of the amortization schedule and the annuity has been reduced to zero.

As a result, you have a triple rounding situation involving the balance along with the principal and interest components on every line of the table. What sometimes happens is that a "missing penny" occurs and the schedule needs to be corrected as per step 12 of the process above. In other words, calculations will sometimes appear to be off by a penny. You can identify the "missing penny" when one of the two standard calculations using the rounded numbers from the schedule becomes off by a penny:

$$1. \quad \text{Previous BAL} - \text{PRN} = \text{current BAL (step 7)}$$

or

$$2. \quad \text{PMT} - \text{INT} = \text{PRN (step 6)}$$

It is important to remember, though, that in actuality there is no "missing penny" in these calculations. It shows up only because you are rounding off numbers. If you were not rounding numbers, this "missing penny" would never occur.

In these instances of the "missing penny", you adjust the schedule as needed to ensure that the math works properly at all times. The golden rule, though, is that the balance in the account (BAL) is always correct and should NEVER be adjusted. Follow this order in making any adjustments:

1. Adjust the PRN if necessary such that the previous $BAL - PRN = \text{current BAL}$.
2. Then adjust INT if necessary such that $\text{PMT} - \text{PRN} = \text{INT}$.

Usually these adjustments come in pairs, meaning that if you need to adjust the PRN up by a penny, somewhere later in the schedule you will need to adjust the PRN down by a penny. Ultimately, these changes in most circumstances have no impact on the total interest (INT) or total principal (PRN) components, since the "missing penny" is nothing more than a rounding error within the schedule.

Your BAI Plus Calculator. The calculator speeds up the repetitive calculations required in the amortization schedule. To create the schedule using the calculator, adapt the steps as follows:

Step 1: Load the calculator with all seven time value of money variables, solving for any unknowns. Ensure that PMT is keyed in with two decimals, and obey the cash flow sign convention.

Steps 2–4: Unchanged.

Steps 5–7: Open the AMORT function. Set the P1 = 1 and P2 = 1. In the appropriate column, record the BAL, PRN, and INT rounded to two decimals.

Step 8: Repeat steps 5–7 by increasing the payment number (P1 and P2) by one each time. Ensure P1 = P2 at all times.

Step 9: Unchanged.

Step 10: Set the P1 and P2 to the final payment number. Record the INT amount.

Steps 11–12: Unchanged.

Step 13: Set the P1 = 1 and P2 = final payment number. Record the INT amount.

Example 13.3A: Payment Plan on a Dishwasher

Tamara purchased a new dishwasher from a department store for \$895.94. By placing it on her store credit card, she can pay off the dishwasher through a special six-month payment plan promotion that charges her 5.9% compounded monthly. Construct the complete amortization schedule for Tamara and total her interest charges.

Plan

Construct a complete amortization schedule for the dishwasher payments along with the total interest paid.

Understand

What You Already Know

Step 1: The timeline for Tamara's purchase appears in the timeline.

$PV_{ORD} = \$895.94$,
 $IY = 5.9\%$, $CY = 12$,
 $PY = 12$, $Years = 0.5$,
 $FV = \$0$

How You Will Get There

Step 1 (continued): Solve for the payment (PMT) using Formulas 9.1, 11.1, and 11.4.

Step 2: Set up the amortization table.

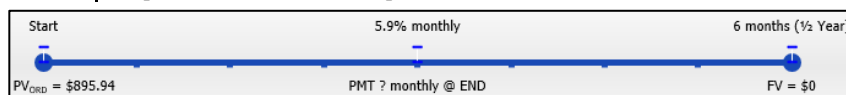
Steps 3 and 4: Fill in the original principal and payment column.

Steps 5 to 8: For each line use Formulas 13.1 and 13.2 and the rearranged Formula 13.3.

Steps 9 to 11: Fill in the principal and balance remaining, calculate the interest using Formula 13.1, and determine the final payment by adding the interest and principal components together.

Step 12: Check for the "missing penny."

Step 13: Sum the interest portion.



Perform

Step 1 (continued): $i = 5.9\% / 12 = 0.491\bar{6}\%$; $N = 12 \times \frac{1}{2} = 6$ payments

$$\$895.94 = PMT \left[\frac{1 - \left[\frac{1}{(1 + 0.00491\bar{6})^{\frac{12}{12}}} \right]^6}{(1 + 0.00491\bar{6})^{\frac{12}{12}} - 1} \right]$$

$$PMT = \frac{\$895.94}{\left[\frac{1 - 0.971001}{0.00491\bar{6}} \right]} = \frac{\$895.94}{[5.898088]} = \$151.90$$

Steps 2 to 11 (with some calculations) are detailed in the table below:

Payment Number	Payment Amount (\$ (PMT))	Interest Portion (\$ (INT))	Principal Portion (\$ (PRN))	Principal Balance Remaining (\$ (BAL))
0—Start				\$895.94
1	\$151.90	(1) \$4.41	(2) \$147.49	(3) \$748.45
2	\$151.90	\$3.68	\$148.22	\$600.22
3	\$151.90	\$2.95	\$148.95	\$451.28
4	\$151.90	\$2.22	\$149.68	\$301.59
5	\$151.90	\$1.48	\$150.42	\$151.18
6	(4) \$151.92	\$0.74	\$151.18	\$0.00
Total				

(1) $INT = \$895.94 \times ((1 + 0.00491\bar{6})^{\frac{12}{12}} - 1) = \$4.405038$

(2) $PRN = \$151.90 - \$4.405038 = \$147.494961$

(3) $BAL_{P2} = \$895.94 - \$147.494961 = \$748.445038$

(4) $Final\ Payment = \$151.177613 + \$0.742386 = \$151.92$

Steps 12 to 13: Adjust for the "missing pennies" (noted in red) and total the interest.

Payment Number	Payment Amount (\$ (PMT))	Interest Portion (\$ (INT))	Principal Portion (\$ (PRN))	Principal Balance Remaining (\$ (BAL))
0—Start				\$895.94
1	\$151.90	\$4.41	\$147.49	\$748.45
2	\$151.90	\$3.67	\$148.23	\$600.22
3	\$151.90	\$2.96	\$148.94	\$451.28
4	\$151.90	\$2.21	\$149.69	\$301.59
5	\$151.90	\$1.49	\$150.41	\$151.18
6	\$151.92	\$0.74	\$151.18	\$0.00
Total	\$911.42	\$15.48	\$895.94	

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
6	5.9	895.94	Answer: -151.903441 Re-keyed as -151.90	0	12	12

Payment	P1	P2	BAL (output)	PRN (output)	INT (output)
1	1	1	748.4450383	147.494961	4.4050383
2	2	2	600.224893	148.22014	3.679854
3	3	3	451.275998	148.948894	2.951105
4	4	4	301.594772	149.681226	2.218773
5	5	5	151.177613	150.417159	1.482840
6	6	6	0.020903	151.156710 *add BAL=151.177613	0.743289
Total	1	6	0.020903	\$95.919096 *add BAL=895.94	15.480903

Present

The complete amortization schedule is shown in the table above. The total interest Tamara is to pay on her dishwasher is \$15.48.

Amortization Schedule Due

In the instance of an annuity due, you require a small modification to the amortization schedule, as illustrated in the next table. Notice that the headers of the second and fifth columns have been modified to clarify the timing of the payment and point in time when the balance is achieved. Each line of the table still represents one payment interval.

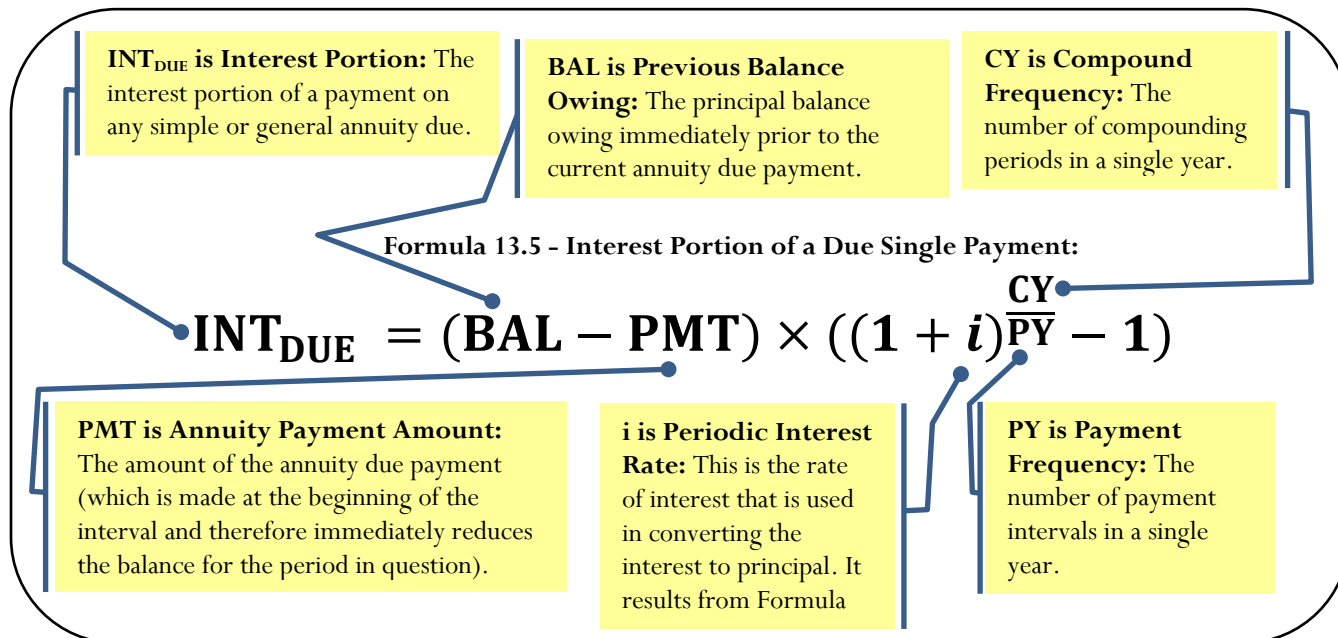
Payment Number	Payment Amount at Beginning of Interval (\$ (PMT))	Interest Portion (\$ (INT))	Principal Portion (\$ (PRN))	Principal Balance Remaining at End of Interval (\$ (BAL))
0—Start	N/A	N/A	N/A	(3)
1	(4)	(5)	(6)	(7)
etc.				
Last payment	(11)	\$0.00 (10)	(9)	\$0.00
Total	*if needed	(13)	*equals original loan balance	N/A



The Formula

Be sure to determine any calculated amounts such as PMT through the appropriate annuity due formulas and not the ordinary annuity formulas. Recall that in an annuity the annuity payment is made at the beginning of the interval, which

immediately reduces the balance eligible for interest. Calculating interest requires Formula 13.1 to be adapted and reintroduced as Formula 13.5.



How It Works

When you work with the amortization of annuities due, you must make the following modifications to the 13-step process:

Step 1: Ensure that you use appropriate annuity due formulas when solving for any unknown variable.

Step 5: Recall that the payment is made at the beginning of the interval, which immediately reduces the balance eligible for interest. Calculating interest requires Formula 13.5, not Formula 13.1.

Step 10: The last annuity due payment reduces the balance to zero. Therefore, no interest is accrued for the last payment interval.



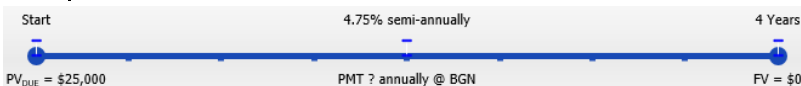
Important Notes

The BAII Plus Calculator. On the calculator, the AMORT function is designed only for ordinary amortization; however, a little adaptation allows you to generate amortization due outputs quickly:

- **P1 and P2.** In an annuity due, since the first payment occurs today (time period 0), the second payment is at time period 1, and so on, the payment number of an annuity due is always one higher than the payment number of an ordinary annuity. To adapt on your calculator, always add 1 to the payment number being calculated. For example, if you are interested in payment seven only, set both P1 and P2 to 8. Or, if you are interested in the payment range 14 through 24, set P1 = 15 and P2 = 25.
- **BAL:** To generate the output, the payment numbers are one too high. This results in the balance being decreased by one extra payment. To adapt the procedure, manually increase the balance by adding one payment (or with BAL on your display, use a shortcut key sequence of $\text{RCL PMT} =$).
- **INT and PRN:** Both of these numbers are correct except for the final payment. If working with the final payment, you need to adjust the PRN by the BAL remaining.

Example 13.3B: An Investment Annuity Due

Maisy just moved to Toronto to attend the University of Toronto. Shortly after she settled into her new apartment, her parents sent her an email on September 1 confirming that they had set up an investment annuity for \$25,000 at 4.75% compounded semi-annually to assist her with tuition fees and buying textbooks. The annuity will deposit the funds to her bank account annually starting today for four years. Construct a complete amortization schedule and calculate the total interest earned.

Plan	Construct a complete amortization schedule for the investment fund along with the total interest earned. Note that this situation presents an investment annuity due.																																						
Understand	What You Already Know Step 1: The timeline for the education fund appears in the timeline. $PV_{DUE} = \$25,000$, $IY = 4.75\%$, $CY = 2$, $PY = 1$, Years = 4, $FV = \$0$	How You Will Get There Step 1 (continued): Solve for the payment (PMT) using Formulas 9.1, 11.1, and 11.5. Step 2: Set up the amortization table for an annuity due. Steps 3 and 4: Fill in the original principal and payment column. Put the first payment on the first line and deduct it immediately with no interest from the principal. Steps 5 to 8: For each line use Formulas 13.5 and 13.2 and the rearranged Formula 13.3. Steps 9 to 11: Fill in the principal and balance remaining, set the interest portion to zero, and determine the final payment by adding the interest and principal component together. Step 12: Check for the "missing penny." Step 13: Total up the interest portion.																																					
																																							
Perform	Step 1 (continued): $i = 4.75\%/2 = 2.375\%$; $N = 1 \times 4 = 4$ payments																																						
	$\$25,000 = PMT \left[\frac{1 - \left[\frac{1}{(1 + 0.02375)^{\frac{2}{1}}} \right]^4}{(1 + 0.02375)^{\frac{2}{1}} - 1} \right] \times (1 + 0.02375)^{\frac{2}{1}} \quad PMT = \frac{\$25,000}{\left[\frac{1 - \left[\frac{1}{1.048064} \right]^4}{0.048064} \right] \times 1.048064} = \$6,696.74$																																						
Steps 2 to 11 (with some calculations) are detailed in the table below:																																							
<table><tr><th>Payment Number</th><th>Payment Amount at Beginning of Interval (\$) (PMT)</th><th>Interest Portion (\$) (INT)</th><th>Principal Portion (\$) (PRN)</th><th>Principal Balance Remaining at End of Interval (\$) (BAL)</th></tr><tr><td>0—Start</td><td></td><td></td><td></td><td>\$25,000.00</td></tr><tr><td>1</td><td>\$6,696.74</td><td>(1)\$879.73</td><td>(2)\$5,817.01</td><td>(3)\$19,182.99</td></tr><tr><td>2</td><td>\$6,696.74</td><td>\$600.14</td><td>\$6,096.60</td><td>\$13,086.39</td></tr><tr><td>3</td><td>\$6,696.74</td><td>\$307.11</td><td>\$6,389.63</td><td>\$6,696.76</td></tr><tr><td>4</td><td>\$6,696.76</td><td>\$0.00</td><td>\$6,696.76</td><td>\$0.00</td></tr><tr><td>Total</td><td>\$26,786.98</td><td>\$1,786.98</td><td>\$25,000.00</td><td></td></tr></table>					Payment Number	Payment Amount at Beginning of Interval (\$) (PMT)	Interest Portion (\$) (INT)	Principal Portion (\$) (PRN)	Principal Balance Remaining at End of Interval (\$) (BAL)	0—Start				\$25,000.00	1	\$6,696.74	(1)\$879.73	(2)\$5,817.01	(3)\$19,182.99	2	\$6,696.74	\$600.14	\$6,096.60	\$13,086.39	3	\$6,696.74	\$307.11	\$6,389.63	\$6,696.76	4	\$6,696.76	\$0.00	\$6,696.76	\$0.00	Total	\$26,786.98	\$1,786.98	\$25,000.00	
Payment Number	Payment Amount at Beginning of Interval (\$) (PMT)	Interest Portion (\$) (INT)	Principal Portion (\$) (PRN)	Principal Balance Remaining at End of Interval (\$) (BAL)																																			
0—Start				\$25,000.00																																			
1	\$6,696.74	(1)\$879.73	(2)\$5,817.01	(3)\$19,182.99																																			
2	\$6,696.74	\$600.14	\$6,096.60	\$13,086.39																																			
3	\$6,696.74	\$307.11	\$6,389.63	\$6,696.76																																			
4	\$6,696.76	\$0.00	\$6,696.76	\$0.00																																			
Total	\$26,786.98	\$1,786.98	\$25,000.00																																				
(1) $INT_{DUE} = (\$25,000 - \$6,696.74) \times ((1 + 0.02375)^{\frac{2}{1}} - 1) = \$879.729032$ (2) $PRN = \$6,696.74 - \$879.729032 = \$5,817.010967$ (3) $BAL_{P2} = \$25,000 - \$5,817.010967 = \$19,182.99$ Step 12: There are no "missing pennies." Step 13: Interest is totalled in the table above.																																							

Calculator Instructions

Mode	N	IY	PV	PMT	FV	P/Y	C/Y
BGN	4	4.75	-25000	Answer: 6,696.744972 Rekeyed as 6,696.74	0	1	2

Payment	P1	P2	BAL (output)	PRN (output)	INT (output)
1	2	2	-12,486.24903 *add PMT = 19,192.99	5,817.010967	879.729032
2	3	3	-6,389.648886 * add PMT = 13,086.39	6,096.600146	600.139853
3	4	4	-0.021369 * add PMT = 6,696.76	6,389.627517 *add BAL = 6,389.648886	307.112483
4	5	5	6,696.717603 * add PMT = -0.021369	6,696.738973 *add BAL = 6,696.76	0.001027
Total	2	5	6,696.717603 * add PMT = -0.021369	24,999.97863 *add BAL = 25,000	1,786.98137

Present

The complete amortization schedule is shown in the table. The total interest earned on the education fund is \$1,786.98.

The Partial Amortization Schedule

Sometimes, businesses are interested only in creating **partial amortization schedules**, which are amortization schedules that show only a specified range of payments and not the entire annuity. This may occur for a variety of reasons. For instance, the complete amortization schedule may be too long (imagine weekly payments on a 25-year loan), or maybe you are solely interested in the principal and interest portions during a specific period of time for accounting and tax purposes.



How It Works

The procedure for partial amortization schedules remains almost the same as a complete schedule with the following notable differences:

Step 2: Set up a partial amortization schedule template as per this next table. The corresponding step numbers are identified in the table.

Payment Number	Payment Amount (\$ (PMT))	Interest Portion (\$ (INT))	Principal Portion (\$ (PRN))	Principal Balance Remaining (\$ (BAL))
Preceding Payment #	N/A	N/A	N/A	(3)
First Payment Number of Partial Schedule	(4)	(5)	(6)	(7)
etc.				
Last Payment Number of Partial Schedule	(11)*	(10)*	(9)*	\$0.00
Totals	*if needed	(13)	(13)	N/A

*Perform these steps only if the final payment in the annuity is involved in the schedule

Step 3: Calculate the balance remaining in the account immediately before the first payment of the partial schedule. Recall that this requires you to calculate the future value of the principal less the future value of all payments made.

Steps 9 to 11: These steps are required only if the final payment is involved in the partial schedule.

Step 13: Sum the principal portion as well as the interest portion.

Example 13.3C: A Partial Loan Amortization Schedule on a Loan

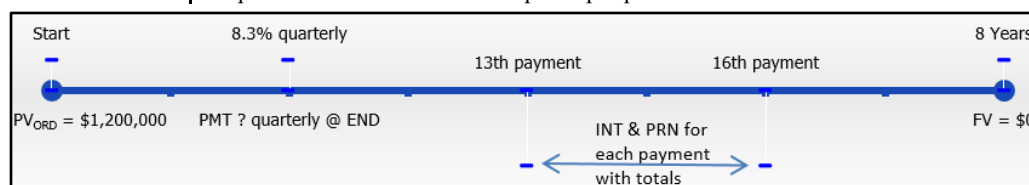
Molson Coors Brewing Company just acquired \$1.2 million worth of new brewing equipment for its Canadian operations. The terms of the loan require end-of-quarter payments for eight years at 8.3% compounded quarterly. For accounting purposes, the company is interested in knowing the principal and interest portions of each payment for the fourth year and also wants to know the total interest and principal paid during the year. Construct the partial amortization schedule.

Plan Construct a partial amortization schedule for the fourth year of the loan along with the total interest and principal paid during the year.

Understand

What You Already Know
 Step 1: The timeline for the equipment loan appears in the timeline.
 $PV_{ORD} = \$1,200,000$,
 $IY = 8.3\%$, $CY = 4$,
 $PY = 4$, Years = 8,
 $FV = \$0$

How You Will Get There
 Step 1 (continued): Solve for the payment (PMT) using Formulas 9.1, 11.1, and 11.4.
 Step 2: Set up the partial amortization table for the ordinary annuity.
 Step 3: Calculate the future value of loan principal for the end of the third year (the 12th payment) using Formulas 9.2 and 9.3.
 Then calculate the future value of the first 12 payments using Formulas 11.1 and 11.2.
 Calculate the principal balance after 12 payments through $BAL = FV - FV_{ORD}$.
 Step 4: Fill in the payment column for the four payments made in the fourth year.
 Steps 5 to 8: For each line of the table, use Formulas 13.1 and 13.2 and the rearranged Formula 13.3.
 Steps 9 to 11: Not needed, since the final payment is not involved.
 Step 12: Check for the "missing penny."
 Step 13: Sum the interest and principal portions.



Perform

Step 1 (continued): $i = 8.3\%/4 = 2.075\%$; $N = 4 \times 8 = 32$ payments

$$\$1,200,000 = PMT \left[\frac{1 - \left[\frac{1}{(1 + 0.02075)^4} \right]^{32}}{(1 + 0.02075)^4 - 1} \right]$$

$$PMT = \frac{\$1,200,000}{\left[\frac{0.481701}{0.02075} \right]} = \$51,691.71$$

Steps 2 to 11 (with some calculations, including Step 3) are detailed in the table below:

Payment Number	Payment Amount (\$) (PMT)	Interest Portion (\$) (INT)	Principal Portion (\$) (PRN)	Principal Balance Remaining (\$) (BAL)
12				(1) \$839,147.91
13	\$51,691.71	(2) \$17,412.32	(3) \$34,279.39	(4) \$804,868.52
14	\$51,691.71	\$16,701.02	\$34,990.69	\$769,877.83
15	\$51,691.71	\$15,974.96	\$35,716.75	\$734,161.08
16	\$51,691.71	\$15,233.84	\$36,457.87	\$697,703.22
Totals				

(1) Step 3: Principal: $N = 4 \times 3 = 12$ compounds; $FV = \$1,200,000(1 + 0.02075)^{12} = \$1,535,373.036$
 Payments: $N = 4 \times 3 = 12$ payments;

$$FV_{ORD} = \$51,691.71 \left[\frac{\left[\frac{(1 + 0.02075)^4}{(1 + 0.02075)^4} - 1 \right]^{12}}{(1 + 0.02075)^4 - 1} \right] = \$51,691.71 \left[\frac{0.279477}{0.02075} \right] = \$696,225.1283$$

Balance: $\$1,535,373.036 - \$696,225.1283 = \$839,147.9072$

(2) $INT = \$839,147.9072 \times ((1 + 0.02075)^4 - 1) = \$17,412.31908$
 (3) $PRN = \$51,691.71 - \$17,412.31908 = \$34,279.39092$
 (4) $BAL_{P2} = \$839,147.9072 - \$34,279.39092 = \$804,868.5163$

Steps 12 to 13: Adjust for the "missing pennies" (noted in red) and total the interest.

Payment Number	Payment Amount (\$) (PMT)	Interest Portion (\$) (INT)	Principal Portion (\$) (PRN)	Principal Balance Remaining (\$) (BAL)
12				(1) \$839,147.91
13	\$51,691.71	(2) \$17,412.32	(3) \$34,279.39	(4) \$804,868.52
14	\$51,691.71	\$16,701.02	\$34,990.69	\$769,877.83
15	\$51,691.71	\$15,974.96	\$35,716.75	\$734,161.08
16	\$51,691.71	\$15,233.85	\$36,457.86	\$697,703.22
Totals	\$206,766.84	\$65,322.15	\$141,444.69	

Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
32	8.3	1200000	Answer: -51,691.71391 Re-keyed as -51691.71	0	4	4

Payment	P1	P2	BAL (output)	PRN (output)	INT (output)
12	12	12	839,147.9072	N/A	N/A
13	13	13	804,868.5163	34,279.39092	17,412.31908
14	14	14	769,877.828	34,990.68829	16,701.02171
15	15	15	734,161.083	35,716.74507	15,974.96493
16	16	16	697,703.2154	36,457.86753	15,233.84247
Total	13	16	697,703.2154	141,444.6918	65,322.14819

Present

The partial amortization schedule for the fourth year is shown in the table above. The total interest paid in the year is \$65,322.15, and the principal portion is \$141,444.69.



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13.4: Special Application: Mortgages

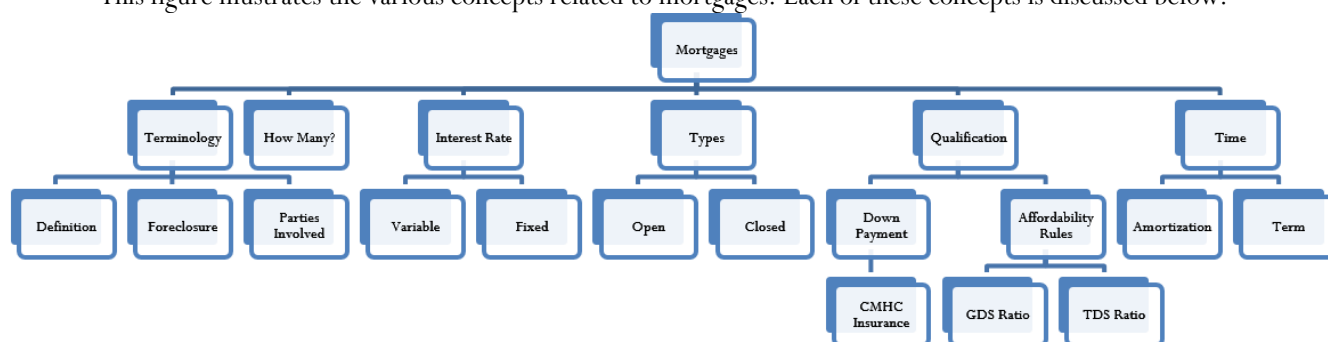
(Your Biggest Purchase)

If you buy a home in Canada, you will almost certainly need to take out a mortgage. After all, the expense is huge: on average a house costs \$784,567 in Vancouver, \$432,576 in Calgary, \$261,666 in Winnipeg, \$503,094 in Toronto, \$272,069 in Halifax, and \$318,363 in the Northwest Territories!¹ As mentioned earlier in this book, the average Canadian home costs approximately \$378,369.² Compare that with Canadian median family gross income of \$69,860,³ and once you consider the income left after all deductions and expenses it becomes obvious that you will have to pay for your home gradually over many years. That means you must make use of a financial institution's money over time, all the while paying for this privilege in the form of interest.

Your home mortgage may well be the biggest personal financial transaction you will ever make, so understanding the factors that determine your mortgage payments can save you a lot of money. This section explains mortgage fundamentals and shows you how to amortize and calculate payments on various mortgages.

Mortgages Explained

This figure illustrates the various concepts related to mortgages. Each of these concepts is discussed below.



Terminology

A **mortgage** is a special type of loan that is collaterally secured by real estate. In essence, the loan has a lien against the property, that is, the right to seize the property for the debt to be satisfied. An individual or business taking out a mortgage is obliged to pay back the amount of the loan with interest based on a predetermined contract. The financial institution, though, has a claim on the real estate property in the event that the mortgage goes into default, meaning that it is not paid as per the agreement. In these instances, financial institutions will pursue **foreclosure** of the property, which allows for the tenants to be evicted and the property to be sold. The proceeds of the sale are then used to pay off the mortgage.

A mortgage always involves two parties. The individual or business that borrows the money is referred to as the **mortgagor**, and the financial institution that lends the money is referred to as the **mortgagee**.

How Many Mortgages

A real estate property can have more than one mortgage. The main mortgage based on the amount of money borrowed to purchase the home is called the **first mortgage**. But property owners can choose to have other mortgages as well. For example, most home equity lines of credit (HELOCs) are secured by a **second mortgage** against the property.

¹ Canadian Real Estate Association (CREA), "Housing Market Stats", <https://www.crea.ca/housing-market-stats/> (accessed July 9, 2011).

² Living In Canada, "Canadian House Prices," www.livingin-canada.com/house-prices-canada.html (accessed July 9, 2011).

³ Statistics Canada, "Median Total Income, by Family Type, by Province and Territory, CANSIM table 111-0009, <http://www40.statcan.ca/101/cst01/famil108a-eng.htm> (accessed October 19, 2010).

Sometimes homeowners borrow money to make home improvements or renovations and have these amounts placed as an additional mortgage against the property. Any number of mortgages against a property is possible, though it is uncommon to have more than three. The order of the mortgages is important. If the mortgage goes into default and foreclosure occurs, the mortgagee with the first mortgage gets access to the proceeds first. If any money is left over after the first mortgage is paid off, the mortgagee of the second mortgage gets the remainder. If anything is left, the third mortgagee gets the balance and so on until all proceeds have been expended. Ultimately, any money left over after all mortgages and costs have been paid belongs to the mortgagor.

Interest Rate

In the mortgage contract, the two parties can agree to either a fixed interest rate or a variable interest rate. Either way, the mortgage always forms an ordinary annuity since interest is not payable in advance.

- Under a fixed interest rate, the principal is repaid through a number of equal payments that cover both the interest and principal components of the loan. The interest portion is highest at the beginning and gradually declines over the amortization period of the mortgage. In Canada, fixed interest rates are either annually or semi-annually compounded; the latter is the prevailing choice.
- In a variable interest rate mortgage, the principal is repaid through an agreed-upon number of unequal payments that fluctuate with changes in borrowing rates. The principal and interest portions of the payment vary as interest rates fluctuate, meaning that the interest portion can rise at any point with any increase in rates. When rates change, a common practice in many financial institutions is to change the variable interest rate as of the first day of the next month. If the rate change does not coincide with a mortgage payment date, then the interest portion is calculated in a way similar to the procedure for a demand loan (see Section 8.5), where the exact number of days at the different rates must be determined. In Canada, variable interest rates are usually compounded monthly.

Types of Mortgages

The mortgage agreement can be open or closed. An **open mortgage** has very few rules and it allows the mortgagor to pay off the debt in full or make additional prepayments at any given point in any amount without penalty. A **closed mortgage** has many rules that determine how the mortgage is to be paid. It does not allow the mortgagor to pay off the debt in full until the loan matures. As a marketing tool, most closed mortgages have “top-up” options that allow the mortgagor to make additional payments (such as an additional 20% per year) against the mortgage without penalty. Any payments exceeding the maximums or early payment of the mortgage are penalized heavily, with a three-month minimum interest charge that could be increased up to a measure called the interest rate differential, which effectively assesses the bank's loss and charges the mortgagor this full amount.

Qualification

Purchasing a home and obtaining a mortgage requires a down payment. You can get a mortgage with a 25% down payment or more; this is known as a loan-to-value ratio of 75% or less. The loan-to-value ratio divides the principal borrowed by the value of the house. If you do not have enough down payment to meet this criterion, you can make the real estate purchase with as little as 5% down (a 95% loan-to-value ratio); however, your mortgage must then be insured by the Canada Mortgage and Housing Corporation (CMHC). The premium charged for the insurance can range from 0.5% to 3.3% of the principal borrowed, depending on the mortgage particulars.

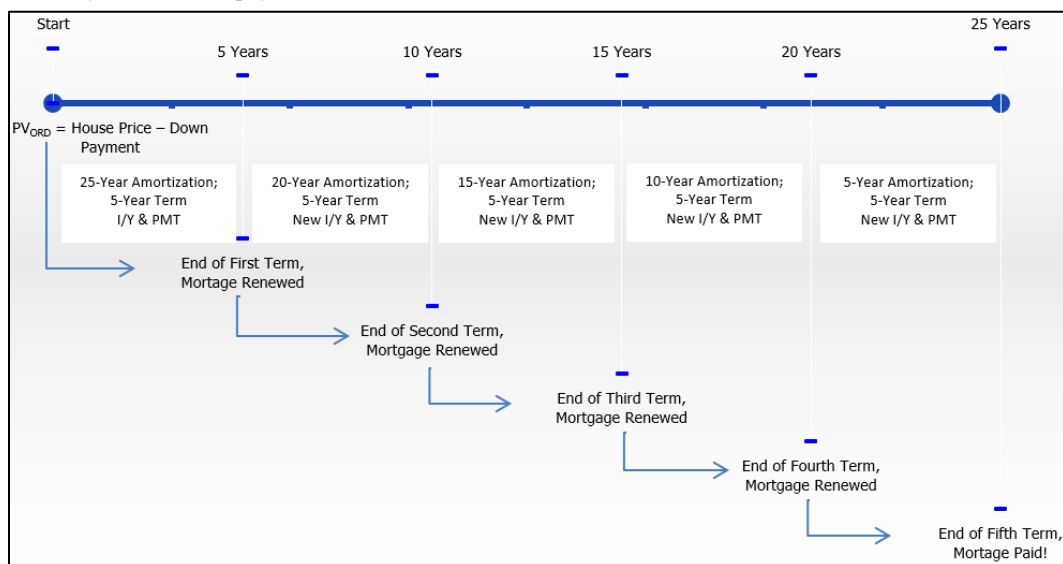
To qualify for a mortgage, you must also meet two affordability rules:

1. The first affordability rule states that your principal, interest, taxes, and heating expenses, or PITH, must not exceed 32% of your gross monthly household income. If the real estate involves a condominium, then 50% of the condo fees are also included in the PITH. This is known as the **gross debt service (GDS) ratio**.
2. The second affordability rule states that the PITH plus all other debt requirements must not exceed 40% of gross monthly household income. Other debt can include any other debt payments such as car loans, credit cards, or student loans. This is known as the **total debt service (TDS) ratio**.

It is important to note that these ratios have some degree of flexibility depending on the financial institution issuing the mortgage. Each of the percentages for the ratios commonly varies by up to $\pm 3\%$ depending on the particular institution's policies.

Time

Finally, due to the large amount of principal borrowed in a real estate purchase, the mortgage may be amortized over an extended time period. The most common amortization is 25 years with a maximum of 30 years. That is a long time frame for any contract. Consequently, mortgages are taken out in shorter terms, usually up to no more than five years, although some institutions do offer seven- or ten-year terms. This allows the financial institution regular opportunities to update the interest rate to reflect current mortgage rates. What this means to the mortgagor is that the mortgage becomes due in full at the end of each term. Most people then renew their mortgage for another term, though the amortization period becomes shorter in accordance with the number of years elapsed since the initial principal was borrowed. The figure below illustrates the typical mortgage amortization process, where the amortization is initially established at 25 years and the mortgagor uses five consecutive five-year terms to pay off the debt.



How Mortgage Interest Rates Are Determined

The actual interest rate you can obtain on a mortgage is determined by five factors in combination:

1. **The Bank of Canada Rate.** This rate sets the basis from which all variable rates are determined. The prime rate (the Bank of Canada rate + 2%) is usually the lowest obtainable interest rate. Some banks, however, discount this rate slightly by a margin between 0.1% and 0.4%.
2. **Bond Market Rates.** Bond market rates set the basis for which all fixed interest rates are determined. Fluctuations in the prime rate do not necessarily affect bond market rates.
3. **The Mortgage Type.** Open mortgages have substantially higher rates than equivalent closed mortgages because of the uncertainty of when the mortgage will be paid off. Banks protect their interests and minimize the risks of open mortgages through the higher rate.
4. **The Term.** As the length of the term increases, the interest rate increases as well. This is due to the uncertainty of future interest rates. The financial institution must protect itself against rising rates in the future.
5. **The Type of Rate.** Variable rates are lower than fixed rates since the financial institution can adjust the rate at any time to match prevailing conditions. Fixed rates do not share this benefit of adjustability and therefore are higher to protect the financial institution.

The table on the next page illustrates some actual posted semi-annually compounded interest rates at the time of writing.

Term	Closed Mortgage		Open Mortgage	
	Variable Rate	Fixed Rate	Variable Rate	Fixed Rate
1 year	2.10%	3.09%	3.14%	6.30%
5 year	3.00%	5.34%	4.00%	8.25%

Calculating the Mortgage Payment

The following mortgage variables are always known:

- Mortgage interest rates are posted by every financial institution. The posted rate is almost always negotiable, and a smart mortgagor may be able to negotiate up to a 1% deduction from the posted rate.
- The mortgagor chooses the amortization period, the term, and the payment frequency and also negotiates these variables with the financial institution.
- The principal is determined by the value of the home purchased less any down payment plus any fees or premiums.
- All mortgages are ordinary annuities.

What is left? The unknown variable is the mortgage payment amount that matches the time value of money variables. When calculating this amount, the most important variable is the amortization period, which determines the length of time over which the loan is repaid. It forms the basis for calculating the payment. Note that the term has no effect on the payment calculation. It dictates only the time frame during which the current mortgage arrangement (interest rate, payment frequency, type, and so on) remains in effect.



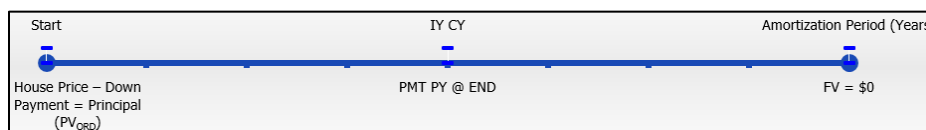
How It Works

Follow these steps to calculate a mortgage payment:

Step 1: Visualize the mortgage by drawing a timeline as illustrated below. Identify all other time value of money variables, including PV_{ORD} , IY , CY , PY , and Years. The future value, FV , is always zero (the mortgage is repaid).

Step 2: Calculate the periodic interest rate (i) using Formula 9.1.

Step 3: Calculate the number of annuity payments (N) using Formula 11.1. Remember to use the amortization period and not the term for the Years variable in this calculation.



Step 4: Calculate the ordinary annuity payment amount using Formula 11.4 and rearranging for PMT . You must round this calculated amount to two decimals since it represents a physical payment amount.



Give It Some Thought

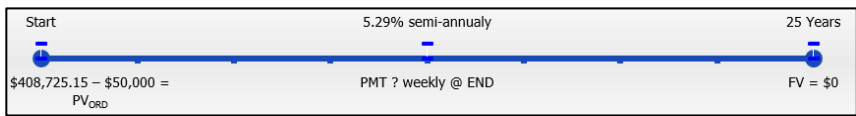
In each of the following situations determine which mortgage results in a higher mortgage payment. Assume all other time value of money variables remain constant.

1. Open or closed
2. Fixed or variable
3. Two-year fixed rate or five-year fixed rate
4. Ten-year amortization or 20-year amortization

Solutions:
 1. Open (open mortgages always have a higher interest rate than closed mortgages).
 2. Fixed rates are always higher to account for future uncertainties.
 3. Five-year fixed rate (the longer the term, the higher the rate).
 4. Ten-year amortization (there is less time to pay off the debt, meaning higher payments are required).

Example 13.4A: What Are the Payments?

The Olivers are looking to purchase a new home from Pacesetter Homes in a northeastern suburb. An model show home can be purchased for \$408,726.15. They are planning on putting \$50,000 as a down payment with a 25-year amortization and weekly payments. If current mortgage rates are fixed at 5.29% compounded semi-annually for a five-year closed term, determine the mortgage payment required.

Plan	Calculate the mortgage payment amount required (PMT).																		
Understand	What You Already Know Step 1: The mortgage timeline appears below. $PV_{ORD} = \\$408,726.15 - \\$50,000 = \\$358,726.15$, $IY = 5.29\%$, $CY = 2$, $PY = 52$, $Years = 25$, $FV = \\$0$	How You Will Get There Step 2: Apply Formula 9.1. Step 3: Apply Formula 11.1. Step 4: Calculate the mortgage payment using Formula 11.4, rearranging for PMT.																	
																			
Perform	Step 2: $i = 5.29\%/2 = 2.645\%$ Step 3: $N = 52 \times 25 = 1,300$ payments Step 4: $\$358,726.15 = PMT \frac{1 - \left[\frac{1}{(1 + 0.02645)^{\frac{52}{2}}} \right]^{1300}}{(1 + 0.02645)^{\frac{52}{2}} - 1}$ $PMT = \frac{\$358,726.15}{\frac{1 - \left[\frac{1}{1.001004} \right]^{1,300}}{0.001004}} = \$494.40$	Calculator Instructions <table border="1"> <thead> <tr> <th>Mode</th><th>N</th><th>I/Y</th><th>PV</th></tr> </thead> <tbody> <tr> <td>END</td><td>1300</td><td>5.29</td><td>358726.15</td></tr> </tbody> </table> <table border="1"> <thead> <tr> <th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr> </thead> <tbody> <tr> <td>Answer: -494.398328</td><td>0</td><td>52</td><td>2</td></tr> </tbody> </table>	Mode	N	I/Y	PV	END	1300	5.29	358726.15	PMT	FV	P/Y	C/Y	Answer: -494.398328	0	52	2	
Mode	N	I/Y	PV																
END	1300	5.29	358726.15																
PMT	FV	P/Y	C/Y																
Answer: -494.398328	0	52	2																
Present	If the Olivers purchase this home as planned, they are mortgaging \$358,726.15 for 25 years. During the first five-year term of their mortgage, they make weekly payments of \$494.40. After the five years, they must renew their mortgage.																		

Renewing the Mortgage

When the term of a mortgage expires, the balance remaining becomes due in full. Typically the balance owing is still quite substantial, so the mortgage must be renewed. As discussed earlier, this means that the mortgagor assumes another mortgage, not necessarily with the same financial institution, and the amortization term is typically reduced by the length of the first term. The length of the second term of the mortgage then depends on the choice of the mortgagor. Other variables such as payment frequency and the interest rate may or may not change.

For example, assume a mortgage is initially taken out with a 25-year amortization and a five-year term. After five years, the mortgage becomes due in full. Unable to pay it, the mortgagor renews the mortgage for the remaining 20-year amortization, and also opts for a three-year term in assuming the new mortgage. When those three years are over, the mortgagor renews the mortgage for the remaining 17-year amortization and again makes another term decision. This process repeats until the debt is ultimately paid off.

**How It Works**

Follow these steps to renew a mortgage:

Steps 1 to 4: The steps for calculating the mortgage payment amount remain unchanged.

Step 5: Determine the balance remaining at the end of the mortgage term. This involves the following:

1. Calculate the future value of the mortgage principal (FV) at the end of the term using Formulas 9.2 (Number of Compounding Periods for Single Payments) and 9.3 (Compound Interest for Single Payments).
2. Calculate the future value of the mortgage payments (FV_{ORD}) made throughout the term using Formulas 11.1 (Number of Annuity Payments) and 11.2 (Future Value of an Ordinary Annuity).
3. Calculate the remaining balance by taking $FV - FV_{ORD} = BAL$.

Step 6: Depending on the information being sought, repeat the above steps as needed for each mortgage renewal using the new amortization remaining, the new interest rate, any changes in payment frequency, and the new term. For example, if you are looking for the mortgage payment in the second term, repeat just steps 2 through 4. If looking for the balance remaining at the end of the second term, repeat step 5 as well.



Important Notes

Most commonly, mortgagors progress along the original path of amortization with each mortgage term renewal. The mortgaging process does not require this, though. With any renewal, the mortgagor may choose either to shorten or to lengthen the amortization period. If shortening the amortization period causes no TDS or GDS ratio concerns, the mortgagor can pay off the mortgage faster. If the mortgagor wishes to lengthen the amortization period, the financial institution may look at the overall time to pay the debt and put an upper cap on how long the amortization period may be increased.



Things To Watch Out For

The most common mistake with mortgages is to confuse the term and the amortization period. Remember:

- When you determine the mortgage payment amount, you use the amortization period to determine the N.
- When you calculate the balance remaining at the end of a mortgage period, you use the term to determine the N.



Paths To Success

When you use your BAII Plus calculator to calculate the remaining balance at the end of the term, you can arrive at this number in one of two ways. Once you have computed the mortgage payment amount and re-entered it into the calculator with only two decimals, you determine the last payment number for the mortgage term and then either

1. Input this value into the N and solve for FV, or
2. Open up the AMORT function and input the last payment number into both P1 and P2. Scroll down to BAL for the solution.



Give It Some Thought

A mortgage was taken out in its first term with a semi-annually compounded interest rate of 6%. If the mortgage remains on the same amortization schedule, what happens to the mortgage payment if the new semi-annually compounded interest rate upon renewal is

1. 6.5%
2. 5.5%
3. 6%

Solutions:
 1. PMT rises, since more interest is charged to the mortgage.
 2. PMT falls, since less interest is charged to the mortgage.
 3. PMT remains the same, since there is no change in the interest charged to the mortgage.

Example 13.4B: A Mortgage Renewal

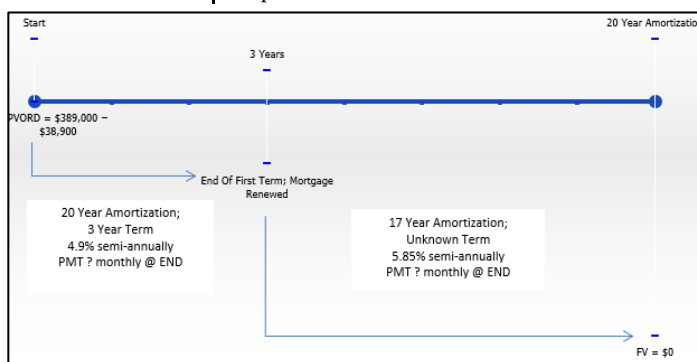
The Chans purchased their home three years ago for \$389,000 less a \$38,900 down payment at a fixed semi-annually compounded rate of 4.9% with monthly payments. They amortized the mortgage over 20 years. The Chans will renew the mortgage on the same amortization schedule at a new rate of 5.85% compounded semi-annually. How much will their monthly payments increase in the second term?

Plan Calculate the original mortgage payment in the first term, or PMT_1 . Then renew the mortgage and recalculate the mortgage payment in the second term, or PMT_2 . The amount by which PMT_2 is higher is their monthly payment increase.

Understand

What You Already Know
 Step 1: The Chans's mortgage timeline appears below.
 First Term:
 $PV_{ORD} = \$389,000 - \$38,900 = \$350,100$,
 $IY = 4.9\%$, $CY = 2$, $PY = 12$, Years = 20,
 $FV = \$0$
 Second Term:
 $PV_{ORD} = \text{BAL after first term}$, $IY = 5.85\%$,
 $CY = 2$, $PY = 12$, Years = 17, $FV = \$0$

How You Will Get There
 Step 2: For the first term, apply Formula 9.1.
 Step 3: For the first term, apply Formula 11.1.
 Step 4: For the first term, calculate the mortgage payment using Formula 11.4, rearranging for PMT .
 Step 5: Calculate the future value of the principal at the end of the first term using Formulas 9.2 and 9.3 for the principal and Formulas 11.1 and 11.2 for the mortgage payments. Determine the remaining balance through $BAL = FV - FV_{ORD}$.
 Step 6: Calculate the new mortgage payment through Formulas 9.1, 11.1, and 11.4, rearranging for PMT .
 Step 7: Calculate the increase from the first to the second payment.



Perform

Step 2: $i = 4.9\% / 2 = 2.45\%$
 Step 3: $N = 12 \times 20 = 240$ payments

Step 4: $\$350,100 = PMT \left[\frac{1 - \left[\frac{1}{(1+0.0245)^{\frac{2}{12}}} \right]^{240}}{(1+0.0245)^{\frac{2}{12}} - 1} \right]$ $PMT = \frac{\$350,100}{\left[\frac{1 - \left[\frac{1}{(1+0.0245)^{\frac{2}{12}}} \right]^{240}}{0.004042} \right]} = \$2,281.73$

Step 5: Principal: $N = 2 \times 3 = 6$ compounds; $FV = \$350,100(1 + 0.0245)^6 = \$404,821.7959$
 Payments: $N = 12 \times 3 = 36$ payments; $FV_{ORD} = \$2,281.73 \left[\frac{\left[(1+0.0245)^{\frac{2}{12}} \right]^{36} - 1}{(1+0.0245)^{\frac{2}{12}} - 1} \right] = \$88,228.30609$
 Balance: $\$404,821.7959 - \$88,228.30609 = \$316,593.49$
 Step 6: $i = 5.85\% / 2 = 2.925\%$; $N = 12 \times 17 = 204$ payments

$\$316,593.49 = PMT \left[\frac{1 - \left[\frac{1}{(1+0.02925)^{\frac{2}{12}}} \right]^{204}}{(1+0.02925)^{\frac{2}{12}} - 1} \right]$ $PMT = \frac{\$316,593.49}{\left[\frac{1 - \left[\frac{1}{(1+0.02925)^{\frac{2}{12}}} \right]^{204}}{(1+0.02925)^{\frac{2}{12}} - 1} \right]} = \$2,440.73$

Step 7: $\$2,440.73 - \$2,281.73 = \$159.00$

Calculator Instructions

Action	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
First-Term Payment	END	240	4.9	350100	Answer: -2,281.730755	0	12	2
Balance at end of First Term	✓	36	c	✓	-2,281.73	Answer: -316,593.4898	✓	✓
Second-Term Payment	✓	204	5.85	316593.49	Answer: -2,440.733592	0	✓	✓

Present

The initial mortgage payment for the three-year term was \$2,281.73. Upon renewal at the higher interest rate, the monthly payment increased by \$159.00 to \$2,440.73.

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Chapter 13 Summary

Key Concepts Summary

Section 13.1: Calculating Interest and Principal Components (Where Did My Money Go?)

- The concept of amortization
- How to calculate interest and principal components for a single payment
- How to calculate interest and principal components for a series of payments

Section 13.2: Calculating the Final Payment (The Bank Wants Every Penny)

- Understanding why the final payment is different than all other payments
- How to calculate the exact amount of the final payment including both interest and principal components
- How to calculate interest and principal components involving the final payment

Section 13.3: Amortization Schedules (How Much Is Mine, How Much Is Theirs?)

- The development of a complete amortization schedule
- The development of a complete amortization schedule due
- The development of a partial amortization schedule

Section 13.4: Special Application: Mortgages (Your Biggest Purchase)

- The language and concepts involved in mortgages
- What determines the mortgage interest rate
- How to calculate the mortgage payment
- The procedure involved in renewing a mortgage

The Language of Business Mathematics

amortization A process by which the principal of a loan is extinguished over the course of an agreed-upon time period through a series of regular payments that go toward both the accruing interest and principal reduction.

amortization period The length of time it will take for the principal of a loan to be reduced to zero.

amortization schedule A table that shows the payment amount, principal component, interest component, and remaining balance for every payment in the annuity.

amortization term The length of time for which the interest rate and payment agreement between the borrower and the lender will remain unchanged.

closed mortgage A type of mortgage that has many strict rules and does not allow the mortgagor to pay off the debt in full until the loan matures. Early payment incurs substantial penalties.

first mortgage A mortgage with first claim to the proceeds of a foreclosure process on a real estate property.

foreclosure A process that allows a financial institution to evict tenants from a mortgaged property and put the property up for sale. The proceeds of the sale are then used to pay off the mortgage.

gross debt service (GDS) ratio On a mortgage, your principal, interest, taxes, and heating expenses (PITH), plus 50% of condo fees if applicable, must not exceed 32% of your gross monthly household income.

mortgage A special type of loan that is collaterally secured by real estate property.

mortgagee The financial institution that lends the money for a mortgage.

mortgagor The individual or business that borrows the money for a mortgage.

open mortgage A type of mortgage that has very few rules and allows the mortgagor to pay off the debt in full or with additional prepayments at any given point without penalty.

partial amortization schedule Amortization schedules that show only a specified range of payments and not the entire annuity.

second mortgage A mortgage with second claim to the proceeds of a foreclosure process on a real estate property. Only the remaining funds after the first mortgage has been paid off are available for the second mortgage to be paid off.

total debt service (TDS) ratio On a mortgage, your principal, interest, taxes, and heating expenses (PITH) plus all other debt requirements must not exceed 40% of your gross monthly household income.

The Formulas You Need to Know

Symbols Used

BAL = principal balance immediately after an annuity payment

BAL_{P1} = principal balance immediately prior to the first payment in a series of annuity payments

BAL_{P2} = principal balance immediately after the last payment in a series of annuity payments

CY = compounding frequency

i = periodic interest rate

INT = interest portion of an ordinary single annuity payment or a series of annuity payments

INT_{DUE} = interest portion of a due single annuity payment

N = number of annuity payments (for annuities) or number of compounding periods (for lump sums)

PMT = annuity payment amount

PRN = principal portion of a single annuity payment or a series of annuity payments

PY = payment frequency

Formulas Introduced

Formula 13.1 Interest Portion of an Ordinary Single Payment: $INT = BAL \times ((1 + i)^{\frac{CY}{PY}} - 1)$ (Section 13.1)

Formula 13.2 Principal Portion of a Single Payment: $PRN = PMT - INT$ (Section 13.1)

Formula 13.3 Principal Portion for a Series of Payments: $PRN = BAL_{P1} - BAL_{P2}$ (Section 13.1)

Formula 13.4 Interest Portion for a Series of Payments: $INT = N \times PMT - PRN$ (Section 13.1)

Formula 13.5: Interest Portion of a Due Single Payment: $INT_{DUE} = (BAL - PMT) \times ((1 + i)^{\frac{CY}{PY}} - 1)$ (Section 13.3)

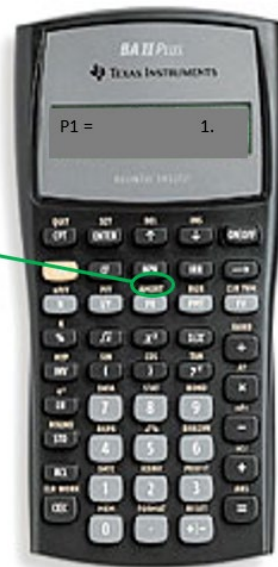
Technology

Calculator

Amortization (AMORT) Function

1. AMORT is located on the 2nd shelf above the PV button, as illustrated in the photo.
2. There are five variables (use ↓ or ↑ to scroll).
 - P1 is the starting payment number. The calculator can work with a single payment or a series of payments.
 - P2 is the ending payment number. This number is the same as P1 when you are concerned with just a single payment. When working with a series of payments, you can set it to a higher number.
 - BAL is the principal balance remaining after the payment number entered into the P2 variable. The cash flow sign is correct as indicated on the calculator display.
 - PRN is the principal portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.
 - INT is the interest portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.
3. To use the Amortization function, the commands are as follows:
 - a. Enter all seven of the time value of money variables accurately (N, I/Y, PV, PMT, FV, P/Y, and C/Y). If PMT was computed, you must re-enter it with only two decimals while retaining the correct cash flow sign convention.
 - b. Press 2nd AMORT.
 - c. Enter a value for P1 and press Enter followed by ↓.

AMORT
Function



- d. Enter a value for P2 and press Enter followed by ↓. Note that the higher the numbers entered in P1 or P2, the longer it will take the calculator to compute the outputs. It is possible that the calculator will go blank and take a few moments before displaying the outputs.
 - e. Using ↓ and ↑, scroll through BAL, PRN, and INT to read the output.
-



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Chapter 14: Bonds and Sinking Funds

(When You Need Big Money)

One of the most powerful ways for an institution to raise large amounts of capital is by issuing marketable bonds. A company or a government will essentially borrow the required financing from investors such as you and me. Individually we cannot provide the needed funds, but the bond issuer can raise a large sum of money from us collectively.

For example, Manitoba Hydro back in 2013 was building new hydro lines extending from the Keewatinooow Converter Station north of Lakes Winnipeg and Manitoba to the southern distribution network. The project, called Bipole III, required environmental approvals and licensing. It needed almost five years to complete the 1,364 kilometre project involving approximately 2,800 45-metre transmission towers. The cost of the project was around \$3.28 billion. Manitoba Hydro's 2011–2012 net income was \$61 million, while its 2012–2013 net income was \$92 million.¹ It obviously cannot just ask a bank for a \$3.28 billion loan! And even in situations when a bank loan is available, financing through bonds is generally more flexible.

One of the largest bond issuers in Canada, with \$312.8 billion in outstanding marketable bonds as of May 2023, is the Government of Canada.² This money is primarily used to fund Canada's national debt.

On the consumer side, marketable bonds represent another investment opportunity you can include in your RRSP portfolio. They feature regular interest payments and a chance to grow your savings through trading on the bond market.

This chapter begins with the basics of how bonds operate. Then you will learn how to calculate the purchase price of a bond and work with bond interest rates. Bond issuance usually comes with a sinking fund provision, which is explained in Section 14.3. Finally, we will explore the financial and accounting responsibilities associated with amortization of bond premiums and accrual of bond discounts.

Outline of Chapter Topics

14.1: Determining the Value of a Bond (What Does It Sell for?)

14.2: Calculating a Bond's Yield (Know When to Hold 'em, Know When to Fold 'em)

14.3: Sinking Fund Schedules (You Need to Show Responsibility)

14.4: Debt Retirement and Amortization (Balancing the Books)

¹ Manitoba Hydro-Electric Board, *62nd Annual Report for the Year Ended March 31, 2013*, www.hydro.mb.ca/corporate/ar/2012/publish/index.html#66-67 (accessed October 2, 2013).

² Statistics Canada, "Bank of Canada Assets and Liabilities," <https://www150.statcan.gc.ca/t1/tbl1/en/tv.action?pid=1010010801> (accessed June 27, 2023).

14.1: Determining the Value of a Bond

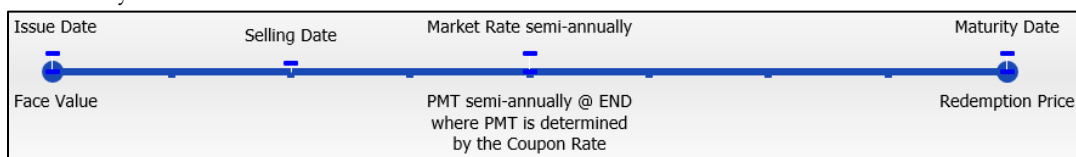
(What Does It Sell for?)

Exactly what is your marketable bond worth? When you inspect the financial section of your local newspaper, you pay particular attention to the quotes on bonds, into which you invested a portion of your RRSP portfolio. Ten years ago, when prevailing bond rates were 5.5%, you purchased 10 Government of Canada \$5,000 face value bonds with a 5% coupon and 20 years remaining to maturity. At the time you paid \$4,699.02 each for them. Today's prevailing bond rates have dropped to 3.35% but are expected to rise in the near future. Should you hold on to those bonds? Should you cash them in by selling them in the bond market?

To help answer those questions, it is critical to know the value, or selling price, of your bonds today. This section explains the concept of a marketable bond along with its important characteristics and terminology. You calculate the bond's selling price first on interest payment dates and then on all other dates, which present a more complex situation.

Bond Basics

A marketable **bond** is a debt that is secured by a specific corporate asset and that establishes the issuer's responsibility toward a creditor for paying interest at regular intervals and repaying the principal at a fixed later date. A **debenture** is the same as a marketable bond, except that the debt is not secured by any specific corporate asset. Mathematically, the calculations are identical for these two financial tools, which this textbook refers to as *bonds* for simplicity. A typical bond timeline looks like the one below. Key terms are discussed as well.



- **Bond Issue Date.** The **bond issue date** is the date that the bond is issued and available for purchase by creditors. Interest accrues from this date.
- **Bond Face Value.** Also called the *par value* or *denomination of the bond*, the **bond face value** is the principal amount of the debt. It is what the investor lent to the bond-issuing corporation. The amount, usually a multiple of \$100, is found in small denominations up to \$10,000 for individual investors and larger denominations up to \$50,000 or more for corporate investors.
- **Bond Coupon Rate.** Also known as the *bond rate* or *nominal rate*, the **bond coupon rate** is the nominal interest rate paid on the face value of the bond. The coupon rate is fixed for the life of the bond. Most commonly the interest is calculated semi-annually and payable at the end of every six-month period over the entire life of the bond, starting from the issue date. All coupon rates used in this textbook can be assumed to be semi-annually compounded unless stated otherwise.
- **Bond Market Rate.** The **bond market rate** is the prevailing nominal rate of interest in the open bond market. Since bonds are actively traded, this rate fluctuates based on economic and financial conditions. On the issue date, the market rate determines the coupon rate that is tied to the bond. Market rates are usually compounded semi-annually, as will be assumed in this textbook unless otherwise stated. Therefore, marketable bonds form ordinary simple annuities, since the interest payments and the market rate are both compounded semi-annually, and the payments occur at the end of the interval.
- **Bond Redemption Price.** Also called the *redemption value* or *maturity value*, the **bond redemption price** is the amount the bond issuer will pay to the bondholder upon maturity of the bond. The redemption price normally equals the face value of the bond, in which case the bond is said to be “redeemable at par” because interest on the bond has already been paid in full periodically throughout the term, leaving only the principal in the account. In some instances a bond issuer may in fact redeem the bond at a premium, which is a price greater than the face value. The redemption price is then stated as a percentage of the face value, such as 103%. For introductory purposes, this text sticks to the most common situation, where the redemption price equals the face value.

- **Bond Maturity Date.** Also known as the *redemption date* or *due date*, the **bond maturity date** is the day upon which the redemption price will be paid to the bondholder (along with the final interest payment), thereby extinguishing the debt.
- **Bond Selling Date.** The date that a bond is actively traded and sold to another investor through the bond market is known as the **bond selling date**. In the timeline, the selling date can appear anywhere on the timeline between the issue date and maturity date, and it may occur more than once as the bond is sold by one investor after another.

Calculating the Bond Price on an Interest Payment Date

Marketable bonds and debentures are nonredeemable, which means the only way to cash these bonds in before the maturity date is to sell them to another investor. Therefore, the key mathematical calculation is what to pay for the bond. The selling date, maturity date, coupon rate, redemption price, and market rate together determine the bond price.

On the bond's issue date, the market rate determines the coupon rate, so these two rates are identical. As a result, the price of the bond equals its face value. After the bond is issued, interest starts to accrue on it, and the market rate begins to fluctuate based on market conditions. This changes the price of the bond.



The Formula

Because the bond pays interest semi-annually, two days of the year are defined as the interest payment dates. To determine a bond's selling price on these two days, you must use the formulas for present value of an ordinary annuity. Once you understand how to perform these basic calculations we will move on to the more complex formulas and techniques required to determine the selling price on the other 363 days of the year. Regardless of the selling date, Formula 14.1 expresses how to determine the price of any bond.

Cash Price is Bond Cash Price: Also known as the *purchase price* or *flat price*, the **bond cash price** is the amount of money an investor must directly pay out to acquire the bond. It represents the total of the market price and any accrued interest.

Formula 14.1 - The Cash Price for Any Bond: $\text{Cash Price} = \text{PRI} + \text{AI}$

PRI is Bond Market Price: Also known as the *quoted price*, the **bond market price** is the actual value of the bond excluding any accrued interest. Since different bonds have different interest payment dates, this price allows investors to understand the true price of the bond itself and compare similar bonds.

AI is Accrued Interest: The **bond accrued interest** is the amount of *simple* interest that the bond has earned but has not yet paid out since the previous interest payment date. This interest belongs to the seller of the bond, since the bond was in the investor's possession during that period. When a bond sells on its interest payment date, the accrued interest has just paid out to the bondholder, thereby making the accrued interest zero, or $\text{AI} = \$0$; the cash price of the bond then equals its market price. Later in this section you will learn how to calculate this amount when the bond does in fact sell between interest payment dates.

To determine the selling price of the bond, you must know the amount of the semi-annual interest payment to the bondholder. You use Formula 14.2 to calculate this amount.

PMT_{BOND} is Bond Coupon Payment: This is the annuity payment that the bondholder receives semi-annually throughout the investment. This interest is not converted into the principal of the bond and therefore does not compound. Instead, this payment is directly paid out to the bondholder. The BOND subscript differentiates this annuity payment from a regular annuity payment since the value of the coupon payment is not determined by the market rate of interest on the bond.

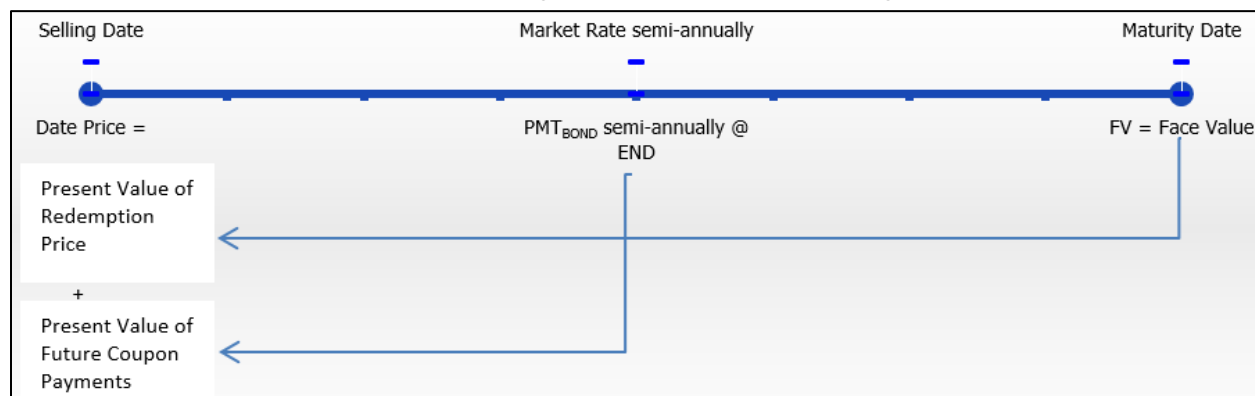
CPN is Bond Coupon Rate: The nominal interest rate paid on the face value of the bond. It is fixed for the life of the bond.

Formula 14.2 - Bond Coupon Annuity Payment Amount:
$$\text{PMT}_{\text{BOND}} = \text{Face Value} \times \frac{\text{CPN}}{\text{CY}}$$

Face Value is Bond Face Value:
The principal amount of the bond.

CY is Bond Coupon Compounding Frequency: The compounding frequency of the coupon rate, which is most commonly semi-annual, that is, $\text{CY} = 2$. Note that by taking the CPN and dividing by CY, you are calculating the periodic coupon rate, or i .

The market price of a bond on its selling date is the present value of all the future cash flows, as illustrated in the figure below. For the bond purchaser, this is a combination of the remaining coupon annuity payments plus the redemption price at maturity (which in this textbook always equals the face value). Formula 14.3 (on the next page) summarizes this calculation, which combines Formulas 9.3 and 11.4 together and simplifies the resulting expression.



The price of a bond fluctuates with the market rate over time. If the bond sells for a price higher than its face value, the difference is known as a **bond premium**. If the bond sells for a price lower than its face value, the difference is known as a **bond discount**. The amount of the premium or discount excludes any accrued interest on the bond. Why does the selling price change like this? Remember that the interest paid by the bond is a fixed rate (the coupon rate) determined at the time of issue.

- Assume a coupon rate of 5%. If the market rate has increased to 6%, it means that investors can buy bonds paying 6%. If you are trying to sell your 5% bond, no one wants to buy it unless you “put it on sale” in an amount that compensates for the 1% difference. Hence, you discount your bond.
- Alternatively, if the market rate decreases to 4%, it means that investors can buy bonds paying 4%. If you are trying to sell your 5% bond, it is very attractive to investors, so you add some extra margin, raising the price by an amount not exceeding the 1% difference. Hence, you sell at a premium price.

The figure after Formula 14.3 illustrates the relationship between the market rate, coupon rate, and the selling price of the bond. Notice that when the coupon rate is higher than the market rate, the selling price rises above its face value. Alternatively, when the coupon rate is lower than the market rate, the selling price falls below its face value. Apply Formula 14.4 (2 pages from here) to calculate the amount of the premium or discount on a bond.

Date Price is Bond Price on the most recent interest payment date: Because the bond sells on its interest payment date, the most recent interest payment date is the selling date. On this date there is no accrued interest, so the market price and the cash price are the same in Formula 14.1. You can then conclude that Date Price = Market Price = Cash Price.

Formula 14.3 - Bond Price on an Interest Payment Date:

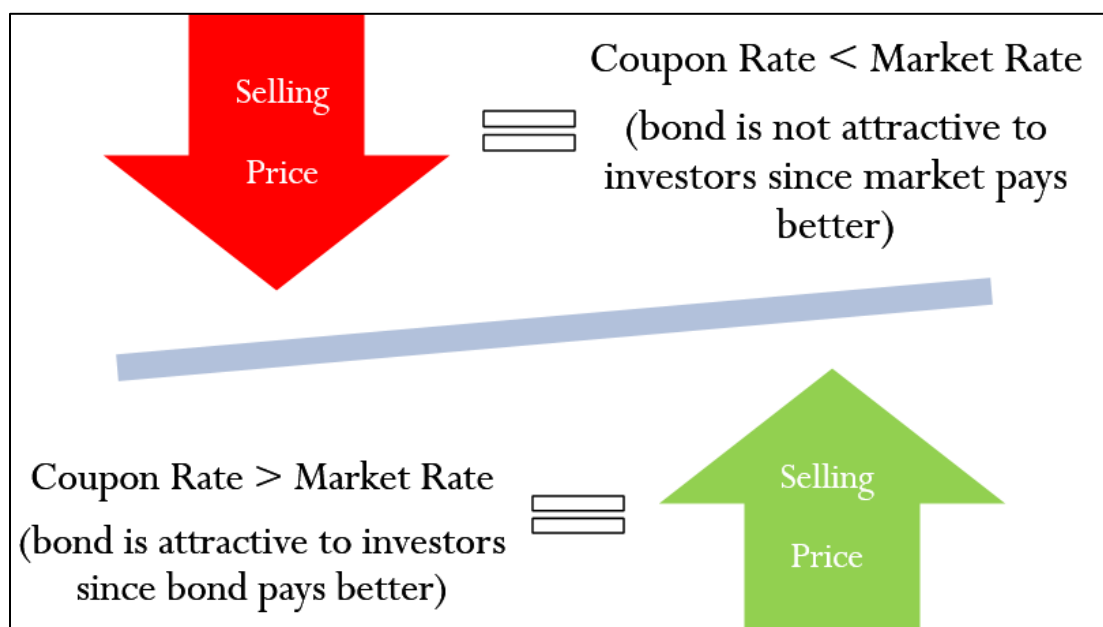
$$\text{Date Price} = \frac{\text{FV}}{(1+i)^N} + \text{PMT}_{\text{BOND}} \left[\frac{1 - \left[\frac{1}{1+i} \right]^N}{i} \right]$$

$\frac{\text{FV}}{(1+i)^N}$ is **Compound interest for single payments:** This is Formula 9.3 rearranged for present value. It calculates the present value of the redemption price (which is assumed to be the same as the face value) on the selling date.

- You calculate the periodic interest rate (i) using Formula 9.1, where the IY is the nominal market rate and the CY is the market rate compounding. Do not use the coupon rate in this calculation.
- You calculate the total number of compounding periods (N) using Formula 9.2, where CY is the market rate compounding and Years is the time remaining until maturity.

$\text{PMT}_{\text{BOND}} \left[\frac{1 - \left[\frac{1}{1+i} \right]^N}{i} \right]$ is **Ordinary Simple**

Annuity Present Value: This is a simplified version of Formula 11.4 in which the market rate and end-of-period coupon payments are both semi-annual, so the coupon payments always form an ordinary simple annuity. As such, you do not need the $\frac{\text{CY}}{\text{PY}}$ exponent, which always divides to one. In a rare event of an ordinary general annuity, you would use the full version of Formula 11.4. Since the payment frequency and compounding frequency match, the periodic rate of interest (i) remains constant throughout the formula.



Premium or Discount: There are two outcomes to this formula:

1. If the result is positive, then the market price of the bond exceeds its face value. This is the amount of the bond premium.
2. If the result is negative, then the market price of the bond has fallen below its face value. This is the amount of the bond discount. Be careful when stating the discount not to include a negative sign (avoid stating "there is a discount of -\$100," since this would form a double negative remark).

Formula 14.4 - Bond Premium or Discount: $\text{Premium or Discount} = \text{PRI} - \text{Face Value}$

PRI is Bond Market Price: The market price is the actual value of the bond excluding any accrued interest. Because the accrued interest has nothing to do with the value of the bond itself, representing only how much of the next interest payment belongs to the seller of the bond, calculations of premiums and discounts use only the market price and not the cash price.

Face Value is Bond Face Value: The principal amount of the bond.



How It Works

Follow these steps to calculate the price of a bond on its interest payment date:

Step 1: Draw a timeline extending from the selling date to the maturity date. Identify all known variables.

Step 2: Using Formula 14.2, calculate the amount of the regular bond interest payment. For future calculations do not round this number.

Step 3: Using Formula 14.3, calculate the date price of the bond. On an interest payment date, the date price is equal to both the market price and cash price.

- Use the market rate for Formula 9.1 (Periodic Interest Rate).
- Typically, to calculate N you need both Formula 9.2 (Number of Compound Periods for Single Payments) for the redemption value and Formula 11.1 (Number of Annuity Payments) for the annuity. However, since compound periods and annuity payments are both semi-annual, then each formula would produce the same value of N. Therefore, you can use just Formula 11.1, recognizing that it represents both the number of compound periods as well as the number of annuity payments.
- The date price from Formula 14.3 equals the market price. Since there is no accrued interest, the cash price is the same as the date price.

Step 4: If required, use Formula 14.4 to calculate any bond premium or discount.



Important Notes

The BOND Function on the BAII Plus Calculator. On an interest payment date, you can solve any bond problem using the regular time value of money buttons on your calculator since bonds represent ordinary simple annuities. However, a built-in function for bonds on the BAII Plus calculator greatly simplifies bond price calculations, particularly when the bond is being sold on a date other than an interest payment date. The BOND function is located on the second shelf above the number 9 key and is accessed by pressing 2nd BOND.

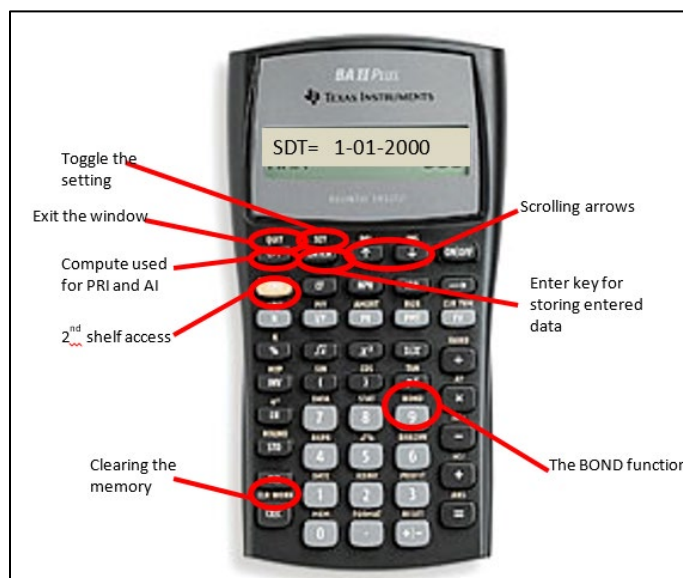
This spreadsheet has nine lines, which you scroll through using the $\downarrow$ and $\uparrow$ arrows. The first seven lines are considered the data entry lines, while the last two are the output lines. Upon opening the window, you should use the 2nd CLR Work function to erase previously loaded data. The data entry lines are as follows:

- SDT is the selling date. It is entered in the standard date format of MM.DDYY where MM is the month number (one or two digits), DD is the two-digit day number, and YY is the last two digits of the year. You must press the ENTER key to store this information. If the question does not involve specific dates, use January 1, 2000, or 1.0100 so as to determine the redemption date more easily.
- CPN is the nominal coupon rate. It is formatted as a percentage but without the percent sign; thus 5.5% is keyed in as 5.5. You must press the ENTER key to store this information.
- RDT is the redemption date or maturity date. It must be entered in the standard date format, and you must press ENTER to store this information. If the question does not involve specific dates, key in the appropriate date based on a selling date of January 1, 2000.
- RV is the redemption value or redemption price expressed as a percentage of the face value. Since the redemption price equals the face value, use the default setting of 100. This textbook never requires you to alter this number.
- ACT/360 is a toggle that you change by pressing 2nd SET. ACT counts the actual number of days in the transaction, while 360 treats every month as having 30 days. In Canada, ACT is the standard, so you should leave the calculator on this setting.
- 2/Y OR 1/Y is a toggle that you change by pressing 2nd SET. 2/Y indicates a semi-annual compound for both the market rate and coupon rate, while 1/Y indicates an annual compound. This textbook always uses the 2/Y setting.
- YLD is the nominal market rate for bonds at the time of sale. It follows the same format as the CPN, that is, a percentage but without the percent sign. You must press the ENTER key to store the information.

The output lines are as follows:

- PRI is the market price of the bond. After scrolling to this line, press the CPT button to calculate this output, which is *not* computed automatically. The output is a percentage of the redemption price (which is the same as the face value). Thus, if you have a \$1,000 face value bond, you need to take the output divided by 100 multiplied by the face value to arrive at the market price.
- AI is the accrued interest of the bond. This output is automatically calculated when you press the CPT button on the PRI line. If you are on an interest payment date, it has a value of zero. If you are in between payments, the output is, just like the PRI, a percentage of the redemption price, so you need to convert it to a value in the same manner as the PRI.

If you are interested in the cash price of the bond, you must add together the values associated with the PRI and AI outputs, as in Formula 14.1. When you are finished with the BOND function, press 2nd QUIT to leave the window.



Things To Watch Out For

When you calculate the price of a bond on the interest payment date, the date price is in fact calculating the market price. Recall that the cash price of the bond is always determined by Formula 14.1, where the market price and accrued interest must be totalled to arrive at the cash price. On interest payment dates, there is no accrued interest, so it always has a value of zero. When working with bonds, get in the habit now of thinking in the manner of Formula 14.1. Later on, when the bond is sold on a noninterest payment date and accrued interest is involved, this habit is handy for figuring out bond prices. With respect to the BAII Plus calculator, always add together the outputs of the PRI and AI windows to arrive at the selling price (cash price) of the bond.

**TOP
SECRET**

Paths To Success

Earlier in this textbook Canada Savings Bonds were discussed. Note that these bonds are fully redeemable at any point, in that you can cash them in at any point with any financial institution before maturity. Therefore, Canada Saving Bonds are not considered marketable bonds and do not operate according to the principles discussed in this section.



Give It Some Thought

1. Explain the relationship between changes in the bond market rate and the price of the bond.
2. What five variables are used in determining the price of a bond?
3. True or false: If a \$1,000 face value bond has a cash price of \$1,125 and a market price of \$1,100, it is selling at a bond premium of \$100.
4. What three variables determine the coupon payment amount?

Solutions:

1. As the market rate rises the bond price decreases, and vice versa.
2. Selling date, maturity date, coupon rate, redemption price, and market rate.
3. True; the bond premium is the difference between the face value and the market price.
4. Face value, nominal coupon rate, and coupon rate compounding frequency; see Formula 14.2.

Example 14.1A: Pricing a Bond with Exact Dates on an Interest Payment Date

A Government of Canada \$50,000 bond was issued on January 15, 2003, with a 25-year maturity. The coupon rate was 10.15% compounded semi-annually. What cash price did the bond have on July 15, 2017, when prevailing market rates were 4.31% compounded semi-annually? What was the amount of the bond premium or discount?

Plan	This bond makes interest payments six months apart, on January 15 and July 15 each year. The bond is being sold on July 15, which is an interest payment date. On an interest payment date, solve for the date price, which is the same as the cash price. Also calculate the premium or discount.	
Understand	What You Already Know Step 1: The timeline for the bond sale appears below. Coupon Interest Payment: $CPN = 10.15\%$, $CY = 2$, Face Value = \$50,000 Bond: $FV = \\$50,000$, $IY = 4.31\%$, $CY = 2$, $PMT_{BOND} = \text{Formula 14.2}$, $PY = 2$, Years Remaining = 10.5	How You Will Get There Step 2: Apply Formula 14.2 to determine the periodic bond interest payment. Step 3: Apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its interest payment date. The cash price in Formula 14.1 equals the date price. Step 4: Apply Formula 14.4 to determine the bond premium or discount.
Perform	Step 2: $PMT_{BOND} = \\$50,000 \times \frac{0.1015}{2} = \\$50,000 \times 0.05075 = \\$2,537.50$ Step 3: $i = 4.31\%/2 = 2.155\%$; $N = 2 \times 10.5 = 21$ (compounds and payments) $\text{Date Price} = \frac{\$50,000}{(1 + 0.02155)^{21}} + \$2,537.50 \left[\frac{1 - \left[\frac{1}{1 + 0.02155} \right]^{21}}{0.02155} \right] = \$74,452.86$ Step 4: Premium or Discount = $\\$74,452.86 - \\$50,000.00 = \\$24,452.86$	

Calculator Instructions

SDT	CPN	RDT	RV	Days	Compound	YLD	PRI	AI
7.1517	10.15	1.1528	100	ACT	2/Y	4.31	Answer: 148.905720	Answer: 0

Transforming the output:

$$\frac{148.905720 + 0}{100} \times \$50,000 = \$74,452.86012$$

Present

The bond has a coupon rate that is substantially higher than the market rate, so it is sold at a premium of \$24,452.86 for a cash price of \$74,452.86. Note that because the sale occurs on the interest payment date, there is no accrued interest, so the market price and cash price are the same.

Example 14.1B: Pricing a Bond with General Dates on an Interest Payment Date

A \$25,000 Government of Canada bond was issued with a 25-year maturity and a coupon rate of 8.92% compounded semi-annually. Two-and-a-half years later the bond is being sold when market rates have increased to 9.46% compounded semi-annually. Determine the selling price of the bond along with the amount of premium or discount.

Plan

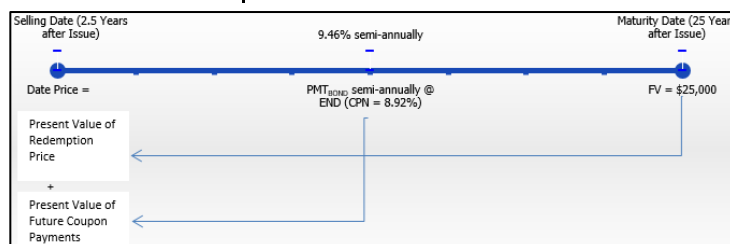
Although there are no specific dates, the coupon is semi-annual, making interest payments every six months. If the bond is being sold 2½ years after issue, this makes the sale date an interest payment date. On an interest payment date, solve for the date price, which is the same as the cash price. Also calculate the premium or discount.

Understand**What You Already Know**

Step 1: The timeline for the bond sale appears below.
 Coupon Interest Payment: CPN = 8.92%, CY = 2, Face Value = \$25,000
 Bond: FV = \$25,000, IY = 9.46%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 22.5

How You Will Get There

Step 2: Apply Formula 14.2 to determine the periodic bond interest payment.
 Step 3: Apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its interest payment date. The cash price in Formula 14.1 equals the date price.
 Step 4: Apply Formula 14.4 to determine the bond premium or discount.

**Perform**

Step 2: $PMT_{BOND} = \$25,000 \times \frac{0.0892}{2} = \$25,000 \times 0.0446 = \$1,115$
 Step 3: $i = 9.46\% / 2 = 4.73\%$; $N = 2 \times 22.5 = 45$ (compounds and payments)

$$\text{Date Price} = \frac{\$25,000}{(1 + 0.0473)^{45}} + \$1,115 \left[\frac{1 - \left[\frac{1}{1 + 0.0473} \right]^{45}}{0.0473} \right] = \$23,751.28$$

Step 4: Premium or Discount = $\$23,751.28 - \$25,000.00 = -\$1,248.72$

Calculator Instructions

SDT	CPN	RDT	RV	Days	Compound	YLD	PRI	AI
1.0100	8.92	7.0122	100	ACT	2/Y	9.46	Answer: 95.005105	Answer: 0

Transforming the output:

$$\frac{95.005105 + 0}{100} \times \$25,000 = \$23,751.27631$$

Present

The bond has a coupon rate that is slightly lower than the market rate, so it is sold at a discount of \$1,248.72 for a cash price of \$23,751.28. Note that because the sale occurs on the interest payment date, there is no accrued interest, so the market price and cash price are the same.

Calculating the Bond Price on a Noninterest Payment Date

There are only two days of the year upon which the cash price and the market price of the bond are the same value. Those days are the interest payment dates, when you determine the bond's value using Formula 14.3. However, what happens if the bond is sold on one of the other 363 days of the year?

On these other dates, the cash price and the market price are not equal. For each day that elapses after an interest payment date, interest for the next payment starts to accrue such that over the next six months, enough interest is available to make the next interest payment.

According to Formula 14.1, when an investor wants to purchase a bond in between interest payment dates, the buyer has to pay the seller a cash price equalling the market price of the bond plus the accrued interest. Why? Assume a bond makes semi-annual interest payments of \$50. When the buyer acquires the bond from the seller, two months have elapsed since the last interest payment date. Since the seller held the bond for two months of the six-month payment interval, it is fair and reasonable for the seller to receive the interest earned during that time frame. However, the bond will not make its next interest payment until four months later, at which time the buyer, who now owns the bond, will receive the full \$50 interest payment for the full six months. Thus, at the time of buying the bond, the buyer has to pay the seller the bond's market price plus the portion of the next interest payment that legally belongs to the seller. In this example, an interest amount representing two of the six months needs to be paid.



The Formula

Arriving at the bond's price in between interest payment dates is a little complex because the cash price is increasing according to a compound interest formula, while in practice the accrued interest on the bond is increasing according to a simple interest formula. If this seems peculiar to you, you would be right, but that is just how bonds happen to work!

You require three important pieces of information: what to pay (the cash price), how much simple interest was included in the cash price (the accrued interest), and what the actual value of the bond was (the market price). To arrive at these three numbers, follow these steps:

1. First calculate the cash price of the bond as shown in Formula 14.5.
2. Calculate the accrued interest included in that price as shown in Formula 14.6.
3. Determine the market price from Formula 14.1.

Cash Price is What The Buyer Pays: This is the amount the buyer of the bond pays, which includes both the market price of the bond and the accrued interest together. This price is determined through compound interest calculations.

Date Price is Bond Price on most recent interest payment date: This is the unrounded bond price on the interest payment date immediately preceding the selling date. It is calculated by Formula 14.3. Thus, if the date of sale occurs $4\frac{3}{4}$ years before maturity, the preceding interest payment date is five years before maturity.

Formula 14.5 - Bond Cash Price on a Noninterest Payment Date:

$$\text{Cash Price} = (\text{Date Price})(1 + i)^t$$

i is Periodic Interest

Rate: The periodic interest rate is calculated using Formula 9.1, where the IY is the nominal market rate and the CY is the market rate compounding frequency.

t is Time Ratio: This is a ratio that determines the exact amount of time that the seller of the bond has held the bond since the last interest payment date. It is calculated as follows:

$$\frac{\text{Number of days since last interest payment date}}{\text{Total number of days in current full payment interval}}$$

For example, assume a bond pays interest on March 1 and September 1. It is sold on May 15. The ratio would be:

$$\frac{\text{March 1 to May 15} = 75 \text{ days}}{\text{March 1 to September 1} = 184 \text{ days}} = \frac{75}{184}$$

AI is Accrued Interest: Accrued interest in between bond interest payments is calculated on a simple interest basis. This formula is a modified version of Formula 8.1 in which $I = Prt$. Recall that PMT_{BOND} is calculated by taking the face value (PV) and multiplying it by the periodic coupon rate ($\frac{CPN}{CY}$), or PVr (matching to Pr in Formula 8.1). You then multiply this by the time component. The result of the calculation determines the proportion of the next interest payment that belongs to the seller of the bond.

Formula 14.6 - Accrued Interest on a Noninterest Payment Date: $AI = PMT_{\text{BOND}} \times t$

PMT_{BOND} is Bond Coupon Payment: This is the annuity payment that the bondholder receives semi-annually throughout the investment. It results from Formula 14.2.

t is Time Ratio: This is the same ratio developed in Formula 14.5. It determines the exact amount of time that the seller of the bond has held the bond since the last interest payment date.



How It Works

Follow these steps to calculate the price of a bond in between interest payment dates:

Step 1: Draw a timeline like the one to the right, extending from the preceding interest payment date to the maturity date. Identify all known variables.

Step 2: Using Formula 14.2, calculate the amount of the regular bond interest payment. For future calculations do not round this number.

Step 3: Using Formula 14.3, calculate the date price of the bond on the interest payment date just preceding the selling date.

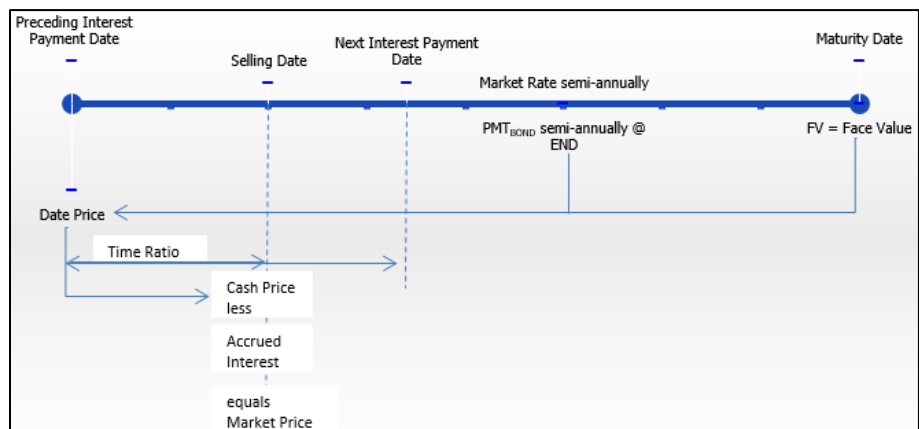
- Use the market rate for Formula 9.1 (Periodic Interest Rate).
- Recall that you use only Formula 11.1 and recognize that it represents both the number of compound periods as well as the number of annuity payments.

Step 4: Calculate the cash price of the bond using Formula 14.5. Calculate the time ratio by determining the exact number of days the seller held the bond as well as the exact number of days involved in the current payment interval.

Step 5: Calculate the accrued interest on the bond using Formula 14.6. Use the time ratio from step 5.

Step 6: Calculate the market price of the bond using Formula 14.1.

Step 7: If required, use Formula 14.4 to calculate the bond premium or discount.



Important Notes

The DATE Function on the BAI Plus Calculator. To create the time ratio, determine the number of days since the last interest payment date as well as the total number of days in the current payment interval. You can compute this through the DATE function. For a full discussion of this function, recall the instructions at the end of Chapter 8. To arrive at the required numbers:

1. Compute the number of days since the last interest payment date by entering the last interest payment date as DT1 and the selling date as DT2. Compute the DBD.
2. Compute the total number of days in the current payment interval by entering the last interest payment date as DT1 and the next interest payment date as DT2. Compute the DBD.



Things To Watch Out For

In calculations of bond premiums and discounts on noninterest-payment dates, the most common mistake is to use the cash price instead of the market price. Remember that the cash price includes both the accrued interest and the market price. The accrued interest does not factor into the value of the bond, since it represents a proportioning of the next interest payment between the seller and the buyer. Therefore, the amount of the bond premium or discount should not include the accrued interest. Use only the market price to determine the premium or discount.

**TOP
SECRET**

Paths To Success

On the BAII Plus, the PRI and AI outputs of the BOND worksheet must be summed together to arrive at the cash price. Both of these outputs represent a percentage of the face value (the same base) and can be summed before converting the percentage into a dollar amount. For example, if PRI = 98% and AI = 2.5%, you could take the total of 100.5% to figure out the cash price. Using the calculator efficiently, you would store the PRI into a memory cell, scroll down, and then add the recall of the memory cell to the AI to arrive at the total percentage of face value.



Give It Some Thought

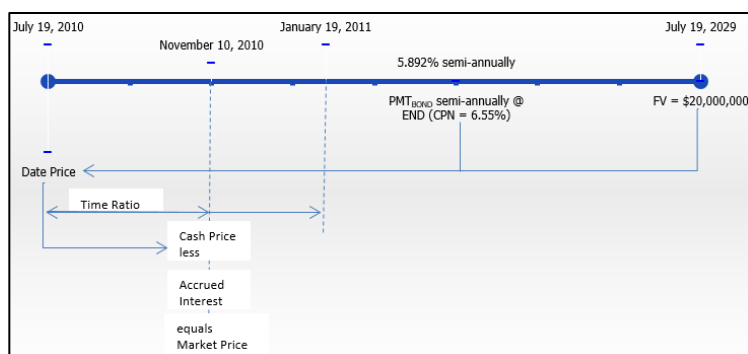
1. Is the cash price or market price higher when a bond is sold in between interest payment dates?
2. Why can you not use Formula 14.3 directly to obtain the market price of a bond selling in between interest payment dates?

Solutions:
 1. Cash price; it includes the accrued interest.
 2. Accrued interest is calculated using simple interest, which results in slightly different numbers than if the bond price used compound interest alone.

Example 14.1C: Bond Price on a Noninterest Payment Date

A \$20 million face value Bell Canada bond was issued on July 19, 1999, with a 30-year maturity and a coupon rate of 6.55% compounded semi-annually. At what cash price did the bond trade on November 10, 2010, when market yields were posted at 5.892% semi-annually? What was the accrued interest? What was the market price? What premium or discount does this represent?

Plan	Notice that this bond makes interest payments six months apart, on January 19 and July 19 of each year. The bond sale date, November 10, is not an interest payment date. Calculate four variables: the cash price, accrued interest (AI), market price (PRI), and the bond premium or discount.	
Understand	What You Already Know Step 1: The timeline of the bond sale appears below. Coupon Interest Payment: CPN = 6.55%, CY = 2, Face Value = \$20 million Bond: FV = \$20 million, IY = 5.892%, CY = 2, PMT _{BOND} = Formula 14.2, PY = 2, Years Remaining = 18.5 years to preceding interest payment date	How You Will Get There Step 2: Apply Formula 14.2 to determine the periodic bond interest payment. Step 3: Apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its preceding interest payment date. Step 4: Apply Formula 14.5 to determine the cash price of the bond. Step 5: Apply Formula 14.6 to determine the accrued interest. Step 6: Apply Formula 14.1 to determine the market price. Step 7: Apply Formula 14.4 to determine the bond premium or discount.



Perform

$$\text{Step 2: } \text{PMT}_{\text{BOND}} = \$20,000,000 \times \frac{0.0655}{2} = \$655,000$$

$$\text{Step 3: } i = 5.892\% / 2 = 2.946\%; N = 2 \times 19 = 38 \text{ (compounds and payments)}$$

$$\text{Date Price} = \frac{\$20,000,000}{(1 + 0.02946)^{38}} + \$655,000 \left[\frac{1 - \left[\frac{1}{1 + 0.02946} \right]^{38}}{0.02946} \right] = \$21,492,511.69$$

$$\text{Step 4: } t = \frac{\text{July 19, 2010 to November 10, 2010}}{\text{July 19, 2010 to January 19, 2011}} = \frac{114}{184}$$

$$\text{Cash Price} = (\$21,492,511.69)(1 + 0.02946)^{\frac{114}{184}} = \$21,882,632.40$$

$$\text{Step 5: } \text{AI} = (\$655,000) \frac{114}{184} = \$405,815.22$$

$$\text{Step 6: } \$21,882,632.40 = \text{PRI} + \$405,815.22 \quad \$21,476,817.18 = \text{PRI}$$

$$\text{Step 7: } \text{Premium or Discount} = \$21,476,817.18 - \$20,000,000.00 = \$1,476,817.18$$

Calculator Instructions

SDT	CPN	RDT	RV	Days	Compound	YLD	PRI	AI
11.1010	6.55	7.1929	100	ACT	2/Y	5.892	Answer: 107.384085	Answer: 2.029076

Transforming the outputs:

$$\text{PRI} = \frac{107.384085}{100} \times \$20,000,000 = \$21,476,817.18$$

$$\text{AI} = \frac{2.029076}{100} \times \$20,000,000 = \$405,815.22$$

Present

The bond had a coupon rate that is higher than the market rate, so it sold at a premium of \$1,476,817.18 for a market price of \$21,476,817.18. The buyer also owed the seller \$405,815.22 of accrued interest; therefore, the cash price was \$21,882,632.40.

Example 14.1D: How Did the Investor Do?

A \$50,000 Government of Ontario bond was issued on March 1, 1995, with a coupon rate of 9.5% compounded semi-annually and 50 years until maturity. Harvey bought the bond on July 17, 1996, when market rates were 8.06% compounded semi-annually, and sold the bond on December 12, 2008, with a semi-annual yield of 3.45%. Based on the market price, what gain or loss did Harvey realize?

Plan

Notice that this bond makes interest payments six months apart, on March 1 and September 1 of each year. Since the bond is being bought on July 17 and sold on December 12, neither date represents an interest payment date. Calculate the market price (PRI) for both dates and then determine the difference.

Understand	What You Already Know Step 1: The timeline for the bond sale appears below. Coupon Interest Payment: CPN = 9.5%, CY = 2, Face Value = \$50,000 Bond Purchase: FV = \$50,000, IY = 8.06%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 49 years to preceding interest payment date Bond Sale: FV = \$50,000, IY = 3.45%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 36.5 years to preceding interest payment date	How You Will Get There Step 2: Apply Formula 14.2 to determine the periodic bond interest payment. For <i>each</i> of the purchase and sale, perform steps 3 through 6: Step 3: Apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its preceding interest payment date. Step 4: Apply Formula 14.5 to determine the cash price of the bond. Step 5: Apply Formula 14.6 to determine the accrued interest. Step 6: Apply Formula 14.1 to determine the market price. Step 7: Determine the difference between the market prices (PRI) from the purchase to the sale.																													
Perform	Step 2: $PMT_{BOND} = \$50,000 \times \frac{0.095}{2} = \$2,375$																														
	<table><tr><th>Step</th><th>Bond Purchase (July 17, 1996)</th><th>Bond Sale (December 12, 2008)</th></tr><tr><td>3</td><td>$i = 8.06\%/2 = 4.03\%; N = 2 \times 49 = 98$ (compounds and payments) $\text{Date Price} = \frac{\\$50,000}{(1 + 0.0403)^{98}} + \\$2,375 \left[\frac{1 - \left[\frac{1}{1 + 0.0403} \right]^{98}}{0.0403} \right]$$= \\$58,747.02738$</td><td>$i = 3.45\%/2 = 1.725\%; N = 2 \times 36.5 = 73$ (compounds and payments) $\text{Date Price} = \frac{\\$50,000}{(1 + 0.01725)^{73}} + \\$2,375 \left[\frac{1 - \left[\frac{1}{1 + 0.01725} \right]^{73}}{0.01725} \right]$$= \\$112,522.6856$</td></tr><tr><td>4</td><td>$t = \frac{\text{March 1, 1996 to July 17, 1996}}{\text{March 1, 1996 to September 1, 1996}} = \frac{138}{184}$ $\text{Cash Price} = (\\$58,747.02738)(1 + 0.0403)^{\frac{138}{184}}$$= \\$60,513.86$</td><td>$t = \frac{\text{September 1, 2008 to December 12, 2008}}{\text{September 1, 2008 to March 1, 2009}} = \frac{102}{181}$ $\text{Cash Price} = (\\$112,522.6856)(1 + 0.01725)^{\frac{102}{181}}$$= \\$113,612.43$</td></tr><tr><td>5</td><td>$AI = (\\$2,375) \frac{138}{184} = \\$1,781.25$</td><td>$AI = (\\$2,375) \frac{102}{181} = \\$1,338.40$</td></tr><tr><td>6</td><td>$\\$60,513.86 = \text{PRI} + \\$1,781.25$ $\\$58,732.61 = \text{PRI}$</td><td>$\\$113,612.43 = \text{PRI} + \\$1,338.40$ $\\$112,274.03 = \text{PRI}$</td></tr></table>	Step	Bond Purchase (July 17, 1996)	Bond Sale (December 12, 2008)	3	$i = 8.06\%/2 = 4.03\%; N = 2 \times 49 = 98$ (compounds and payments) $\text{Date Price} = \frac{\$50,000}{(1 + 0.0403)^{98}} + \$2,375 \left[\frac{1 - \left[\frac{1}{1 + 0.0403} \right]^{98}}{0.0403} \right]$ $= \$58,747.02738$	$i = 3.45\%/2 = 1.725\%; N = 2 \times 36.5 = 73$ (compounds and payments) $\text{Date Price} = \frac{\$50,000}{(1 + 0.01725)^{73}} + \$2,375 \left[\frac{1 - \left[\frac{1}{1 + 0.01725} \right]^{73}}{0.01725} \right]$ $= \$112,522.6856$	4	$t = \frac{\text{March 1, 1996 to July 17, 1996}}{\text{March 1, 1996 to September 1, 1996}} = \frac{138}{184}$ $\text{Cash Price} = (\$58,747.02738)(1 + 0.0403)^{\frac{138}{184}}$ $= \$60,513.86$	$t = \frac{\text{September 1, 2008 to December 12, 2008}}{\text{September 1, 2008 to March 1, 2009}} = \frac{102}{181}$ $\text{Cash Price} = (\$112,522.6856)(1 + 0.01725)^{\frac{102}{181}}$ $= \$113,612.43$	5	$AI = (\$2,375) \frac{138}{184} = \$1,781.25$	$AI = (\$2,375) \frac{102}{181} = \$1,338.40$	6	$\$60,513.86 = \text{PRI} + \$1,781.25$ $\$58,732.61 = \text{PRI}$	$\$113,612.43 = \text{PRI} + \$1,338.40$ $\$112,274.03 = \text{PRI}$															
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Step 7: Gain = \$112,274.03 – \$58,732.61 = \$53,541.42																															
Calculator Instructions																															
<table><tr><th>Bond</th><th>SDT</th><th>CPN</th><th>RDT</th><th>RV</th><th>Days</th><th>Compound</th><th>YLD</th><th>PRI</th><th>AI</th></tr><tr><td>Purchase</td><td>7.1796</td><td>9.5</td><td>3.0145</td><td>100</td><td>ACT</td><td>2/Y</td><td>8.06</td><td>Answer: 117.465216</td><td>Answer: = 3.5625</td></tr><tr><td>Sale</td><td>12.1208</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td>3.45</td><td>Answer: 224.548072</td><td>Answer: = 2.676795</td></tr></table>		Bond	SDT	CPN	RDT	RV	Days	Compound	YLD	PRI	AI	Purchase	7.1796	9.5	3.0145	100	ACT	2/Y	8.06	Answer: 117.465216	Answer: = 3.5625	Sale	12.1208	✓	✓	✓	✓	✓	3.45	Answer: 224.548072	Answer: = 2.676795
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<table><tr><th>Bond Purchase (July 17, 1996)</th><th>Bond Sale (December 12, 2008)</th></tr><tr><td>Transforming the output: $PRI = \frac{117.465216}{100} \times \\$50,000 = \\$58,732.60843$</td><td>Transforming the output: $PRI = \frac{224.548072}{100} \times \\$50,000 = \\$112,274.0362$ (Difference of \$0.01 is due to rounding.)</td></tr></table>		Bond Purchase (July 17, 1996)	Bond Sale (December 12, 2008)	Transforming the output: $PRI = \frac{117.465216}{100} \times \$50,000 = \$58,732.60843$	Transforming the output: $PRI = \frac{224.548072}{100} \times \$50,000 = \$112,274.0362$ (Difference of \$0.01 is due to rounding.)																										
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Present	Harvey acquired the bond for a market price of \$58,732.61 and sold the bond approximately 12.5 years later for \$112,274.03 because of the very low market rates in the bond market. As a result, the gain on his bond amounts to \$53,541.42.																														



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14.2: Calculating a Bond's Yield

(Know When to Hold 'em, Know When to Fold 'em)

Should you always hold onto a bond until maturity? Is there a best time to sell? In the opener to the last section, you invested in 10 Government of Canada \$5,000 face value bonds with a 5% coupon and 20 years remaining to maturity. For these you paid \$4,699.02 each when prevailing bond rates were 5.5%. Ten years after you purchased those bonds, prevailing bond market rates have dropped to 3.35% but are expected to rise in the near future. Is now the time to sell those bonds?

This is a complex decision with many variables; however, to help make that decision you must know at least three critical pieces of information: the selling price of the bond, the yield you would realize on your investment if you held onto the bond until maturity, and the yield if you sold the bond today.

This section integrates the calculations of bond prices and bond yields so that you better understand your bond investments. These guidelines apply whether you are investing personally or on behalf of your company.

Yield to Maturity

The **yield to maturity**, also known as *basis*, is a bond's overall rate of return when purchased at a market price and held until maturity. It includes both the semi-annual interest that the bondholders earn on their investment along with the gain or loss resulting from the difference between the market price on the selling date and the redemption price. This yield to maturity is exactly equal to the market rate of return on the date of purchase. Thus, in the example above, if you hang on to those bonds until maturity, you will realize a yield to maturity of 5.5%.

In this section, you are turning around the calculations from Section 14.1, where you answered the question, "Knowing the market rate of return, what will you pay?" Now you are asked, "Knowing what you pay, what is the market rate of return?" Thus, rewording the opening example you would have: "If you paid \$4,699.02 for a \$5,000 face value bond with 20 years to maturity having a 5% coupon, what yield to maturity would you realize?"



The Formula

You need no new formulas to calculate a bond's yield to maturity. The goal is to solve for the nominal rate of interest, or *Y*. You must work with Formulas 14.2, 14.3, and 9.1. Recall that Formula 14.2 determines the semi-annual bond coupon interest payment amount. You substitute this amount into Formula 14.3, which calculates the price of the bond on an interest payment date. However, in this case you must solve Formula 14.3 not for the date price but for the periodic rate of interest, or *i*. Once you know the periodic interest rate, you can substitute it into Formula 9.1 and solve for the nominal interest rate, or *Y*.

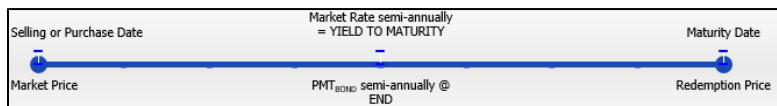
One problem in using the formula approach is that it is impossible to algebraically isolate the periodic interest rate in Formula 14.3. You must turn to technology such as the BAII Plus calculator or Excel.



How It Works

Follow these steps to calculate a bond's yield to maturity:

Step 1: Draw a timeline like the one presented here, extending from the selling date to the maturity date. Identify all known variables.



Step 2: Using Formula 14.2, calculate the amount of the bond interest payment.

Step 3: As in Section 14.1, use Formula 11.1 to calculate the *N*. Since the market rate and the coupon rate are both semi-annual, the *N* is used as both the total number of compounds and the total number of payments.

- If using a manual method, substitute all known numbers into Formula 14.3 and attempt to solve for the periodic interest rate (*i*). Once you know this, convert it to a nominal interest rate using Formula 9.1.

- Alternatively, if you are using technology such as the calculator, input all known variables and solve directly for the nominal interest rate (IY).



Important Notes

Note two conditions when you calculate yield to maturity:

- Trading takes place only on interest payment dates.
- The bondholder reinvests all coupon payments at the same rate of interest.

Calculations not meeting these conditions are beyond the introductory scope of this textbook.



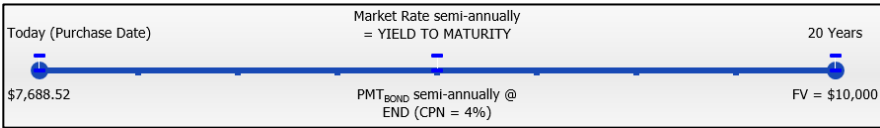
Give It Some Thought

If you purchase a bond with a coupon rate of 4% when the market rate is 5%, what yield to maturity will you realize?

Solution:
5%; your yield to maturity depends on the market rate, not the coupon rate.

Example 14.2A: A Basic Yield to Maturity

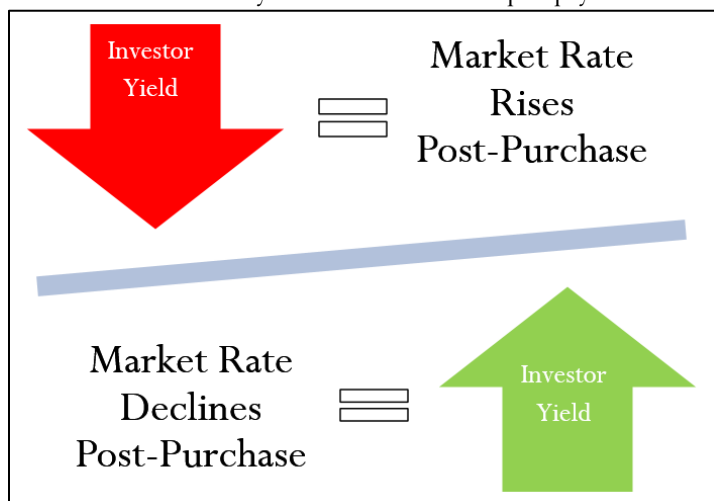
Suppose a \$10,000 face value bond is purchased for \$7,688.52 with 20 years until maturity and has a coupon rate of 4% semi-annually. What yield will the bondholder realize if she holds onto it until maturity?

Plan	The yield to maturity represents the nominal rate of interest on the bond, or IY.																
Understand	What You Already Know Step 1: The timeline for the bond sale appears below. Coupon Interest Payment: CPN = 4%, CY = 2, Face Value = \$10,000 Bond: Date Price = \$7,688.52, FV = \$10,000, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 20 	How You Will Get There Step 2: Apply Formula 14.2 to determine the periodic bond interest payment. Step 3: Apply Formulas 11.1, 14.3, and 9.1 to determine the yield to maturity (IY). Solve Formulas 14.3 and 9.1 using the calculator.															
	Step 2: PMT_{BOND} = \$10,000 × $\frac{0.04}{2}$ = \$200 Step 3: N = 2 × 20 = 40 (compounds and payments) $\$7,688.52 = \frac{\$10,000}{(1 + i)^{40}} + \$200 \left[\frac{1 - \left[\frac{1}{1 + i} \right]^{40}}{i} \right] \quad i = \frac{IY}{2}$ Calculator output = IY = 6%	Calculator Instructions <table><tr><td>Mode</td><td>N</td><td>I/Y</td><td>PV</td></tr><tr><td>END</td><td>40</td><td>Answer: 6.000002</td><td>-7688.52</td></tr></table> <table><tr><td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>200</td><td>10000</td><td>2</td><td>2</td></tr></table>	Mode	N	I/Y	PV	END	40	Answer: 6.000002	-7688.52	PMT	FV	P/Y	C/Y	200	10000	2
Mode	N	I/Y	PV														
END	40	Answer: 6.000002	-7688.52														
PMT	FV	P/Y	C/Y														
200	10000	2	2														
Present	If she holds onto the bond for the next 20 years, she will realize a yield to maturity of 6% compounded semi-annually.																

The Investor's Yield

The yield-to-maturity calculation requires that the bond be held until its maturity date, at which point the future value redemption price is known and equal to its face value. However, this condition may not hold. The bondholder may sell the bond at any point before the maturity date. If the bond is sold, the future value is based on the prevailing bond rate at that time, and the price is generally not equal to its face value. What then is the bond's yield for the investor?

The investor's yield consists of the coupon payments that have been received while the investor possesses the bond



along with the difference achieved between the purchase price and the selling price. The next figure illustrates what happens to the investor's yield by relating the post-purchase market rate and the investor's yield.

- When the market rate rises after the bond has been purchased, the investor experiences a lower yield than the original yield to maturity since the future bond selling price is reduced. Suppose that investors purchase the bond at a 5% market rate:
 - If they hold it until maturity, they achieve a 5% yield to maturity.
 - If they instead sell the bond before maturity when the market rate is 6%, the investor's yield is less than 5%.
- When the market rate declines after the bond has been purchased, the investor experiences a higher yield

than the original yield to maturity, since the future bond selling price is increased. Suppose again that investors purchase the bond at a 5% market rate:

- If they hold it until maturity, they of course achieve a 5% yield to maturity.
- If they sell the bond when the market rate is 4%, the investor's yield is more than 5%.



The Formula

The existing bond formulas are sufficient for calculating the investor's yield. You need Formula 14.2 to determine the bond coupon payment amount. You use Formula 14.3 to calculate both the purchase price and the selling price. You then use it a third time to solve for the periodic interest rate. When you solve for i , the future value of the bond that you substitute into the formula must be the selling price of the bond at the future date, not the redemption price.



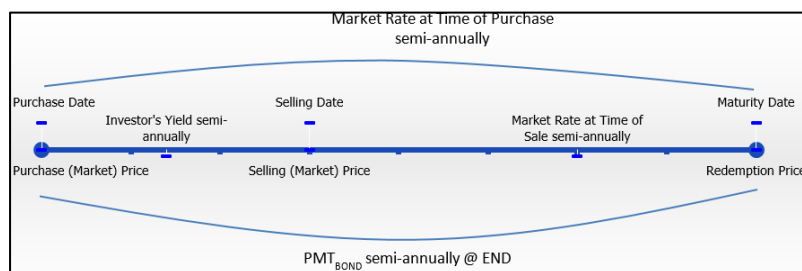
How It Works

Follow these steps to calculate the investor's yield:

Step 1: Draw a timeline like the one depicted here, extending from the purchase date to the maturity date. Clearly indicate the selling date. Identify all known variables.

Step 2: Using Formula 14.2, calculate the amount of the bond interest payment.

Step 3: If needed, calculate the purchase price using Formula 14.3. As in previous procedures, use Formula 11.1 to calculate the semi-annually based N representing both the total number of compounds and the total number of payments. Use the market rate at the time of purchase.



Step 4: If needed, calculate the selling price using Formula 14.3 in the same manner. Use the market rate at the time of sale.

Step 5: Solve for the nominal rate of interest (IY) between the purchase date and the selling date. The purchase price is the date price, and the selling price is the future value (FV).

- If using a manual method, substitute all known numbers into Formula 14.3 and solve for the periodic interest rate (i). Convert it into a nominal interest rate using Formula 9.1.
- Alternatively, if using technology such as a calculator, then input all known variables and solve directly for the nominal interest rate (IY).



Important Notes

The same two requirements that applied to calculating yield to maturity persist when you calculate investor's yield:

1. Trading takes place only on interest payment dates.
2. The bondholder reinvests all coupon payments at the same rate of interest.



Give It Some Thought

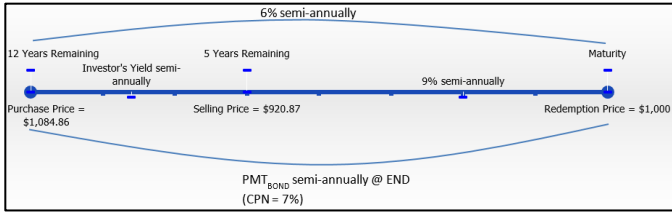
In each of the following, determine whether the investor's yield increases, decreases, or remains the same.

1. A bond is purchased when the market yield is 5% and then sold when the market rate is 4%.
2. A bond is purchased when the market yield is 5% and then sold when the market rate is 5%.
3. A bond is purchased when the market yield is 5% and then sold when the market rate is 6%.

Solutions:
 1. Increases; a lower market rate causes the selling price to rise.
 2. Same; no change in the market rate results in no change to the yield.
 3. Decreases; a higher market rate causes the selling price to decline.

Example 14.2B: The Investor's Yield

A \$1,000 face value bond with a 7% coupon and 12 years to maturity was purchased for \$1,084.68 when market rates were 6%. It sold seven years later for \$920.87, when market rates were 9%. What yield did the bondholder realize?

Plan	The investor's yield represents the nominal rate of interest on the bond during the time the investor owned the bond, or IY.																		
Understand	What You Already Know Step 1: The timeline for the purchase and sale of the bond appears below. Coupon Interest Payment: CPN = 7%, CY = 2, Face Value = \$1,000 Purchase/Sale of Bond: PV = \$1,084.68, FV = \$920.87, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Held = 7	How You Will Get There Step 2: Apply Formula 14.2 to determine the periodic bond interest payment. Step 3: Determine the bond purchase price. It is known at \$1,084.68 and does not need to be calculated. This is the date price for step 5. Step 4: Determine the bond selling price. It is known at \$920.87 and does not need to be calculated. This is the future value (FV) for step 5. Step 5: Apply Formulas 11.1, 14.3, and 9.1 to determine the investor's yield (IY). Solve Formulas 14.3 and 9.1 using your calculator.																	
Perform	Step 2: $PMT_{BOND} = \\$1,000 \times \frac{0.07}{2} = \\$35$ Step 5: $N = 2 \times 7 = 14$ (compounds and payments) $\$1,084.68 = \frac{\$920.87}{(1+i)^{14}} + \$35 \left[\frac{1 - \left[\frac{1}{1+i} \right]^{14}}{i} \right]$ Calculator output = IY = 4.6003% $i = \frac{IY}{2}$	Calculator Instructions <table border="1"><tr><td>Mode</td><td>N</td><td>I/Y</td><td>PV</td></tr><tr><td>END</td><td>14</td><td>Answer: 4.600320</td><td>-1084.68</td></tr></table> <table border="1"><tr><td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>35</td><td>920.87</td><td>2</td><td>2</td></tr></table>	Mode	N	I/Y	PV	END	14	Answer: 4.600320	-1084.68	PMT	FV	P/Y	C/Y	35	920.87	2	2	
Mode	N	I/Y	PV																
END	14	Answer: 4.600320	-1084.68																
PMT	FV	P/Y	C/Y																
35	920.87	2	2																
Present	The bond market price dropped significantly because of the 3% rise in the market rate over the seven-year time period. Thus, the investor does not realize the original yield to maturity of 6% (the market rate), and in fact the investor's yield has dropped to 4.6003% compounded semi-annually.																		

Example 14.2C: What Yield Do You Realize if You Sell Your Bond Now?

In the opening discussions to Sections 14.1 and 14.2, you had invested in 10 Government of Canada \$5,000 face value bonds with a 5% coupon and 20 years remaining to maturity, paying \$4,699.02 each when prevailing bond rates were 5.5%. You notice today, 10 years later, that prevailing bond rates have dropped to 3.35%. What investor's yield would you realize if you sold your bonds today?

Plan	You need to calculate the yield for just one of the 10 bonds since, of course, the yield is the same across all bonds. The investor's yield represents the nominal rate of interest on the bond during the time you own the bond, or IY.																												
Understand	What You Already Know Step 1: The timeline for purchase and sale of the bond appears below. Coupon Interest Payment: $CPN = 5\%$, $CY = 2$, Face Value = \$5,000 Purchase of Bond: $PV = \\$4,699.02$ Sale of Bond: $FV = \\$5,000$, $IY = 3.35\%$, $CY = 2$, $PMT_{BOND} = \text{Formula 14.2}$, $PY = 2$, Years Remaining = 10 Purchase/Sale of Bond: $PV = \\$4,699.02$, $FV = \text{Sale Price}$, $CY = 2$, $PMT_{BOND} = \text{Formula 14.2}$, $PY = 2$, Years Held = 10	How You Will Get There Step 2: Apply Formula 14.2 to determine the periodic bond interest payment. Step 3: Determine the bond purchase price. It is known at \$4,699.02 and does not need to be calculated. This is the date price for step 5. Step 4: Determine the bond selling price by applying Formulas 9.1, 11.1, and 14.3. This is the future value (FV) for step 5. Step 5: Apply Formulas 11.1, 14.3, and 9.1 to determine the investor's yield (IY). Solve Formulas 14.3 and 9.1 using the calculator.																											
Perform	Step 2: $PMT_{BOND} = \\$5,000 \times \frac{0.05}{2} = \\$125$ Step 4: $i = 3.35\%/2 = 1.675\%$; $N = 2 \times 10 = 20$ (compounds and payments) $\text{Date Price} = \frac{\$5,000}{(1 + 0.01675)^{20}} + \$125 \left[\frac{1 - \left[\frac{1}{1 + 0.01675} \right]^{20}}{0.01675} \right] = \$5,696.14$ Step 5: $N = 2 \times 10 = 20$ (compounds and payments) $\$4,699.02 = \frac{\$5,696.14}{(1 + i)^{20}} + \$125 \left[\frac{1 - \left[\frac{1}{1 + i} \right]^{20}}{i} \right] \quad i = \frac{IY}{2}$ Calculator output = $IY = 6.8338\%$ Calculator Instructions <table><tr><th>Step</th><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>4</td><td>END</td><td>20</td><td>3.35</td><td>Answer: 5,696.138252</td><td>125</td><td>5000</td><td>2</td><td>2</td></tr><tr><td>5</td><td>✓</td><td>✓</td><td>Answer: 6.833820</td><td>-4,699.02</td><td>✓</td><td>5696.14</td><td>✓</td><td>✓</td></tr></table>		Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	4	END	20	3.35	Answer: 5,696.138252	125	5000	2	2	5	✓	✓	Answer: 6.833820	-4,699.02	✓	5696.14	✓	✓
Step	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y																					
4	END	20	3.35	Answer: 5,696.138252	125	5000	2	2																					
5	✓	✓	Answer: 6.833820	-4,699.02	✓	5696.14	✓	✓																					
Present	The bond market price increased significantly over the 10-year time period because the market rate dropped from 5.5% to 3.35%. Thus, the investor realizes more than the original yield to maturity of 5.5% (the market rate), and in fact the investor's yield has risen to 6.8338% compounded semi-annually.																												



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14.3: Sinking Fund Schedules

(You Need to Show Responsibility)

Individuals and businesses should always plan to save toward their future goals. A sinking fund represents one way of accomplishing this, earning interest while regular contributions build up, all to reach a specified target at the end of the period.

For instance, let's say you are a production manager. In your desk inbox a consultant's report warns about the worrisome state of your production facilities and warehousing operations. Because your current production machinery is really showing its age, you need to replace it within three years. Meanwhile, increasing demand makes a new warehouse in Scarborough necessary within five years. The costs of each project are forecast at \$1 million and \$3 million, respectively. You see in the financial reports for the company that your company averages annual net profits of about \$750,000. Your company needs to start setting aside the required funds now for these urgent projects.

On your way home, you pass by a newly constructed residential neighbourhood. You have your eye on a starter home there with a list price of approximately \$325,000. The 5% down payment required by the mortgage lender is \$16,250. You have been making end-of-month contributions of \$239 for the past four years into a savings plan earning 5% compounded monthly. On a whim, you head into the development, where you find a show home currently on the market for \$250,000. You are quite smitten with the home and know that it will not take long to sell. Before putting in an offer, you wonder if your savings plan has enough funds to meet the 5% down payment requirement today.

Sinking Funds

A **sinking fund** is a special account into which an investor, whether an individual or a business, makes annuity payments such that sufficient funds are on hand by a specified date to meet a future savings goal or debt obligation. In its simplest terms, it is a financial savings plan. As the definition indicates, it has either of two main purposes:

1. **Capital Savings.** When your goal is to acquire some form of a capital asset by the end of the fund, you have a capital savings sinking fund. What is a capital asset? It is any tangible property that is not easily converted into cash. Thus, saving up to buy a home, car, warehouse, or even new production machinery qualifies as capital savings.
2. **Debt Retirement.** When your goal is to pay off some form of debt by the end of the fund, then you have a debt retirement sinking fund. Perhaps as a consumer if you were able to get a 0% interest plan with no payments for one year; you might want to make monthly payments into your own savings account such that you would have the needed funds to pay off your purchase when it comes due. Businesses usually set up these funds for the retirement of stocks, bonds, and debentures.

Whether the sinking fund is for capital savings or debt retirement, the mathematical calculations and procedures are identical. Now, why discuss sinking funds in the chapter about bonds? Many bonds carry a sinking fund provision. Once the bond has been issued, the company must start regular contributions to a sinking fund because large sums of money have been borrowed over a long time frame; investors need assurance that the bond issuer will be able to repay its debt upon bond maturity. For example, in the chapter opener the profits needed to repay the financing for the Bipole III project will not just miraculously appear in the company's coffers. Instead, over a period of time the company will accumulate the funds through saved profits.

To provide further assurance to bondholders, the sinking fund is typically managed by a neutral third party rather than the bond-issuing company. This third-party company ensures the integrity of the fund, working toward the debt retirement in a systematic manner according to the provisions of the sinking fund. Investors much prefer bonds or debentures that are backed by sinking funds and third-party management because they are less likely to default.

In the case of bonds or debentures, sinking funds are most commonly set up as ordinary simple annuities that match the timing of the bond interest payments. Thus, when a bond issuer makes an interest payment to its bondholders, it also makes an annuity payment to its sinking fund. In other applications, any type of annuity is possible, whether ordinary or due and general or simple.

Complete Ordinary Sinking Fund Schedules

A complete **sinking fund schedule** is a table that shows the sinking fund contribution, interest earned, and the accumulated balance for every payment in the annuity. It is very similar to an amortization schedule except that (1) the balance increases instead of decreasing, and (2) the interest is being earned instead of being paid.



The Formula

To complete the sinking fund schedule, you must calculate the interest earned and thus apply Formula 13.1, which calculates the interest portion of a single payment:

$$\text{INT} = \text{BAL} \times ((1 + i)^{\frac{\text{CY}}{\text{PY}}} - 1)$$

Aside from that formula, the table is mathematically simple, requiring only basic addition.



How It Works

Follow these steps to develop a complete sinking fund schedule:

Step 1: Draw a timeline. Identify all of your time value of money variables (N , IY , FV_{ORD} , PMT , PV , PY , CY). If either N or PMT is unknown, solve for it using an appropriate formula. Remember to round PMT to two decimals.

Step 2: Set up a sinking fund schedule following the template below, (the cells are marked with the step numbers that follow). The number of payment rows in your table is equal to the number of payments in the annuity (N).

Payment Number	Payment Amount at End(\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)
0—Start	N/A	N/A	(3)
1	(4)	(5)	(6)
...			
Last payment			
Total	(9)	(9)	N/A

Step 3: Fill in the present value of the annuity (PV). You typically start with a zero balance when you save toward a future goal, although sometimes you may make an immediate lump-sum deposit to start the account.

Step 4: Fill in the rounded annuity payment (PMT) all the way down the column, including the final payment row.

Step 5: Calculate the interest portion of the sinking fund's current balance (INT) using Formula 13.1. Round the number to two decimals for the table, but keep track of all decimals throughout.

Step 6: Calculate the new balance by taking the previous unrounded balance on the line above and adding both the annuity payment and the unrounded interest on the current row. Round the result to two decimals for the table, but keep track of all decimals for future calculations.

Step 7: Repeat steps 5 and 6 for each annuity payment until the schedule is complete.

Step 8: Since all numbers are rounded to two decimals throughout the table, check the table for the "missing penny." This is the same "missing penny" concept as discussed in the Chapter 13 amortization schedules. Ensure that previous balances plus current payments and interest equal the new balance. If not, adjust the interest earned amount as needed. Do not adjust the balance in the account or the payment. Make as many penny adjustments as needed; they typically appear in pairs such that when you must increase one line by one penny, another subsequent line reduces the interest by one penny.

Step 9: Total up the principal payments made and interest earned for the entire schedule.



Important Notes

When you work with sinking fund schedules, remember three key considerations with respect to the annuity type, final payment adjustment, and the rounding of the sinking fund payment:

1. You can use the ordinary sinking fund schedule for both simple and general annuities.
 - Under a simple sinking fund, the payments and interest always occur together on the same line of the schedule, which raises no complications of realized versus accrued interest.
 - However, for the general annuity, although you could create a table where interest is converted to principal between each annuity payment, it makes no fundamental or mathematical difference in the balance of the fund at any given time. Therefore, this textbook's sinking fund schedules reflect the interest earned regardless of whether it has been converted to principal or remains in accrued format as of each annuity payment date.
2. You are not legally required to be precise when saving up for a future goal. You do not need to adjust the final payment to bring the sinking fund to an exact balance. If the account balance is marginally below or above the goal, you can either top it up as needed or use the extra funds for another purpose.

For the above reason, the sinking fund annuity payment amount is frequently rounded off to a convenient whole number. In this textbook you will be clearly instructed if such rounding is to occur; otherwise, use the exact annuity payment rounded to two decimals.

Your BAH Plus Calculator. Although the calculator has no function called "sinking fund," sinking funds have the same characteristics as amortization schedules. Therefore, use the AMORT function located on the 2nd shelf above the PV key to create the sinking fund schedule. You can find full instructions for the AMORT function in Chapter 13. The key difference in using this function for sinking funds is that the principal grows instead of declines.

With respect to the cash flow sign convention, your PV (if not zero) and PMT are negative, since money is being invested into the account. The FV is a positive number, since it can be withdrawn in the future. As in Chapter 13, you need to input all time value of money buttons accurately before accessing this function.

The BAL and INT outputs are accurate and true to definition. The BAL represents the balance in the fund. The INT represents the interest earned for the specified payments. The PRN output is also accurate, but its definition is changed to represent the total of the annuity payments made and the interest earned. It represents the total increase in the balance of the fund over the course of the specified payments.



Give It Some Thought

In an amortization schedule, the interest portion for each payment steadily declines with each annuity payment. What do you think happens to the interest earned with each payment in a sinking fund?

Solution:
It steadily gets larger since the principal balance is increasing instead of decreasing.

Example 14.3A: A Sinking Fund for a Bond

The City of Winnipeg issued a \$500,000 face value bond with three years until maturity. It will make contributions at the end of every six months to a sinking fund earning 5.8% compounded semi-annually. Construct a complete sinking fund schedule and calculate the total principal contributions and interest components.

Plan

You are to construct a complete sinking fund schedule for the bond. The schedule needs to provide the total principal contributions, which is the same as the total payments (PMT) made to the fund, as well as the total interest (INT) earned.

Understand

What You Already Know

Step 1: The timeline for the City of Winnipeg’s sinking fund appears below.
FV_{ORD} = \$500,000,
IY = 5.8%, CY = 12,
PY = 2, Years = 3,
PV = \$0

How You Will Get There

Step 1 (continued): Solve for the ordinary sinking fund annuity payment (PMT) using Formulas 9.1, 11.1, and 11.2 (rearranging for PMT).
Step 2: Set up the sinking fund schedule.
Steps 3 and 4: Fill in the original principal with zero (since this is the opening balance) and payment column with the PMT from step 1.
Steps 5 to 7: Use Formula 13.1 to calculate the interest and add the row to get the new balance for each line.
Step 8: Check for the "missing penny."
Step 9: Sum the interest portion as well as the total payments for the principal contribution.

Today

5.8% semi-annually

3 Years

PV = \$0

PMT ? semi-annually @ END

FV_{ORD} = \$500,000

Perform

Step 1 (continued): $i = 5.8\%/2 = 2.9\%$; $N = 2 \times 3 = 6$ payments

$$\$500,000 = PMT \left[\frac{(1 + 0.029)^{\frac{2}{1}}^6 - 1}{(1 + 0.029)^{\frac{2}{1}} - 1} \right]$$

$$PMT = \frac{\$500,000}{\left[\frac{(1 + 0.029)^{\frac{2}{1}}^6 - 1}{(1 + 0.029)^{\frac{2}{1}} - 1} \right]} = \$77,493.07$$

Steps 2 to 7 (with some calculations) are detailed in the table below:

Payment Number	Payment Amount at End(\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)
0—Start			\$0.00
1	\$77,493.07	(1) \$0.00	(2) \$77,493.07
2	\$77,493.07	(3) \$2,247.30	(4) \$157,233.44
3	\$77,493.07	(5) \$4,559.77	(6) \$239,286.28
4	\$77,493.07	\$6,939.30	\$323,718.65
5	\$77,493.07	\$9,387.84	\$410,599.56
6	\$77,493.07	\$11,907.39	\$500,000.02
Total	\$464,958.42	\$35,041.60	

(1) $INT = \$0 \times ((1 + 0.029)^{\frac{2}{1}} - 1) = \$0$

(2) $New\ Balance = \$0.00 + \$77,493.07 + \$0.00 = \$77,493.07$

(3) $INT = \$77,493.07 \times ((1 + 0.029)^{\frac{2}{1}} - 1) = \$2,247.298945$

(4) $New\ Balance = \$77,493.07 + \$77,493.07 + \$2,247.298945 = \$157,233.4389$

(5) $INT = \$157,233.4389 \times ((1 + 0.029)^{\frac{2}{1}} - 1) = \$4,559.769729$

(6) $New\ Balance = \$157,233.4389 + \$77,493.07 + \$4,559.769729 = \$239,286.28$

Steps 8 to 9: There are no rounding errors and the table is correct. Total the interest and principal contributions.

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
END	6	5.8	0	Answer: -77,493.06705 Rekeyed as -77493.07	500000	2	2

Payment	P1	P2	BAL (output)	INT (output)
1	1	1	-77,493.07	0
2	2	2	-157,233.439	2,247.29903
3	3	3	-239,286.2788	4,559.769732
4	4	4	-323,718.6508	6,939.302084
5	5	5	-410,599.5617	9,387.840875
6	6	6	-500,000.019	11,907.38729
Total	1	6	-500,000.019	35,041.59901

Note: Total contributions are $PMT \times 6$

Present

The complete sinking fund schedule is shown in the table above. The total interest earned by the City of Winnipeg is \$35,041.60 in addition to the \$464,958.42 of principal contributions made. Note that the fund has an extra \$0.02 in it after the three years.

Partial Ordinary Sinking Fund Schedules

For the same reasons as with amortization schedules, sometimes you are interested in creating partial sinking fund schedules for a specified range of payments and not the entire sinking fund. This may be because of any of the following reasons:

- The sinking fund schedule is too long.
- Your sole interest is in the interest portion during a specific period of time for accounting and tax purposes.
- You are assisting in the financial planning of the individual or organization at a particular point in time.



How It Works

For a partial sinking fund schedule, follow almost the same procedure as for a complete sinking fund schedule, with the following notable differences:

Step 2: Set up the partial sinking fund schedule according to the template provided here. The table identifies the corresponding step numbers.

Step 3: Calculate the balance accumulated in the account immediately before the first payment of the partial schedule. Recall the three calculations required to determine this value:

Payment Number	Payment Amount at End (\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)
Preceding Payment #	N/A	N/A	(3)
First Payment Number of Partial Schedule	(4)	(5)	(6)
...			
Last Payment Number of Partial Schedule			
Total	(9)	(9)	N/A

- Calculate the future value of the original principal. Use Formulas 9.1 (Periodic Interest Rate), 9.2 (Number of Compounding Periods for Single Payments), and 9.3 (Compound Interest for Single Payments).
- Calculate the future value of all annuity payments already made. Apply Formulas 11.1 (Number of Annuity Payments) and 11.2 (Ordinary Annuity Future Value).
- Sum the above two calculations to arrive at the principal balance (BAL) immediately before the first payment required in the partial sinking fund schedule.

Example 14.3B: A Partial Sinking Fund Schedule

The Alberta Capital Finance Authority (ACFA) issued a \$200,000 face value bond with five years until maturity. The sinking fund provision requires payments to be made at the end of every quarter into a fund earning 4.4% compounded quarterly. Create a partial sinking fund schedule for the third year. Calculate the total interest earned and total contributions made in the third year.

Plan

You are to construct a partial sinking fund schedule for the third year of the loan along with the total interest earned and total contributions for the year.

Understand	What You Already Know Step 1: The timeline for the bond appears below. $FV_{ORD} = \\$200,000$, $IY = 4.4\%$, $CY = 4$, $PY = 4$, Years = 5, $PV = \\$0$	How You Will Get There Step 1 (continued): Solve for the ordinary sinking fund annuity payment (PMT) using Formulas 9.1, 11.1, and 11.2 (rearranging for PMT). Step 2: Set up the partial sinking fund schedule according to the template. Step 3: There is no single payment at the beginning ($PV = \\$0$). Calculate just the future value of the first eight annuity payments using Formulas 11.1 and 11.2. The balance in the account prior to the third year is equal to the FV_{ORD}. Step 4: Fill in the payment column for the four payments made in the third year. Steps 5 to 7: Use Formula 13.1 to calculate interest and add the row to get the new balance for each line. Step 8: Check for the "missing penny." Step 9: Total up the interest portion as well as the total payments for the principal contribution.																																																		
	Today 4.4% quarterly 5 Years PV = \$0 PMT ? quarterly @ END FV_{ORD} = \$200,000																																																			
Perform	Step 1 (continued): $i = 4.4\%/4 = 1.1\%$; $N = 4 \times 5 = 20$ payments $\\$200,000 = PMT \left[\frac{(1 + 0.011)^4^{20} - 1}{(1 + 0.011)^4 - 1} \right]$ $PMT = \frac{\\$200,000}{\left[\frac{(1 + 0.011)^4^{20} - 1}{(1 + 0.011)^4 - 1} \right]} = \\$8,994.98$ Steps 2 to 7 (with some calculations, including step 3) are detailed in the table below: <table><tr><th>Payment Number</th><th>Payment Amount at End(\$) (PMT)</th><th>Interest Earned or Accrued (\$) (INT)</th><th>Principal Balance Accumulated at End of Payment Interval (\$) (BAL)</th></tr><tr><td>8</td><td></td><td></td><td>(1) \$74,792.09</td></tr><tr><td>9</td><td>\$8,994.98</td><td>(2) \$822.71</td><td>(3) \$84,609.78</td></tr><tr><td>10</td><td>\$8,994.98</td><td>\$930.71</td><td>\$94,535.47</td></tr><tr><td>11</td><td>\$8,994.98</td><td>\$1,039.89</td><td>\$104,570.34</td></tr><tr><td>12</td><td>\$8,994.98</td><td>\$1,150.27</td><td>\$114,715.59</td></tr><tr><td>Total</td><td>\$35,979.92</td><td>\$3,943.58</td><td></td></tr></table> (1) Step 3: $N = 4 \times 2 = 8$ payments; $FV_{ORD} = \\$8,994.98 \left[\frac{(1 + 0.011)^4^8 - 1}{(1 + 0.011)^4 - 1} \right] = \\$8,994.98 \left[\frac{0.091463}{0.011} \right] = \\$74,792.0893$ (2) $INT = \\$74,792.0893 \times ((1 + 0.011)^4 - 1) = \\$822.712982$ (3) New Balance = $\\$74,792.0893 + \\$8,994.98 + \\$822.712982 = \\$84,609.78229$ Steps 8 to 9: In this case there are no rounding errors and the table above is correct. Total the interest and principal contributions.		Payment Number	Payment Amount at End(\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)	8			(1) \$74,792.09	9	\$8,994.98	(2) \$822.71	(3) \$84,609.78	10	\$8,994.98	\$930.71	\$94,535.47	11	\$8,994.98	\$1,039.89	\$104,570.34	12	\$8,994.98	\$1,150.27	\$114,715.59	Total	\$35,979.92	\$3,943.58																							
	Payment Number	Payment Amount at End(\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)																																																
8			(1) \$74,792.09																																																	
9	\$8,994.98	(2) \$822.71	(3) \$84,609.78																																																	
10	\$8,994.98	\$930.71	\$94,535.47																																																	
11	\$8,994.98	\$1,039.89	\$104,570.34																																																	
12	\$8,994.98	\$1,150.27	\$114,715.59																																																	
Total	\$35,979.92	\$3,943.58																																																		
Calculator Instructions <table><tr><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>END</td><td>20</td><td>4.4</td><td>0</td><td>Answer: -8,994.980862 Re-keyed as -8994.98</td><td>200000</td><td>4</td><td>4</td></tr></table> <table><tr><th>Payment</th><th>P1</th><th>P2</th><th>BAL (output)</th><th>INT (output)</th></tr><tr><td>8</td><td>8</td><td>8</td><td>-74,792.0893</td><td>Not needed</td></tr><tr><td>9</td><td>9</td><td>9</td><td>-84,609.78228</td><td>822.712982</td></tr><tr><td>10</td><td>10</td><td>10</td><td>-94,535.46989</td><td>930.707605</td></tr><tr><td>11</td><td>11</td><td>11</td><td>-104,570.3401</td><td>1,039.890169</td></tr><tr><td>12</td><td>12</td><td>12</td><td>-114,715.5938</td><td>1,150.273741</td></tr><tr><td>Total</td><td>9</td><td>12</td><td>-114,715.5938</td><td>3943.584497</td></tr></table> Note: Total contributions are $PMT \times 4$		Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	END	20	4.4	0	Answer: -8,994.980862 Re-keyed as -8994.98	200000	4	4	Payment	P1	P2	BAL (output)	INT (output)	8	8	8	-74,792.0893	Not needed	9	9	9	-84,609.78228	822.712982	10	10	10	-94,535.46989	930.707605	11	11	11	-104,570.3401	1,039.890169	12	12	12	-114,715.5938	1,150.273741	Total	9	12	-114,715.5938	3943.584497
Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y																																													
END	20	4.4	0	Answer: -8,994.980862 Re-keyed as -8994.98	200000	4	4																																													
Payment	P1	P2	BAL (output)	INT (output)																																																
8	8	8	-74,792.0893	Not needed																																																
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10	10	10	-94,535.46989	930.707605																																																
11	11	11	-104,570.3401	1,039.890169																																																
12	12	12	-114,715.5938	1,150.273741																																																
Total	9	12	-114,715.5938	3943.584497																																																
Present	The partial sinking fund schedule for the third year is shown in the table above. ACFA contributed \$35,979.92 to the fund and earned total interest of \$3,943.58.																																																			

Sinking Funds Due Schedules

While ordinary sinking funds are typical for bonds, capital savings sinking funds can take any form. Whether saving personally for the down payment on a house or saving at work for the acquisition of a warehouse, the investor determines the timing of the annuity payments.

Sinking funds due require a small modification to the headers in the sinking fund due schedule, as illustrated in the table. Recall that the payment occurs at the beginning of the payment interval.

Payment Number	Payment Amount at BGN (\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)
0—Start	N/A	N/A	(3)
1	(4)	(5)	(6)
...			
Last payment			
Total	(9)	(9)	N/A

1. Ensure that any unknown variables like PMT are calculated by an annuity due formula.
2. The first payment occurs at the beginning of the first time period. Hence, this first payment receives interest, unlike the first payment in an ordinary annuity, which receives no interest during the first time period because it is paid at the end.

In accordance with the practice for previous schedules, do not distinguish between earned and accrued interest for general sinking funds due. Both amounts are represented in the third column of the table. If you need a partial sinking fund due schedule, its structure is the same as the table for ordinary sinking funds, although the headers change to match the table above.



The Formula

Calculating the interest portion in a sinking fund due schedule requires a slight modification to Formula 13.1 for the interest portion of a single payment. In an annuity due, the payment is deposited at the beginning of the payment interval and therefore accrues interest during the current payment interval. This means that you must add the annuity payment to the previous balance before calculating interest.

INT_{SFDUE} is Interest Portion: The interest portion of any sinking fund annuity payment. Recall that the formula accommodates both simple and general annuities.

PMT is Sinking Fund Due Annuity Payment Amount: This is the annuity due payment amount regularly deposited into the sinking fund due account.

CY is Compound Frequency: The number of compounding periods per year.

Formula 14.7 - Interest Portion of a Sinking Fund Single Payment Due:

$$\text{INT}_{\text{SFDUE}} = (\text{BAL} + \text{PMT}) \times ((1 + i)^{\frac{\text{CY}}{\text{PY}}} - 1)$$

BAL is Previous Balance Owing: This is the principal balance prior to the deposit of the current annuity due payment.

i is Periodic Interest Rate: This is the periodic rate of interest used in converting the interest to principal. It results from Formula 9.1.

PY is Payment Frequency: The number of payment intervals per year.



How It Works

The steps for creating either a complete or partial sinking fund due schedule remain much the same as for an ordinary sinking fund schedule. You require the following adaptations:

- Step 2: Use the sinking fund due table instead of ordinary sinking fund table.
- Step 5: Use Formula 14.7 instead of Formula 13.1.



Important Notes

Your BAI Plus Calculator. Probably the most significant change occurs here. The AMORT function is designed only for ordinary amortization, but you can easily adapt it to sinking funds due. These changes are similar to the adaptations required for amortization schedules due.

- **P1 and P2:** In a sinking fund due, since the first payment occurs today (time period 0), the second payment is at time period 1, and so on. So the payment number of a sinking fund due is always one higher than the payment number of an ordinary sinking fund; always add 1 to the payment number being calculated. For example, if interested only in payment seven, set both P1 and P2 to 8. Or if interested in the payment range 14 through 24, set P1 = 15 and P2 = 25.
- **BAL:** To generate the output, the payment numbers are one too high. This results in the balance being increased by one extra payment. To adapt, manually decrease the balance by removing one payment (or with BAL on your display, use the shortcut key sequence of — RCL PMT =).
- **INT and PRN:** Both of these numbers are correct. Recall that PRN reflects the sinking fund payment (PMT) and interest amount (INT) together.

Example 14.3C: Saving for a Vehicle Down Payment

In an effort to be more environmentally friendly, Bernard is considering leasing an electric vehicle. The lease terms require a down payment of \$2,000. To save up, Bernard starts making quarterly contributions today for the next year into a fund earning 5.3% semi-annually. Create a sinking fund due schedule and calculate his total payments and interest.

Plan	You are to construct a complete sinking fund due schedule for car savings along with the total interest (INT) and total principal contributions, or the total payments (PMT) made to the fund.	
Understand	What You Already Know Step 1: The timeline for the car down payment savings plan appears below. $FV_{\text{DUE}} = \\$2,000$, $IY = 5.3\%$, $CY = 2$, $PY = 4$, $\text{Years} = 1$, $PV = \\$0$	How You Will Get There Step 1 (continued): Solve for the sinking fund due annuity payment (PMT) using Formulas 9.1, 11.1, and 11.3 (rearranging for PMT). Step 2: Set up the sinking fund due schedule. Steps 3 and 4: Fill in the original principal with zero (since this is the opening balance) and the payment column with the PMT from step 1. Steps 5 to 7: Use Formula 14.7 to calculate interest and add the row to get the new balance for each line. Step 8: Check for the "missing penny." Step 9: Total up the interest portion as well as the total payments for the principal contribution.

Step 1 (continued): $i = 5.3\%/2 = 2.65\%$; $N = 4 \times 1 = 4$ payments

$$\$2,000 = \text{PMT} \left[\frac{\left[(1 + 0.0265)^{\frac{2}{4}} \right]^4 - 1}{(1 + 0.0265)^{\frac{2}{4}} - 1} \right] \times (1 + 0.0265)^{\frac{2}{4}} \quad \text{PMT} = \frac{\$2,000}{\left[\frac{\left[(1 + 0.0265)^{\frac{2}{4}} \right]^4 - 1}{(1 + 0.0265)^{\frac{2}{4}} - 1} \right] \times (1 + 0.0265)^{\frac{2}{4}}} = \$483.87$$

Steps 2 to 7 (with some calculations) are detailed in the table below:

Payment Number	Payment Amount at BGN (\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)
0–Start			\$0.00
1	\$483.87	(1) \$6.37	(2) \$490.24
2	\$483.87	(3) \$12.82	(4) \$986.93
3	\$483.87	\$19.36	\$1,490.16
4	\$483.87	\$25.98	\$2,000.02
Total			

(1) $\text{INT}_{\text{SFDUE}} = (\$0 + \$483.87) \times ((1 + 0.0265)^{\frac{2}{4}} - 1) = \$6.369356$

(2) New Balance = $\$0.00 + \$483.87 + \$6.369356 = \$490.239356$

(3) $\text{INT}_{\text{SFDUE}} = (\$490.239356 + \$483.87) \times ((1 + 0.0265)^{\frac{2}{4}} - 1) = \$12.822555$

(4) New Balance = $\$490.239356 + \$483.87 + \$12.822555 = \$986.931911$

Steps 8 to 9: Adjust for the "missing pennies" (noted in red) and total the payments and interest.

Payment Number	Payment Amount at BGN (\$) (PMT)	Interest Earned or Accrued (\$) (INT)	Principal Balance Accumulated at End of Payment Interval (\$) (BAL)
0–Start			\$0.00
1	\$483.87	(1) \$6.37	(2) \$490.24
2	\$483.87	(3) \$12.82	(4) \$986.93
3	\$483.87	\$19.36	\$1,490.16
4	\$483.87	\$25.99	\$2,000.02
Total	\$1,935.48	\$64.54	

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	4	5.3	0	Answer: -483.865761 Re-keyed as -483.87	2000	4	2

Payment	P1	P2	BAL (output) (You must press – RCL PMT = to correct)	INT (output)
1	2	2	-490.239356	6.369356
2	3	3	-986.931911	12.822555
3	4	4	-1,490.162611	19.360699
4	5	5	-2,000.017519	25.984907
Total	2	5	-2,000.017519	64.537518

Perform

Present

The complete sinking fund due schedule is shown in the table above. The total interest earned by Bernard is \$64.54 in addition to the \$1,935.48 of principal contributions made. Note that the fund has an extra \$0.02 in it because of rounding.



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14.4: Debt Retirement and Amortization

(Balancing the Books)

Issuing, buying, and selling bonds results in financial obligations and accounting responsibilities. For instance, your firm is about to issue marketable bonds to finance a major venture in the near future. These bonds require a sinking fund provision to ensure investor confidence. However, the company first needs to foresee its financial obligations if it issues the bonds. How much interest will the company need to pay out to its bondholders annually? What annual sum will it deposit into the sinking fund to satisfy the provision? How does the liability side of the company's balance sheet reflect the fund's provisions? All these questions need to be answered so that you can make an informed decision.

Meanwhile, the finance department reports that your company invested in marketable bonds purchased at a discount. This means your company will benefit from the future bond interest payments and also realize the bond's redemption price upon maturity. Your firm's accounting records must show capital gains being realized over the years, in the form of the difference between the face value and the discounted amount at which the bonds were purchased.

As you can see, the decision to issue or invest in a bond has many financial and tax implications that a company or an individual investor must address. This section explores the financial obligations associated with retiring bond debt and accurate recording of the debt. Also examined is the accounting for premiums and discounts, both of which have tax implications for bondholders and issuers.

Debt Retirement

When an organization issues a bond, the three primary financial implications involve the bond's interest payments, the sinking fund payments, and the balance sheet liability tied to the bond.

1. **Annual Bond Interest Payments.** As discussed earlier in this chapter, most bonds compound semi-annually with semi-annual interest payments, thus $CY = PY$. Because companies manage their books by their fiscal year, they are more interested in the annual total payment commitment than in the amount of each payment, PMT_{BOND} . Therefore, the annual total payment commitment is represented by

$$\text{Annual Bond Interest Payment} = PMT_{\text{BOND}} \times PY$$

Substituting Formula 14.2 in place of PMT_{BOND} :

$$\text{Annual Bond Interest Payment} = \text{Face Value} \times \frac{CPN}{CY} \times PY$$

Since $CY = PY$, simplifying the above formula produces the most direct method of arriving at the annual payment amount:

$$\text{Annual Bond Interest Payment} = \text{Face Value} \times CPN$$

2. **Annual Bond Sinking Fund Payments.** Sinking fund payments are most commonly set up to match the timing of the bond's interest payments. Hence, these payments are semi-annual. Again, the focus is on the annual sinking fund obligation, which you calculate by multiplying the sinking fund payment by the payment frequency, or $PMT \times PY$. The total of the annual bond interest payments and the annual sinking fund payments is referred to as the **annual cost of the bond debt** or the periodic cost of the bond debt.
3. **Balance Sheet Liability.** The bond represents a debt for the issuing organization. Through the sinking fund, the company saves up money to extinguish that debt. The amount of debt that is outstanding is the difference between the money owing and the money saved at any given point in time. Thus, the **book value of the bond debt** represents the difference between the principal amount owing on the bond and the accumulated balance in the sinking fund at any point in time. For example, if the company issued \$10 million in bonds and has accumulated \$1 million in its sinking fund, the book value of the debt is \$9 million. This would be the balance owing if the company used the sinking fund to retire a portion of the bond debt at face value.



The Formula

To calculate the annual cost of the bond debt, you combine both the annual bond interest payments and annual bond sinking fund payments into a single formula.

ACD is Annual Cost of the Bond Debt: The total annual financial obligation associated with a bond issuance.

Formula 14.8 - The Annual Cost of the Bond Debt:

$$\text{ACD} = (\text{Face Value} \times \text{CPN}) + (\text{PMT} \times \text{PY})$$

(Face Value × CPN) is Annual Bond Interest Payments: The total annual interest paid to bondholders, which reflects the face value of the bond outstanding (Face Value) and the nominal coupon rate of interest paid (CPN).

(PMT × PY) is Annual Bond Sinking Fund Payments: The total annual sinking fund contributions, which reflect the sinking fund annuity payment amount (PMT) multiplied by the sinking fund payment frequency (PY). You calculate the PMT through Formula 11.4.

Formula 14.9 lets you calculate the book value of the bond debt.

BVD is Book Value of the Bond Debt: For any particular point in time, this is the net balance on the bond; it reflects the principal owing less the amount of money saved.

Formula 14.9 - The Book Value of the Bond Debt:

$$\text{BVD} = \text{Bonds Outstanding} - \text{BAL}$$

Bonds Outstanding is Debt Principal: The principal amount of the debt. Recall that you assume the redemption price equals the face value of the bond. You also assume that no bonds are retired (bought back from investors) prior to maturity. Hence, this amount is the face value of the bond.

BAL is Sinking Fund Balance: For any given point in time, this is the balance that has accumulated in the sinking fund. Calculate this future value amount through Formula 11.2.



How It Works

Follow these steps to calculate the annual cost of the bond debt:

Step 1: Identify the face value of the bond and the coupon rate.

Step 2: If the sinking fund payment (PMT) and payment frequency (PY) are known, skip to step 4. Otherwise, draw a timeline for the sinking fund and identify known variables.

Step 3: Calculate the sinking fund payment using Formula 11.4.

Step 4: Calculate the annual cost of the bond debt using Formula 14.8.

Follow these steps to calculate the book value of the bond debt:

Step 1: Identify the face value of the bond.

Step 2: If the balance in the sinking fund (BAL) is known, skip to step 5. Otherwise, draw a timeline for the sinking fund and identify known variables.

Step 3: Calculate the sinking fund payment using Formula 11.4.

Step 4: Calculate the future value of the sinking fund at the point of interest using Formula 11.2. This is the sinking fund balance, or BAL.

Step 5: Calculate the book value of the bond debt using Formula 14.9.



Give It Some Thought

Suppose a company decides not to wait until maturity and retires some of its bond debt early on an interest payment date. What happens to the BVD if the debt is retired when:

- the market rate equals the coupon rate?
- the market rate is higher than the coupon rate?
- the market rate is lower than the coupon rate?

(Hint: Think about what happens to the market price of the bond.)

Solutions:

a. The BVD is unchanged. If the two rates are the same, then the bonds sell at par value. Money is transferred dollar for dollar from the sinking fund to directly reduce the bond debt.

b. The BVD decreases. If the market rate is higher, the bonds sell at a discount. The company acquires more bonds at the lower bond price, reducing the number of outstanding bonds.

c. The BVD increases. If the market rate is lower, the bonds sell at a premium. The company acquires fewer bonds at the higher bond price, leaving a larger number of outstanding bonds than by redeeming at par.

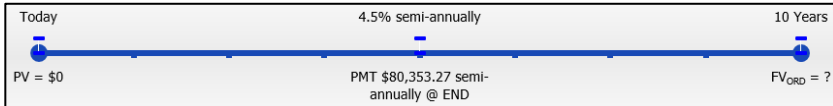
Example 14.4A: The Annual Cost of the Bond Debt

The Bank of Montreal issued a \$10,000,000 face value bond carrying a 5.1% coupon with 30 years until maturity. The bond has a matching sinking fund provision for which monies are invested at 4.5%. Calculate the annual cost of the bond debt. Assume all compounding and payments are semi-annual.

Plan	You need to calculate the annual cost of the bond debt (ACD).																		
Understand	What You Already Know Step 1: Face Value = \$10,000,000, CPN = 5.1% Step 2: The timeline for the sinking fund appears below. $FV_{ORD} = \\$10,000,000$, $IY = 4.5\%$, $CY = 2$, $PY = 2$, Years = 30, $PV = \\$0$	How You Will Get There Step 3: Solve for the ordinary sinking fund annuity payment (PMT) using Formulas 9.1, 11.1, and 11.2 (rearranging for PMT). Step 4: Calculate the annual cost of the bond debt using Formula 14.8.																	
Perform	Step 3: $i = 4.5\%/2 = 2.25\%$; $N = 2 \times 30 = 60$ payments $\$10,000,000 = PMT \left[\frac{(1 + 0.0225)^{60} - 1}{(1 + 0.0225)^2 - 1} \right]$ $PMT = \frac{\$10,000,000}{\left[\frac{(1 + 0.0225)^{60} - 1}{(1 + 0.0225)^2 - 1} \right]} = \$80,353.27$ Step 4: $ACD = (\\$10,000,000 \times 0.051) + (\\$80,353.27 \times 2)$ $= \\$510,000 + \\$160,706.54 = \\$670,706.54$	Calculator Instructions <table border="1"> <tr> <td>Mode</td><td>N</td><td>I/Y</td><td>PV</td></tr> <tr> <td>END</td><td>60</td><td>4.5</td><td>0</td></tr> </table> <table border="1"> <tr> <td>PMT</td><td>FV</td><td>P/Y</td><td>C/Y</td></tr> <tr> <td>Answer: -80,353.27482</td><td>10000000</td><td>2</td><td>2</td></tr> </table>		Mode	N	I/Y	PV	END	60	4.5	0	PMT	FV	P/Y	C/Y	Answer: -80,353.27482	10000000	2	2
Mode	N	I/Y	PV																
END	60	4.5	0																
PMT	FV	P/Y	C/Y																
Answer: -80,353.27482	10000000	2	2																
Present	Each year, the Bank of Montreal pays \$510,000 in interest to its bondholders. Additionally, it deposits \$160,706.54 to its sinking fund. Thus, the annual cost of the bond debt is \$670,706.54 every year for the next 30 years.																		

Example 14.4B: The Book Value of the Bond Debt

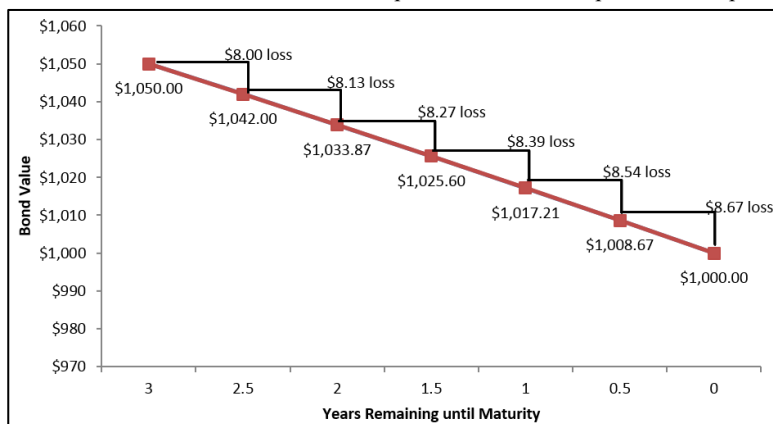
Using Example 14.4A, calculate the book value of the bond debt after 10 years.

Plan	You must calculate the book value of the bond debt (BVD) after 10 years.																		
Understand	What You Already Know Step 1: Face Value = \$10,000,000 Steps 2 and 3: The sinking fund balance is unknown. From the previous example, the sinking fund payment is $PMT = \\$80,353.27$. The timeline for calculating the sinking fund after 10 years appears below. $FV_{ORD} = \\$10,000,000$, $IY = 4.5\%$, $CY = 2$, Years = 10, $PMT = \\$80,353.27$, $PY = 2$, $PV = \\$0$ 	How You Will Get There Step 4: Solve for future value of the ordinary sinking fund, or FV_{ORD}, after 10 years using Formulas 9.1, 11.1, and 11.2. Step 5: Calculate the book value of the bond debt using Formula 14.9.																	
Perform	Step 4: $i = 4.5\%/2 = 2.25\%$; $N = 2 \times 10 = 20$ payments made over 10 years $FV_{ORD} = \$80,353.27 \left[\frac{(1 + 0.0225)^{20} - 1}{(1 + 0.0225)^2 - 1} \right] = \$2,001,722.10$ Step 5: $BVD = \\$10,000,000 - \\$2,001,722.10 = \\$7,998,277.90$	Calculator Instructions <table><tr><td>Mode</td><td>N</td><td>I/Y</td><td>PV</td><td>PMT</td></tr><tr><td>END</td><td>20</td><td>4.5</td><td>0</td><td>-80353.27</td></tr></table> <table><tr><td>FV</td><td>P/Y</td><td>C/Y</td></tr><tr><td>Answer: 2,001,722.095</td><td>2</td><td>2</td></tr></table>		Mode	N	I/Y	PV	PMT	END	20	4.5	0	-80353.27	FV	P/Y	C/Y	Answer: 2,001,722.095	2	2
Mode	N	I/Y	PV	PMT															
END	20	4.5	0	-80353.27															
FV	P/Y	C/Y																	
Answer: 2,001,722.095	2	2																	
Present	After 10 years, the Bank of Montreal will accumulate \$2,001,722.10 in its sinking fund. With a \$10,000,000 debt, the book value of the bond debt remaining equals \$7,998,277.90.																		

Amortization of Bond Premiums and Accrual of Bond Discounts

When a bond sells for a price different from its face value, the bond premium or bond discount has accounting implications. Take a look at each scenario.

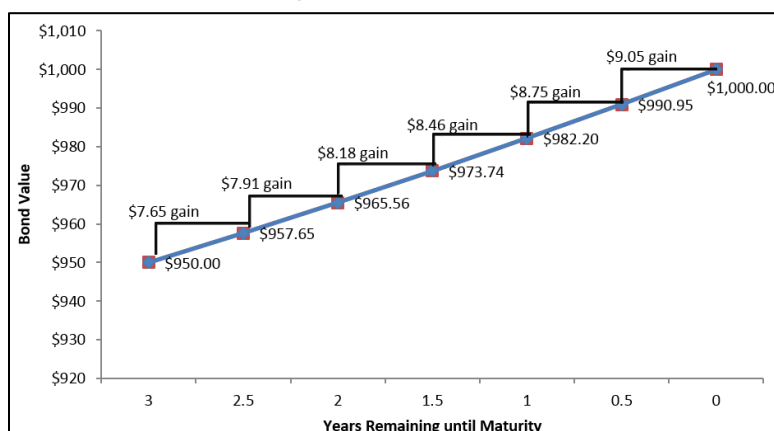
Bond Premium. In these situations, the investor pays more for the bond, say \$1,050 for a \$1,000 bond. If the investor holds onto the bond until maturity, only the redemption price of \$1,000 is returned. The bond premium of \$50 represents a capital loss for the investor. A **capital loss** is the amount by which the current value of an asset falls short of the original purchase price. Commonly accepted practices allow the investor to amortize the \$50 capital loss over the period of time that the bond is held and not just in the period during which the capital loss actually occurs (at maturity). The figure illustrates the amortization of a capital loss assuming three years remain until maturity on a \$1,000 bond carrying a 5% coupon purchased when the market rate was 3.23775%. Note that the total capital loss of \$50 is spread throughout the entire three-year time frame.



Bond Discount. In these situations, the investor pays less for the bond, say \$950 for a \$1,000 bond. If the investor holds onto the bond until maturity, the investor receives the full redemption price of \$1,000. The bond discount of \$50 represents a

capital gain for the investor. A **capital gain** is the amount by which the current value of an asset exceeds the original purchase price. Commonly accepted practice allows the investor to accrue the \$50 capital gain over the period of time that the bond is held and not just in the period during which the capital gain actually occurs (at maturity). For example, assuming three years remain until maturity on a \$1,000 bond carrying a 5% coupon purchased when the market rate was 6.8729%, the figure illustrates the accrual of a capital gain of \$50. Note that the total gain is spread throughout the three-year time frame.

In either situation, the gain or loss has tax implications for the investor. These amounts appear on tax forms and either raise the amount of taxes paid by the investor (for gains) or lower the amount of taxes (for losses). As well, for companies these amounts appear on financial statements.



The Formula

To allow the investor to understand these capital gains or losses, you must develop either an amortization table for premiums or an accrual table for discounts. The table here illustrates the standard format of either table. You should note the following in the table:

Payment Interval Number (at end)	Bond Interest Payment Amount (PMT_{BOND})	Interest at Yield Rate (PMT)	Amortized Premium or Discount Accrued	Bond Value (Cash Price)
0	N/A	N/A	N/A	Purchase Price
1	Formula 14.2	Formula 10.1	Premium: $PMT_{\text{BOND}} - PMT$ Discount: $PMT - PMT_{\text{BOND}}$	Premium: Previous Value – Premium Discount: Previous Value + Discount
...				
Last Payment				
Total	Total Interest Received	Net Income Realized	Premium: Capital Loss Discount: Capital Gain	N/A

- A suitable header of discount or premium is used for the fourth column
- In the last two columns, the method of calculation depends on whether the table is for a premium or discount.



How It Works

Follow these steps to develop a premium amortization table or discount accrual table:

Step 1: Draw a timeline for the bond. Identify all of the time value of money variables for the bond (N , IY , FV , PMT_{BOND} , PY , CY). Use Formulas 14.2 and 14.3 as needed to solve for any unknowns.

Step 2: Set up a table, using the appropriate headers depending on the bond discount or premium. The number of payment intervals is the N identified in step 1.

Step 3: Fill in the purchase price of the bond under the “Bond Value” column for payment 0.

Step 4: Fill in the bond annuity payment, or PMT_{BOND} , for every payment in the “Bond Interest Payment Amount” column.

Step 5: Based on the market yield at which the bond was purchased, calculate the interest portion of the current bond payment based on the prior “Bond Value” using Formula 10.1. Round the number to two decimals for the table but retain all decimals for future calculations.

Step 6: Calculate the amortized premium or discount accrued using one the two techniques shown below. Either way, round the result to two decimals for the table but retain all decimals for future calculations.

- For a bond premium, calculate the amortized premium of the current bond payment by taking the second column and subtracting the third column, or $PMT_{\text{BOND}} - PMT$.
- For a bond discount, calculate the discount accrued by taking the third column and subtracting the second column, or $PMT - PMT_{\text{BOND}}$.

Step 7: Calculate the new bond value for the current payment using one of the two techniques shown below. Either way, round the result to two decimals for the table but retain all decimals for future calculations.

- For a bond premium, calculate the new bond value by taking the previous unrounded “Bond Value” on the line above and subtracting the unrounded amortized amount from step 6.
- For a bond discount, add the two numbers to calculate the new bond value.

Step 8: Repeat steps 5 through 7 for each payment interval until you reach bond maturity.

Step 9: Since all numbers are rounded to two decimals throughout the table, check the table for the “missing penny” using the same method as for amortization tables.

- The previous bond values plus or minus the amortized or discount amount must equal the new bond value.
- The amortized or discount amount plus the interest portion must equal the bond payment amount.

Note that on the final line of the table the balance should equal the face value. If necessary, adjust the penny to exactly match the face value.

Step 10: Sum the bond interest payments received. This is the total interest payments received by the investor.

Step 11: Sum the bond interest earned at the yield rate.

Step 12: Sum the amortized premium or discount accrued to arrive at the total capital loss or capital gain, respectively. You calculate a nominal net income by taking the total bond interest and either subtracting the total capital loss or adding the total capital gain.



Important Notes

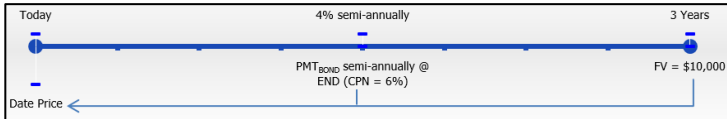
This section introduces how to spread the capital gain or capital loss on a bond across different time periods. Therefore, it sticks to premium amortization tables and discount accrual tables where the bond is purchased on its interest payment date. If the bond is purchased on some other date, this adds complications that are better left for more in-depth texts.

Example 14.4C: Bond Premium Amortization

When posted market rates were 4%, Baseline Industries acquired a \$10,000 bond carrying a 6% coupon rate with three years remaining until maturity. Develop a complete bond premium amortization table. Assume that all payments and interest are semi-annual.

Plan

You need to construct a complete bond premium amortization table.

Understand	What You Already Know Step 1: The timeline of the bond purchase appears below. Coupon Interest Payment: CPN = 6%, CY = 2, Face Value = \$10,000 Bond: FV = \$10,000, IY = 4%, CY = 2, PMT_{BOND} = Formula 14.2, PY = 2, Years Remaining = 3	How You Will Get There Step 1 (continued): Apply Formula 14.2 to determine the periodic bond interest payment. Then apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its interest payment date. Step 2: Set up the bond premium amortization table. Steps 3 and 4: Fill in the purchase price and the bond payment amount column. Steps 5 to 8: Use Formula 10.1 and for each line calculate the last two columns. Step 9: Check for the "missing penny." Steps 10 to 12: Total up the bond payments, interest payments, and amortized amounts.																																															
																																																	
Perform	Step 1: $PMT_{BOND} = \\$10,000 \times \frac{0.06}{2} = \\$300$ $i = 4\%/2 = 2\%$; $N = 2 \times 3 = 6$ payments Date Price = $\frac{\\$10,000}{(1 + 0.02)^6} + \\$300 \left[\frac{1 - \left[\frac{1}{1 + 0.02} \right]^6}{0.02} \right] = \\$10,560.14$ Steps 2 to 8 (with some calculations) are detailed in the table below:																																																
	<table><tr><th>Payment Interval Number (at end)</th><th>Bond Interest Payment Amount (PMT_{BOND})</th><th>Interest at Yield Rate (PMT)</th><th>Amortized Premium</th><th>Bond Value (Cash Price)</th></tr><tr><td>0</td><td></td><td></td><td></td><td>\$10,560.14</td></tr><tr><td>1</td><td>\$300</td><td>(1) \$211.20</td><td>(2) \$88.80</td><td>(3) \$10,471.34</td></tr><tr><td>2</td><td>\$300</td><td>\$209.43</td><td>\$90.57</td><td>\$10,380.77</td></tr><tr><td>3</td><td>\$300</td><td>\$207.61</td><td>\$92.38</td><td>\$10,288.39</td></tr><tr><td>4</td><td>\$300</td><td>\$205.77</td><td>\$94.23</td><td>\$10,194.15</td></tr><tr><td>5</td><td>\$300</td><td>\$203.88</td><td>\$96.12</td><td>\$10,098.04</td></tr><tr><td>6</td><td>\$300</td><td>\$201.96</td><td>\$98.04</td><td>\$10,000.00</td></tr><tr><td>Total</td><td></td><td></td><td></td><td></td></tr></table>				Payment Interval Number (at end)	Bond Interest Payment Amount (PMT _{BOND})	Interest at Yield Rate (PMT)	Amortized Premium	Bond Value (Cash Price)	0				\$10,560.14	1	\$300	(1) \$211.20	(2) \$88.80	(3) \$10,471.34	2	\$300	\$209.43	\$90.57	\$10,380.77	3	\$300	\$207.61	\$92.38	\$10,288.39	4	\$300	\$205.77	\$94.23	\$10,194.15	5	\$300	\$203.88	\$96.12	\$10,098.04	6	\$300	\$201.96	\$98.04	\$10,000.00	Total				
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Total																																																	
(1) $\\$10,560.14 \times 0.02 = \\$211.2028$ (2) $\\$300 - \\$211.2028 = \\$88.7972$ (3) $\\$10,560.14 - \\$88.7972 = \\$10,471.3428$																																																	
Steps 9 to 12: Adjust for the "missing pennies" (noted in red) and total the bond payment amount, interest at yield rate, and amortized premiums.																																																	
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	END	6	4	Answer: -10,560.14309	300	10000	2	2																																									
Present	By purchasing the bond at a premium price of \$10,560.14 and holding it until maturity, when it has a redemption price of \$10,000, Baseline Industries takes a \$560.14 capital loss. It receives \$1,800 in bond payments, loses \$560.14, and realizes nominal net income of \$1,239.86.																																																

Example 14.4D: Bond Discount Accrual

Recalculate Example 14.4C by changing the market rate to 8%. Construct a bond discount accrual table.

Plan

You need to construct a complete bond discount accrual table.

Understand**What You Already Know**

Step 1: The timeline for the bond purchase appears below.

Coupon Interest Payment:

CPN = 6%, CY = 2,

Face Value = \$10,000

Bond: FV = \$10,000,

IY = 8%, CY = 2,

PMT_{BOND} = Formula 14.2,

PY = 2, Years Remaining = 3

How You Will Get There

Step 1 (continued): Apply Formula 14.2 to determine the periodic bond interest payment. Then apply Formulas 9.1, 11.1, and 14.3 to determine the price of the bond on its interest payment date.

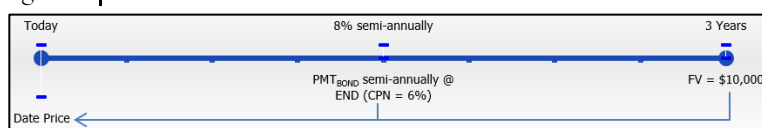
Step 2: Set up the bond discount accrual table.

Steps 3 and 4: Fill in the purchase price and the bond payment amount column.

Steps 5 to 8: Use Formula 10.1 and for each line calculate the last two columns.

Step 9: Check for the "missing penny."

Steps 10 to 12: Total up the bond payments, interest payments, and accrued amounts.

**Perform**Step 1: $PMT_{BOND} = \$10,000 \times \frac{0.06}{2} = \$300$ $i = 8\%/2 = 4\%$; $N = 2 \times 3 = 6$ payments

$$\text{Date Price} = \frac{\$10,000}{(1 + 0.04)^6} + \$300 \left[\frac{1 - \left[\frac{1}{1 + 0.04} \right]^6}{0.04} \right] = \$9,475.79$$

Steps 2 to 8 (with some calculations) are detailed in the table below:

Payment Interval Number (at end)	Bond Interest Payment Amount (PMT _{BOND})	Interest at Yield Rate (PMT)	Discount Accrued	Bond Value (Cash Price)
0				\$9,475.79
1	\$300	(1) \$379.03	(2) \$79.03	(3) \$9,554.82
2	\$300	\$382.19	\$82.19	\$9,637.01
3	\$300	\$385.48	\$85.48	\$9,722.50
4	\$300	\$388.90	\$88.90	\$9,811.39
5	\$300	\$392.46	\$92.46	\$9,903.85
6	\$300	\$396.15	\$96.15	\$10,000.00
Total				

(1) $\$9,475.79 \times 0.04 = \$379.0316$ (2) $\$379.0316 - \$300 = \$79.0316$ (3) $\$9,475.79 + \$79.0316 = \$9,554.8216$

Steps 9 to 12: Adjust for the "missing pennies" (noted in red) and total the bond payment amount, interest at yield rate, and discounts accrued.

Payment Interval Number (at end)	Bond Interest Payment Amount (PMT _{BOND})	Interest at Yield Rate (PMT)	Discount Accrued	Bond Value (Cash Price)
0				\$9,475.79
1	\$300	\$379.03	\$79.03	\$9,554.82
2	\$300	\$382.19	\$82.19	\$9,637.01
3	\$300	\$385.49	\$85.49	\$9,722.50
4	\$300	\$388.89	\$88.89	\$9,811.39
5	\$300	\$392.46	\$92.46	\$9,903.85
6	\$300	\$396.15	\$96.15	\$10,000.00
Total	\$1,800	\$2,324.21	\$524.21	

Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
END	6	8	Answer: -9,475.786314	300	10000	2	2

Present

By purchasing the bond at a discounted price of \$9,475.79 and holding it until maturity, when it has a redemption price of \$10,000, Baseline Industries earns a \$524.21 capital gain. It receives \$1,800 in bond payments, gains \$524.21, and realizes nominal net income of \$2,324.21.



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Chapter 14 Summary

Key Concepts Summary

Section 14.1: Determining the Value of a Bond (What Does It Sell for?)

- Bond definition, characteristics, and key terminology
- Calculating a marketable bond price when the selling date is the interest payment date
- Calculating premiums and discounts
- Calculating a marketable bond price when the selling date is a noninterest payment date

Section 14.2: Calculating a Bond's Yield (Know When to Hold 'em, Know When to Fold 'em)

- Calculating the yield on the bond to maturity
- Calculating the yield realized by an investor if a bond is purchased and sold before maturity

Section 14.3: Sinking Fund Schedules (You Need to Show Responsibility)

- Sinking funds and their purposes
- How to construct complete ordinary sinking fund schedules
- How to construct partial ordinary sinking fund schedules
- Adapting any ordinary sinking fund into an annuity fund due schedule

Section 14.4: Debt Retirement and Amortization (Balancing the Books)

- The financial implications of retiring bond debt
- How to amortize a bond premium or accrue a bond discount

The Language of Business Mathematics

annual cost of the bond debt The annual total of the bond interest payments and the bond sinking fund payments.

bond A debt that is secured by a specific corporate asset and that establishes the issuer's responsibility toward a creditor for paying interest at regular intervals and repaying the principal at a fixed later date.

bond accrued interest The amount of interest that the bond has earned but has not yet paid out since the previous interest payment date.

bond cash price Also known as the *purchase price* or *flat price*, this is the amount of money an investor must directly pay out to acquire the bond. It represents the total of the market price and the accrued interest.

bond coupon rate Also known as the *bond rate* or *nominal rate*, this is the nominal interest rate paid on the face value of the bond.

bond discount The amount by which a bond's selling price falls short of its face value.

bond face value Also called the *par value* or *denomination of the bond*, this is the principal amount of the debt that the investor lent to the bond-issuing corporation.

bond issue date The date that a bond is issued and available for purchase by creditors

bond market price Also known as the *quoted price*, this is the actual value of the bond excluding any accrued interest.

bond market rate This is the prevailing nominal rate of interest in the open bond market.

bond maturity date Also known as the *redemption date* or *due date*, this is the day upon which the redemption price will be paid to the bondholder (along with the final interest payment), thereby extinguishing the debt.

bond premium The amount by which a bond's selling price exceeds its face value.

bond redemption price Also called the *redemption value* or *maturity value*, this is the amount the bond issuer will pay to the bondholder upon maturity of the bond.

bond selling date This is the date that a bond is actively traded and sold to another investor through the bond markets.

book value of the bond debt The difference between the principal amount owing on the bond and the accumulated balance in the sinking fund at any point in time.

capital gain The amount by which the current value of an asset exceeds the original purchase price.

capital loss The amount by which the current value of an asset falls short of the original purchase price.

debenture A debt that is not secured by a specific corporate asset and that establishes the issuer's responsibility toward a creditor for paying interest at regular intervals and repaying the principal at a fixed later date.

sinking fund A special account into which an investor, whether an individual or a business, makes annuity payments such that sufficient funds will be on hand by a specified date to meet a future savings goal or debt obligation.

sinking fund schedule A table that shows the sinking fund contribution, interest earned, and the accumulated balance for every payment in the annuity.

yield to maturity A bond's overall rate of return when purchased at a market price and held until maturity. It includes both the semi-annual interest that the bondholders earn on their investment along with the gain or loss resulting from the difference between the market price and the redemption price.

The Formulas You Need to Know

Symbols Used

ACD = Annual cost of the bond debt

AI = Accrued interest

BAL = Principal balance

BVD = Book value of the bond debt

Cash Price = Price paid for a bond including the market price and accrued interest

CPN = Nominal bond coupon rate of interest

CY = Compounding per year or compounding frequency

Date Price = Price of the bond on the interest payment date preceding the sale date

Discount = Bond discount amount

Face Value = Bond face value

i = Periodic interest rate

INT_{SFDUE} = Interest portion of any single annuity due sinking fund payment

N = Number of annuity payments

PMT = Annuity payment amount

PMT_{BOND} = The bond's annuity payment as determined by the coupon rate

Premium = Bond premium amount

PRI = Market price

t = Time ratio (using the actual number of days)

Formulas Introduced

Formula 14.1 The Cash Price for Any Bond: $Cash\ Price = PRI + AI$ (Section 14.1)

Formula 14.2 Bond Coupon Annuity Payment Amount: $PMT_{BOND} = Face\ Value \times \frac{CPN}{CY}$ (Section 14.1)

Formula 14.3 Bond Price on an Interest Payment Date: $Date\ Price = \frac{FV}{(1+i)^N} + PMT_{BOND} \left[\frac{1 - \left[\frac{1}{1+i} \right]^N}{i} \right]$ (Section 14.1)

Formula 14.4 Bond Premium or Discount: $Premium\ or\ Discount = PRI - Face\ Value$ (Section 14.1)

Formula 14.5 Bond Cash Price on a Noninterest Payment Date: $Cash\ Price = (Date\ Price)(1+i)^t$ (Section 14.1)

Formula 14.6 Accrued Interest on a Noninterest Payment Date: $AI = PMT_{BOND} \times t$ (Section 14.1)

Formula 14.7 Interest Portion of a Sinking Fund Single Payment Due: $INT_{SFDUE} = (BAL + PMT) \times ((1+i)^{\frac{CY}{PY}} - 1)$ (Section 14.3)

Formula 14.8 The Annual Cost of the Bond Debt: $ACD = (Face\ Value \times CPN) + (PMT \times PY)$ (Section 14.4)

Formula 14.9: The Book Value of the Bond Debt: $BVD = Bonds\ Outstanding - BAL$ (Section 14.4)

Technology

Calculator

The DATE function was used in this chapter. See the end of Chapter 8 for a full discussion of this function.

The Bond Function

1. The BOND function is located on the second shelf above the number 9 key and accessed by pressing 2nd BOND.
2. Scroll through using the ↓ and ↑ arrows.
3. There are nine lines in the function. The first seven lines are considered the data entry lines, while the last two are the output lines.
4. Upon opening the window, you should use the 2nd CLR Work function to erase previously entered data.

The input lines are as follows:

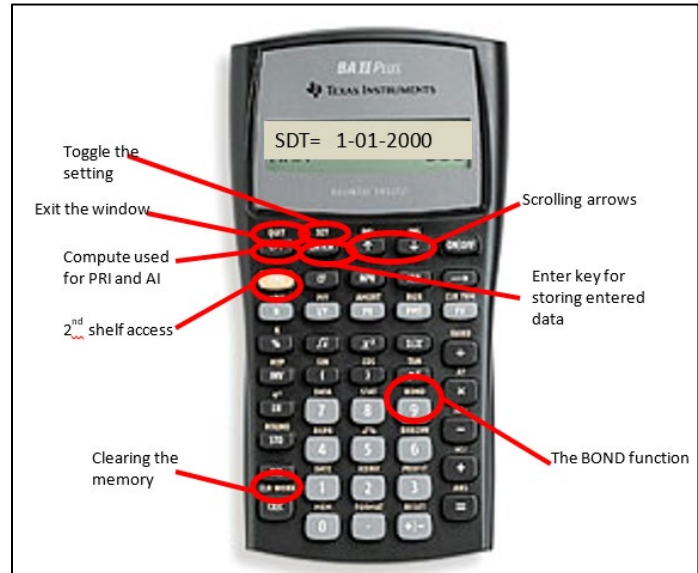
5. SDT is the selling date. Enter it in the standard date format of MM.DDYY and press the ENTER key to store the information.
6. CPN is the nominal coupon rate. It is formatted as a percentage but without the percent sign, so you key in 5.5% as 5.5. You must press the ENTER key to store the information.
7. RDT is the redemption date or maturity date. Enter it in the standard date format and press ENTER to store the information.
8. RV is the redemption value or redemption price expressed as a percentage of the face value. Since the redemption price equals the face value, use the default setting of 100.
9. ACT/360 is a toggle that you change by pressing 2nd SET. ACT counts the actual number of days in the transaction, while 360 treats every month as having 30 days. In Canada, ACT is the standard.
10. 2/Y or 1/Y is a toggle that you change by pressing 2nd SET. 2/Y indicates a semi-annual compound for both the market rate and coupon rate, while 1/Y indicates an annual compound.
11. YLD is the nominal market rate for bonds at the time of sale. It follows the same format as the CPN. Press the ENTER key to store the information.

The output lines are as follows:

12. PRI is the market price of the bond. Press the CPT button to calculate for this solution. The output is a percentage of the redemption price (which is the same as the face value).
13. AI is the accrued interest of the bond. It is automatically calculated when you press the CPT button in the PRI line. The output is a percentage of the redemption price.

Sinking Funds Using AMORT

14. The AMORT function was used in this chapter. See the end of Chapter 13 for a full discussion of this function. Specific sinking fund comments and adaptations are listed below.
 - Ordinary sinking funds: The principal grows instead of declining.
 - With respect to the cash flow sign convention, the PV (if not zero) and PMT are negative, since money is being invested into the account. The FV is a positive number since it can be withdrawn in the future.
 - The BAL and INT outputs are accurate and true to definition. The PRN output is also accurate, but its definition is changed to represent the total of the annuity payments made and the interest earned.
 - Sinking funds due: You must adapt the payment number.
 - *P1 and P2*: Always add one to the payment number(s) desired.
 - *BAL*: It has one payment too many. To correct for this, decrease the balance by removing one payment (or with BAL on your display, press - RCL PMT =).
 - *INT and PRN*: Both of these numbers are correct.





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Chapter 15: Making Good Decisions

(I Would Like a Raise and a Promotion!)

You must make financial decisions throughout your professional career and everyday personal life. Some of these decisions are easy. To make the best choice you need a little intuition and simple calculation. Other decisions are very challenging, confronting you with a great array of competing options, each of which is bundled with numbers projected from now until ... whenever. If you tried just to wing it in these latter scenarios, you could make a catastrophic mistake.

For example, let's say you are the production manager for a company that soon needs to replace a critical machine that is nearing the end of its useful life. At your invitation, salespeople from three competing companies have paid you a visit this week. Each of them showed you an impressive replacement machine. The three machines appear to be equal in design and performance, but each carries a different price tag. Each also differs widely in operating costs such as power consumption, consumables, and labour costs. Maintenance costs follow different timetables. In all cases, you can either purchase the machines or lease them through the supplier's leasing plan.

These machines are not cheap. Because the one you select represents such a significant investment on your company's part, good financing for it is an integral part of the decision. Will the money to pay for it come from a bank loan, a bond issuance, or by issuing some common shares? Perhaps the finance department can withdraw some money from your organization's current investments. Each of these funding sources is associated with a different interest rate.

Your company relies on you to choose the best machine at the lowest possible cost. From a strictly financial perspective, and assuming that all machines are equally productive, which of the three machines is your best choice?

This investment decision should not be guesswork. In previous chapters you have already learned many of the fundamental financial skills required to make effective monetary decisions. In this chapter you will amalgamate and apply your existing skills in dealing with compound interest, ordinary annuities, annuities due, leases, loans, and much more. Section 15.1 introduces the concept of net present value and its application to decision making. You also see a new concept known as *cash flows*. Using these techniques you will work through the basic kinds of decision-making scenarios. Section 15.2 then examines other decision characteristics along with an important decision-making measure called the internal rate of return.

Outline of Chapter Topics

15.1: Net Present Value (What Is the Value of My Decision?)

15.2: Other Measures for Making Decisions (How Else Can I Do It?)

15.1: Net Present Value

(What Is the Value of My Decision?)

If you decide to pursue a course of action, what is the net benefit or cost of making that decision? How can you tell systematically which choice is financially superior?

Let's say that your marketing department thinks it has stumbled onto the next hot product to hit the market. The estimated research and development costs are \$10 million today with an expected \$5 million revitalization cost in five years. The product hits shelves in two years. Estimated year-end profits are \$4 million in the first three years and \$3 million for the subsequent two years. At that time, the rapid pace of technology will end the life cycle of the new product. This last year will earn an estimated \$1 million in profits. The funds needed for this project will be obtained at 12% compounded annually. Does this project make financial sense? Your co-worker argues that the decision is clear: You would spend \$15 million to earn \$19 million, so the project should go forward. What do you think?

Your co-worker's analysis summed the nominal numbers. This clearly violates the fundamental concept of time value of money. In this section, we will explore different types of investment decisions and study techniques for making optimal financial decisions.

What Is Involved in Making Monetary Business Decisions?

Investment decisions must consider both the type of decision being made along with the source of funding, which influences the interest rate used in financial calculations.

What Type of Decision Is Being Made? While there is no such thing as a "standard" or "normal" decision, many business decisions seem to fall into the same three categories:

1. **Decision #1:** *Making One Choice from Multiple Options.* When you face a variety of options that will solve your problem, which option is financially the best?
2. **Decision #2:** *Deciding Whether to Pursue One Course of Action.* When you face only one particular course of action, do you do it or not?
3. **Decision #3:** *Making Multiple Choices under Constraints.* If you have limited resources and many different opportunities from which you can choose more than one, how do you select which opportunities to pursue and which to let pass by?

What Monetary Source Is Being Used and What Does It Cost? The structure of many decisions follows the old adage that you have to spend money to make money. Many courses of action require an investment up front (costs) to receive the reward (profits) in the future. This means you need to tap a source of money before you can go forward with any course of action. Individuals and corporations can access many of the same monetary sources, but businesses have additional options that are not available to individuals.

- **Personal Funding Sources.** You have limited options for obtaining money. Typically, you can either use debt financing such as a loan from a bank or a line of credit, or you can withdraw money from an investment fund or savings account.
 1. If you borrow, the financial cost of the project is the interest rate that is being charged on the borrowed funds. Therefore, if you fund the project using a loan at 8.5% compounded annually, the loan's interest rate becomes the nominal rate you should use in the required time value of money calculations.
 2. If you withdraw money from an investment, the cost of this project is the interest rate you forego on those funds. For example, if you take the money out of a GIC earning 4% compounded annually, you should use the GIC's interest rate in the decision calculations.
- **Business Funding Sources.** Businesses have substantially more choices than individuals when obtaining financing. In addition to debt financing and current investments, businesses also can issue bonds or use equity financing, such as offering company shares. Bartering with business partners is also possible. The same principle applies, though: Whatever interest rate the funding source involves carries forward into any decision-making calculations.

The fundamental concept of time value of money states that all money must be moved to the same date before you can compare alternatives fairly. As mentioned above, the source of funding you use determines the interest rate in these calculations. If you use multiple sources to gather the needed funds, your interest rate is a weighted average of all the debt and equity financing rates, and of course if you use a single source then the weighted average is simply the one source's rate. The weighted interest rate is known as the **cost of capital**. This textbook assumes the cost of capital in calculations is compounded annually unless otherwise specified.

How to Make the Decision

Businesses most commonly make decisions by performing one of two methods: net present value or internal rates of return. Net present value methods are based on all of the cash flows of a decision expressed in today's dollars. This is the focus of this section. Internal rates of return methods are based on the percentage gains of an investment or the yield of the project, which is discussed in the next section.

The **net present value (NPV)** method calculates the difference between all benefits and costs for any given project in today's dollars. This decision-making technique calculates the value of all future income and expense streams brought back to today (which serves as the focal date). The word *net* means that the technique considers the invested money, all costs or expenses, along with all rewards such as profits in its calculations. Remember these criteria for making smart NPV financial decisions:

1. If the net present value is greater than or equal to \$0, the project makes financial sense since it will add value to the company or at minimum break even. It is financially logical to pursue these types of projects.
2. If the net present value is less than \$0, the project does not make financial sense since it will lose money for the company. A company has no financial reason to pursue these types of projects.

Of course, businesses often have many nonfinancial reasons for pursuing or not pursuing certain courses of action. These reasons are beyond the scope of this textbook.

Observe how NPV relates to each of the three common types of business decision categories:

1. **Decision #1: Making One Choice from Multiple Options.** You should choose the project with the best NPV. In this section you will learn a technique that helps you choose between projects of identical length. Section 15.2 introduces another technique for choosing between projects of differing length. For example, you cannot simply use NPV to compare project #1, which has a four-year life, against project #2, which has a six-year life.
2. **Decision #2: Choosing Whether to Pursue One Course of Action.** You should pursue the project if the NPV is positive or at a minimum zero.
3. **Decision #3: Making Multiple Choices under Constraints.** You should choose the projects that maximize the total NPV.

This table summarizes the relationship between the three decision types and the financial criteria.

Decision Type	Financial Criteria
Making One Choice from Multiple Options	Best NPV
Pursuing One Course of Action	NPV must be \$0 or higher
Making Multiple Choices under Constraints	Combination of choices that efficiently maximizes total NPV



The Formula

There is no one formula for calculating the net present value. However, you can represent the NPV conceptually through Formula 15.1.

NPV is Net Present Value: The formula nets out all costs and benefits. Here is how to interpret the NPV:

1. A zero NPV means that the project repays the amounts of its investments plus the required cost of capital. In other words, it breaks even.
2. A positive NPV means that the project provides returns that exceed its investments and the cost of capital.
3. A negative NPV means that the project does not provide enough return to pay for its investments and the cost of capital.

Formula 15.1 - Net Present Value:

$$\text{NPV} = (\text{Present Value of All Future Cash Flows}) - (\text{Initial Investment})$$

Present Value of All Future Cash Flows is All Inflows and Outflows: Take all future cash inflows (such as profits or savings) and outflows (such as investments or costs) and bring the amounts back to today using the cost of capital.

Initial Investment is Money Spent Today: Since most projects require advance funding, this amount represents a cash outflow today, against which you must net the present value of all future cash flows.



How It Works

Every decision is different—there is no one definable method to solve a problem. If the solution was obvious, everyone would always make smart decisions! Instead, three rules for moving money will help you choose well:

1. **Rule #1:** If there is no regularity to the payments, then each amount represents a single payment. You can calculate the present value of all future cash flows only through the techniques you learned in Chapter 9 for compound interest present value for single payments. Example 15.1A demonstrates this rule.
2. **Rule #2:** If there are regular payments continuously in the same amount, you have an annuity situation. You can calculate the present value of all future cash flows using the techniques from Chapter 11 for compound interest present value for either ordinary annuities or annuities due. Example 15.1A also demonstrates this rule.
3. **Rule #3:** If there are regular payments continuously but in differing amounts, then to use a formula you would have to compute the present value of compound interest single payments as per Rule #1. What distinguishes Rule #3 from Rule #1, though, is the "regularity" characteristic. Under this condition, and assuming that you have access to a financial calculator or spreadsheet, this particular situation presents a shortcut technique called cash flows. Examples 15.1B and 15.1C illustrate these cash flows, which are explained later in this section.

Rule
#1

- Apply PV compound interest single payment formulas (Chapter 9)

Rule
#2

- Apply PV compound interest annuity formulas (Chapter 11)

Rule
#3

- Apply PV compound interest cash flow technique (Chapter 15)



Important Notes

When you calculate net present value, most financial forecasts of costs and benefits are best estimates. Therefore, you do not need to be precise to the penny in the final solution. Depending on the dollar amounts involved, companies may commonly round these calculations to the nearest hundred or thousand dollars. While there is no standard practice, this textbook rounds final solutions to the nearest dollar.

Example 15.1A: Decision #1: Choosing between Options through NPV

Saggitt Accounting Services needs to regularly update its existing computer workstations. If it leases the workstations for two years, it will make beginning-of-month payments of \$1,125 with no residual value. If it purchases the workstations, it will need to pay \$26,500 today and it can sell the equipment after two years for 10% of the purchase price. It will need to borrow these funds for two years at 7.75% compounded monthly. Strictly on a financial basis, which option should Saggitt Accounting pursue? How much better is that option in today's dollars?

Plan

Since you need to decide today, you should calculate the net present value (NPV) for each alternative. Each of these choices represents a cost, so you choose the option with the lowest NPV.

Understand	What You Already Know The timeline for the purchase appears in the first timeline below. With respect to the bank loan itself, present value calculations are not necessary since the present value of all loan payments equals the original loan principal. $PV_{ORD} = \\$26,500$, $IY = 7.75\%$, $CY = 12$, $FV = 10\% \times \\$26,500 = \\$2,650$, $Years = 2$ The timeline for the lease appears in the second timeline below. Use the interest rate from the loan in this calculation since Saggitt Accounting would otherwise have to get a loan if it chose not to lease. Therefore, the loan rate represents an appropriate cost of capital. $PMT = \\$1,125$, $PY = 12$, $IY = 7.75\%$, $CY = 12$, $FV = \\$0$, $Years = 2$	How You Will Get There Purchase Timeline: Step 1: You only need to move the revenue from the end of the second year to today. This is decision Rule #1. Therefore, apply Formulas 9.1, 9.2, and 9.3 rearranging for PV. Step 2: Calculate NPV using Formula 15.1. Lease Timeline: Step 3: Take the annuity due and move it to today. This is decision Rule #2. Apply Formulas 11.1 and 11.5. Step 4: There is no initial investment. Formula 15.1 is then $NPV = PV_{DUE}$. Step 5: Calculate the savings by choosing the better option.																										
	Today 7.75% monthly 2 Years $PV_{ORD} = \text{Pay } \\$26,500$ PMT monthly @ END FV = Receive \$2,650 Today 7.75% monthly 2 Years $PV_{DUE} = ?$ \$1,125 monthly @ BGN FV = \$0																											
Perform	Step 1: $i = 7.75\%/12 = 0.6458\bar{3}\%$; $N = 12 \times 2 = 24$ compounds $\\$2,650 = PV(1 + 0.06458\bar{3})^{24}$ $PV = \\$2,650 \div (1 + 0.06458\bar{3})^{24} = \\$2,270.63$ Step 2: $NPV = \\$2,270.63 - \\$26,500.00 = -\\$24,229.37$ or a cost of \$24,229.37 Step 3: $N = 12 \times 2 = 24$ payments $PV_{DUE} = \\$1,125 \left[\frac{1 - \left[\frac{1}{(1 + 0.006458\bar{3})^{12}} \right]^{24}}{(1 + 0.006458\bar{3})^{12}} \right] \times (1 + 0.006458\bar{3})^{12} = \\$25,098.24$ Step 4: $NPV = \text{cost of } \\$25,098.24$ Step 5: $\\$25,098.24 - \\$24,229.37 = \\$868.87$ savings																											
	Calculator Instructions <table><tr><th>Transaction</th><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>Purchase</td><td>END</td><td>24</td><td>7.75</td><td>Answer: -2,270.631567</td><td>0</td><td>2650</td><td>12</td><td>12</td></tr><tr><td>Lease</td><td>BGN</td><td>✓</td><td>✓</td><td>Answer: -25,098.23508</td><td>-1125</td><td>0</td><td>✓</td><td>✓</td></tr></table>	Transaction	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	Purchase	END	24	7.75	Answer: -2,270.631567	0	2650	12	12	Lease	BGN	✓	✓	Answer: -25,098.23508	-1125	0	✓	✓
Transaction	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y																				
Purchase	END	24	7.75	Answer: -2,270.631567	0	2650	12	12																				
Lease	BGN	✓	✓	Answer: -25,098.23508	-1125	0	✓	✓																				
Present	If Saggitt Accounting Services purchases the machine, it will cost \$24,299.37 in today's dollars. If it leases the machine, it will cost \$25,098.24 in today's dollars. The smart choice is to purchase the machine, realizing a savings of \$868.87 in today's dollars.																											

A Regular Stream of Cash Flows

A **cash flow** is a movement of money into or out of a particular project. Therefore, any time you incur a cost or receive a benefit on a particular project, you have a cash flow. Under the three rules for moving money, both single payments and annuities are cash flows. You can move money from a future date to a present date using the existing formulas or calculator techniques.

The third decision rule is for the situation of continuous regular payments that are variable in amount. As indicated in the rule, to solve the problem by formula you would need to calculate compound interest present value (Rule #1) repeatedly. This procedure requires many repetitive calculations in which you calculate N and PV for each cash flow. This is quite cumbersome, since every cash flow introduces two more calculations and another present value to net out. The calculator will simplify this process, reducing both the effort and the chance of error.

As you work with these continuous cash flows, where to put an amount on the timeline depends on whether it is a cost or a benefit.

- **Costs.** Unless otherwise noted, capital costs are spent at the beginning of any time segment. Thus, if there are capital costs in year two, you can assume you incur them at the beginning of year two, which is the same as the end of year one.
- **Benefits.** Revenues, net profits (product sales minus product costs), or other benefits are realized at the end of any time segment. Thus, if there are net profits in year two, you can assume you realize them at the end of year two.



How It Works

Follow these steps when working with regular cash flows of varying amounts:

Step 1: Draw a timeline illustrating all of the cash flows, cost of capital, and any initial investment. Always draw your timeline in an ordinary annuity format with all transactions occurring at the end of each time interval. Thus, any costs you incur at the beginning of a time segment are placed at the end of the prior time segment (unless noted otherwise).

Step 2: For any particular point on the timeline at which more than one cash flow exists, net all cash flows into a single number.

Step 3: Move each cash flow to its present value. If using formulas, calculate the periodic interest rate from Formula 9.1. Follow this calculation by repeatedly applying Formulas 9.2 (Number of Compound Periods for Single Payments) and 9.3 (Compound Interest for Single Payments). Alternatively, if using technology then apply the technique discussed in the appropriate section below.

Step 4: Net the results using Formula 15.1 to arrive at the NPV.



Important Notes

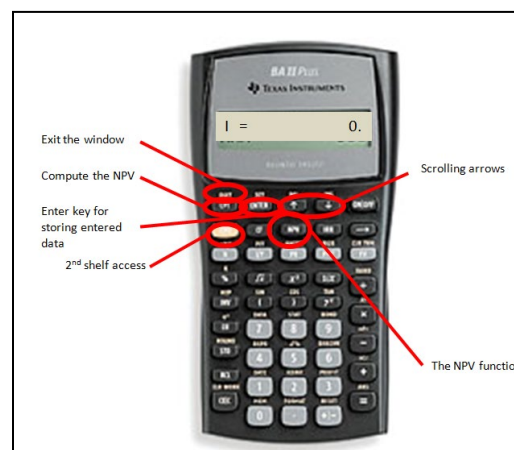
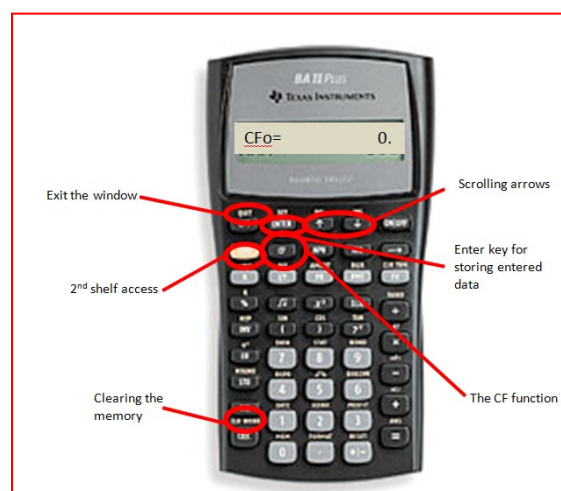
Using the BAII Plus Cash Flow Worksheet. The cash flow function of the calculator requires the cash flows to be regular (such as every month or every quarter or every year). The payment amounts can be the same or variable. If the regularity assumption is not met, do not use the cash flow function; instead solve the problem using Rule #1 for compound interest single payments.

The photo illustrates the buttons used in the cash flow function of your BAII Plus calculator. Access the cash flow function by pressing CF on the keypad. Immediately clear the memory using 2nd CLR Work to delete any previously entered data. Use the $\uparrow$ and $\downarrow$ arrows to scroll through the various lines. You must strictly obey the cash flow sign convention and press Enter after keying in any data. Here are the various lines appearing in the cash flow function:

- CF_0 = any cash flow today
- CXX = a particular cash flow, where XX is the cash flow number starting with 01. You must key in the cash flows in order from the first time segment to the last. You cannot skip a time segment even if it has a value of zero. Each time segment is a placeholder on the timeline.
- FXX = the frequency of a particular cash flow, where XX is the cash flow number. It is how many times *in a row* the corresponding cash flow amount occurs. This allows you to enter recurring amounts instead of keying them in separately. By default, the calculator sets this variable to 1.

Once you have entered all cash flows, you must access other functions to generate the output. For net present value, press NPV on the keypad as illustrated in the photo. This window has two lines:

- I = the matching periodic interest rate for the interval of each time segment. If the timeline is drawn with yearly intervals, then this must



be the annual cost of capital. If the timeline is drawn with semi-annual intervals, this needs to be the matching semi-annual periodic cost of capital.

- NPV = the net present value. Press CPT to calculate this amount. The output nets all future cash flows against any initial investment. This key executes Formula 15.1.



Things To Watch Out For

The most common mistake in using the BAII Plus calculator is failing to enter the matching periodic interest rate in the NPV window. Always check the following:

1. The compounding on the cost of capital must match the interval between cash flows. If not, convert the rate to match. For example, if your intervals are semi-annual but your cost of capital is annual, you must convert the cost of capital to its equivalent semi-annually compounded rate.
2. Apply Formula 9.1, being sure to enter the periodic rate and not the nominal rate of interest.



Paths To Success

Since cash flows can be inflows or outflows, it avoids confusion to write all numbers with their cash flow sign conventions onto the timeline. This helps you keep track of the different flows and minimizes the possibility of incorrectly netting out all present values.



Give It Some Thought

If a negative net present value is calculated, does that mean the project is unprofitable?

Solution: No, it means the project cannot recover its investments and the cost of capital. If the company can sufficiently lower its cost of capital, then the net present value may in fact become positive.

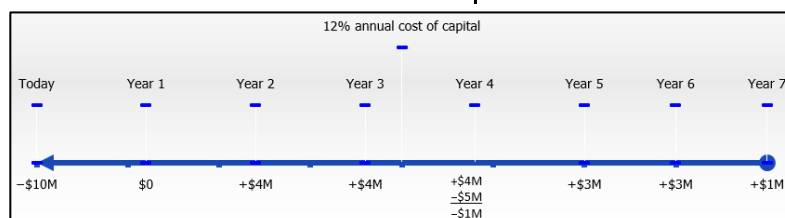
Example 15.1B: Decision #2: Pursuing a Particular Option

Revisit the new product launch discussed in the opening scenario in this section. Recall that research and development costs are \$10 million today with an expected revitalization cost of \$5 million in the fifth year. The product is expected to go on sale starting in two years and is forecast to produce year-end profits of \$4 million for the first three years and another \$3 million per year for the subsequent two years. Its last year will earn a projected \$1 million in profit. The cost of capital for this project is 12% compounded annually. From a financial perspective only, should this new product be launched?

Plan You have cash flows on many different dates. In order to decide today, you should move all monies to today and calculate the net present value (NPV).

Understand **What You Already Know**
Steps 1 and 2: The timeline for the proposed product launch appears below.
 $IY = 12\%$, $CY = 1$, $PV = -\$10M$, Future cash flows are shown in the timeline. Note that the cost in the fifth year occurs at the beginning of the year, which is the same point in time as the end of the fourth year.

How You Will Get There
Step 3: Calculate the periodic interest rate by using Formula 9.1. For each cash flow, calculate the present value using Formulas 9.2 and 9.3, rearranging for PV.
Step 4: Calculate NPV by using Formula 15.1.

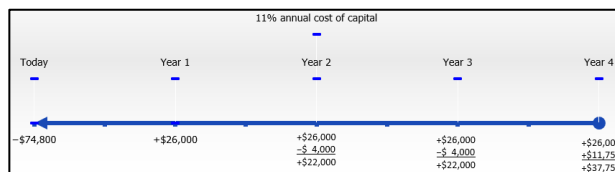
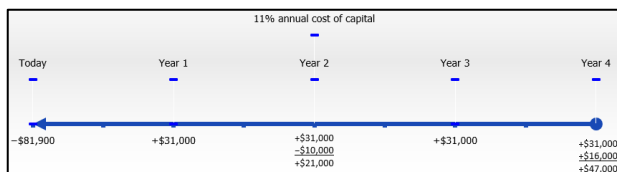


Perform	Step 3: $i = 12\%/1 = 12\%$																									
	Cash Flow 1: $N = 1 \times 2 = 2$ compounds; $PV = \$4,000,000 \div (1 + 0.12)^2 = \$3,188,775.51$																									
	Cash Flow 2: $N = 1 \times 3 = 3$ compounds; $PV = \$4,000,000 \div (1 + 0.12)^3 = \$2,847,120.991$																									
	Cash Flow 3: $N = 1 \times 4 = 4$ compounds; $PV = -\$1,000,000 \div (1 + 0.12)^4 = -\$635,518.0784$																									
	Cash Flow 4: $N = 1 \times 5 = 5$ compounds; $PV = \$3,000,000 \div (1 + 0.12)^5 = \$1,702,280.567$																									
	Cash Flow 5: $N = 1 \times 6 = 6$ compounds; $PV = \$3,000,000 \div (1 + 0.12)^6 = \$1,519,893.364$																									
	Cash Flow 6: $N = 1 \times 7 = 7$ compounds; $PV = \$1,000,000 \div (1 + 0.12)^7 = \$452,349.2153$																									
	Step 4: $NPV = (\$3,188,775.51 + \$2,847,120.991 - \$635,518.0784 + \$1,702,280.567 + \$1,519,893.364 + \$452,349.2153) - \$10,000,000 = \$9,074,901.569 - \$10,000,000 = -\$925,098.43 \Rightarrow -\$925,098$																									
	Calculator Instructions																									
	<table><tr><th colspan="3">Cash Flows</th></tr><tr><th>Cash Flow</th><th>Amount (CXX)</th><th>Frequency (FXX)</th></tr><tr><td>CF0</td><td>-10000000</td><td>N/A</td></tr><tr><td>C01 & F01</td><td>0</td><td>1</td></tr><tr><td>C02 & F02</td><td>4000000</td><td>2</td></tr><tr><td>C03 & F03</td><td>-1000000</td><td>1</td></tr><tr><td>C04 & F04</td><td>3000000</td><td>2</td></tr><tr><td>C05 & F05</td><td>1000000</td><td>1</td></tr></table>			Cash Flows			Cash Flow	Amount (CXX)	Frequency (FXX)	CF0	-10000000	N/A	C01 & F01	0	1	C02 & F02	4000000	2	C03 & F03	-1000000	1	C04 & F04	3000000	2	C05 & F05	1000000
Cash Flows																										
Cash Flow	Amount (CXX)	Frequency (FXX)																								
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C04 & F04	3000000	2																								
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Present	<table><tr><th colspan="2">NPV</th></tr><tr><td>I</td><td>NPV</td></tr><tr><td>12</td><td>Answer: -925,098.4309</td></tr></table>			NPV		I	NPV	12	Answer: -925,098.4309																	
	NPV																									
	I	NPV																								
12	Answer: -925,098.4309																									
If the forecasted costs and profits are reasonably accurate, then the new product should not be launched under the current financial structure since its net present value is $-\$925,098$. This means that the profits are unable to recover the investments required along with the cost of capital.																										

Example 15.1C: Decision #1: Choosing between Options with Cash Flows

Lethbridge Community College is considering purchasing new industrial photocopier equipment and has narrowed the choices down to two comparable Xerox and Canon machines. The Xerox machine can be acquired for \$81,900 today and is expected to have a life of four years with savings of \$31,000 in labour and materials each year. A \$10,000 maintenance package is expected to be purchased at the beginning of year three, and the machine can be sold for \$16,000 after the four years. The Canon machine retails for \$74,800 and is forecasted to save \$26,000 in labour and materials each year. A \$4,000 maintenance procedure is needed at the beginning of both years three and four. The salvage value of the machine is estimated at \$11,750. The cost of capital is 11%. Which machine should be purchased, and how much money will it save over the alternative?

Plan	For each photocopier, you have cash flows on many different dates. In order to decide today, you should move all monies to today and calculate the net present value (NPV) for each photocopier.	
Understand	What You Already Know Steps 1 and 2: The timelines for each photocopier appear below (Xerox on left, Canon on right). Xerox: $IY = 11\%$, $CY = 1$, $PV = -\\$81,900$, future cash flows are shown in the timeline. Canon: $IY = 11\%$, $CY = 1$, $PV = -\\$74,800$, future cash flows are shown in the timeline. Note that costs occurring at the beginning of the year are recorded at the end of the previous year.	How You Will Get There Step 3: Calculate the periodic interest rate by using Formula 9.1. For each photocopier, calculate the present value of each cash flow using Formulas 9.2 and 9.3, rearranging for PV. Step 4: For each photocopier, calculate NPV by using Formula 15.1. Step 5: Calculate the savings by choosing the better option.



Perform	Step 3: $i = 11\%/1 = 11\%$																																								
	Xerox:																																								
	Cash Flow 1: $N = 1 \times 1 = 1$ compounds; $PV = \$31,000 \div (1 + 0.11)^1 = \$27,927.\overline{927}$																																								
	Cash Flow 2: $N = 1 \times 2 = 2$ compounds; $PV = \$21,000 \div (1 + 0.11)^2 = \$17,044.0711$																																								
	Cash Flow 3: $N = 1 \times 3 = 3$ compounds; $PV = \$31,000 \div (1 + 0.11)^3 = \$22,666.93282$																																								
	Cash Flow 4: $N = 1 \times 4 = 4$ compounds; $PV = \$47,000 \div (1 + 0.11)^4 = \$30,960.35578$																																								
	Step 4: $NPV = (\$27,927.\overline{927} + \$17,044.0711 + \$22,666.93282 + \$30,960.35578) - \$81,900$ $= \$98,599.28763 - \$81,900 = \$16,699.29 \Rightarrow \$16,699$																																								
	Canon:																																								
	Cash Flow 1: $N = 1 \times 1 = 1$ compounds; $PV = \$26,000 \div (1 + 0.11)^1 = \$23,423.\overline{423}$																																								
	Cash Flow 2: $N = 1 \times 2 = 2$ compounds; $PV = \$22,000 \div (1 + 0.11)^2 = \$17,855.69353$																																								
Cash Flow 3: $N = 1 \times 3 = 3$ compounds; $PV = \$22,000 \div (1 + 0.11)^3 = \$16,086.21039$																																									
Cash Flow 4: $N = 1 \times 4 = 4$ compounds; $PV = \$37,750 \div (1 + 0.11)^4 = \$24,867.09427$																																									
Step 4: $NPV = (\$23,423.\overline{423} + \$17,855.69353 + \$16,086.21039 + \$24,867.09427) - \$74,800$ $= \$82,232.42162 - \$74,800 = \$7,432.42 \Rightarrow \$7,432$																																									
Step 5: Xerox has higher savings by $\$16,699 - \$7,432 = \$9,267$																																									
Calculator Instructions																																									
<table><tr><th colspan="5">Cash Flows</th></tr><tr><th></th><th colspan="2">Xerox</th><th colspan="2">Canon</th></tr><tr><th>Cash Flow</th><th>Amount (CXX)</th><th>Frequency (FXX)</th><th>Amount (CXX)</th><th>Frequency (FXX)</th></tr><tr><td>CF0</td><td>-81900</td><td>N/A</td><td>-74800</td><td>N/A</td></tr><tr><td>C01 & F01</td><td>31000</td><td>1</td><td>26000</td><td>1</td></tr><tr><td>C02 & F02</td><td>21000</td><td>1</td><td>22000</td><td>2</td></tr><tr><td>C03 & F03</td><td>31000</td><td>1</td><td>37750</td><td>1</td></tr><tr><td>C04 & F04</td><td>47000</td><td>1</td><td></td><td></td></tr></table>		Cash Flows						Xerox		Canon		Cash Flow	Amount (CXX)	Frequency (FXX)	Amount (CXX)	Frequency (FXX)	CF0	-81900	N/A	-74800	N/A	C01 & F01	31000	1	26000	1	C02 & F02	21000	1	22000	2	C03 & F03	31000	1	37750	1	C04 & F04	47000	1		
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C01 & F01	31000	1	26000	1																																					
C02 & F02	21000	1	22000	2																																					
C03 & F03	31000	1	37750	1																																					
C04 & F04	47000	1																																							
<table><tr><th colspan="3">NPV</th></tr><tr><th></th><th>Xerox</th><th>Canon</th></tr><tr><td>I</td><td>11</td><td>11</td></tr><tr><td>NPV</td><td>Answer: 16,699.28763</td><td>Answer: 7,432.421617</td></tr></table>		NPV				Xerox	Canon	I	11	11	NPV	Answer: 16,699.28763	Answer: 7,432.421617																												
NPV																																									
	Xerox	Canon																																							
I	11	11																																							
NPV	Answer: 16,699.28763	Answer: 7,432.421617																																							
Present	Based on the estimated costs, maintenance, and savings, the Xerox photocopier should be purchased because it results in an expected savings of \$9,267 in today's dollars. The Xerox machine represents a current cash flow of \$16,699, while the Canon is \$7,432.																																								

Working with Limited Resources

In an ideal financial world, you should always pursue any project with a positive net present value since that decision results in a financial gain. However, no one, whether an individual person or a large corporation or a government, actually has access to unlimited resources to fund all its potential projects. If such unlimited resources existed, the world's problems could be solved in a day!

There are usually many paths in life that you can choose, each with its own merits. These paths are not necessarily mutually exclusive, so you could choose more than one. However, having limited resources means that you cannot pursue all of these paths. For example, perhaps PepsiCo has learned through market research that it could launch 15 new flavours and types of soft drinks. Each of these flavours and types requires vast production resources, labour demands, and marketing support to bring it to market. But even a company as huge as PepsiCo does not have the resources to pursue all of these opportunities. Which one or ones does it select?

A process called **hard capital rationing** is used to decide. This process requires you to calculate the net present value of each possible project and then allocate your limited capital resources to the projects efficiently to maximize the combined net present value. From a financial perspective, you do not consider any projects with a negative net present value.



The Formula

The goal in choosing multiple options is to maximize the net present value. In selecting these various options, it is unfair to compare projects of different magnitudes simply by looking at their NPV. One would expect that a project requiring a \$10 million investment should produce a much larger NPV than a project requiring a \$10,000 investment. Instead, it is proper to examine the efficient usage of these limited resources, or to look at the "bang for your buck." This requires you to calculate the net present value ratio shown in Formula 15.2.

NPV_{RATIO} is Net Present Value Ratio: By comparing the net present value to the initial investment required, you achieve a measure of the effectiveness of your option. For example, if you calculate a ratio of 1.55, then you learn that for every dollar invested today in the project you will see a net benefit of \$1.55 expressed in today's funds. Rounding this number to two decimals is appropriate.

Formula 15.2 - Net Present Value Ratio:

$$\text{NPV}_{\text{RATIO}} = \frac{\text{NPV}}{\text{Cfo}}$$

Cfo is Initial Cash Flow: This is the initial investment required to pursue the project. You must have access to these required funds from an available financing source today. When you select which projects to pursue, the total of all these initial cash flows for the projects chosen must be within your budgetary limitations.

NPV is Net Present Value: The value of all current and future cash flows related to any one option for the project. This number is calculated from Formula 15.1.



How It Works

Follow these steps to perform hard capital rationing:

Step 1: Identify the budget constraints.

Step 2: If not already known, calculate the net present value (NPV) for each of the available projects using Formula 15.1.

Step 3: Calculate the net present value ratio for each project using Formula 15.2.

Step 4: Rank the projects from the highest to lowest ratios, noting the initial cash flow required under each project.

Step 5: Starting with the project with the highest net present value ratio and progressing through the ranked list of projects, assess different combinations of projects and their total NPV such that the total initial cash flows remain within the budget constraint.

Step 6: Choose the combination that maximizes the NPV.



Important Notes

This technique makes the assumption that each of the projects being considered is independent of the others. In other words, choosing one particular project or path does not prevent another project or path from being chosen.

There are much more advanced techniques for optimizing a list of projects. These techniques include management sciences or linear programming methods. However, these techniques are beyond the scope of this textbook, which focuses simply on the basic principles behind this decision.



Paths To Success

When completing step 5 of the process, always select the highest NPV_{RATIO} projects in order until you reach a point where the remaining budget becomes an issue. In other words, only at the point where the selection of the next ranked project eliminates a lower-ranked project from consideration do you need to start examining the different possible combinations. In most typical situations, the optimal combination includes all projects selected before this critical point.

Example 15.1D: Decision #3: Working with Limited Resources

The Gap has identified several projects that it can pursue, as listed in the table below. The current finances of the company permit a capital budget of \$1.25 million. Which projects should be undertaken?

Project		Initial Cash Flow	Net Present Value						
Automate a production process		\$428,000	\$214,000						
Launch new fashion A		\$780,000	\$550,000						
Replace machinery in distribution centre		\$150,000	\$175,000						
Launch new fashion B		\$395,000	\$610,000						
Expand a retail outlet		\$145,000	\$245,000						
Revise delivery processes		\$282,000	\$420,000						
Plan	You need to perform hard capital rationing to maximize your NPV within your limited budget. This requires you to calculate the net present value ratio (NPV_{RATIO}) for each project.								
Understand	What You Already Know		How You Will Get There						
	Step 1: The total budget available is \$1.25 million. Step 2: The net present values are provided in the table.		Step 3: Calculate the net present value ratio for each project using Formula 15.2. Steps 4 to 6: Rank the NPV_{RATIO} from highest to lowest, assess the budget and combinations available, and then choose the possibility that maximizes NPV.						
Perform	Steps 3 to 5: The NPV_{RATIO} is shown in the table below. Projects are sorted into descending order. A few calculations listed below the table show how the NPV_{RATIO} is calculated. Note that after you select the first project, if you were to select the second project then this would eliminate the fifth project since its initial cash flow exceeds the remaining budget ($\$1,250,000 - \$145,000 - \$395,000 = \$710,000$ remaining). Therefore, you always include the first project in your combinations and must explore various arrangements of the next five projects.								
				Some Possible Combinations					
	Project	Initial Cash Flow	Net Present Value	Net Present Value Ratio	1	2	3	4	5
	Expand a retail outlet	\$145,000	\$245,000	(1) 1.69	•	•	•	•	•
	Launch new fashion B	\$395,000	\$610,000	(2) 1.54	•	•	•		
	Revise delivery processes	\$282,000	\$420,000	1.49	•	•		•	•
	Replace machinery in distribution centre	\$150,000	\$175,000	1.17	•		•	•	
	Launch new fashion A	\$780,000	\$550,000	0.71					•
	Automate a production process	\$428,000	\$214,000	0.50		•	•	•	
	(1) $NPV_{RATIO} = \frac{\$245,000}{\$145,000} = 1.69$								
(2) $NPV_{RATIO} = \frac{\$610,000}{\$395,000} = 1.54$									
		Combinations							
		1	2	3	4	5			
Total Budget	(1) \$972,000	\$1,250,000	\$1,118,000	\$1,005,000	\$1,207,000				
Total NPV	(2) \$1,450,000	\$1,489,000	\$1,244,000	\$1,054,000	\$1,215,000				
(1) $\$145,000 + \$395,000 + \$282,000 + \$150,000 = \$972,000$									
(2) $\$245,000 + \$610,000 + \$420,000 + \$175,000 = \$1,450,000$									
Step 6: Combination #2 offers the highest NPV of \$1,489,000.									
Present	After the projects are ranked from highest to lowest NPV ratio, the "Expand a retail outlet" offered the highest ratio and is selected. Combination #2, consisting of "Expand a retail outlet," "Launch new fashion B," "Revise delivery processes," and "Automate a production process" offers the highest NPV of \$1,489,000 and uses the full \$1,250,000 capital budget.								



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15.2: Other Measures for Making Decisions

(How Else Can I Do It?)

The previous section explored the three main types of decisions and introduced various techniques for making smart investment decisions. This section revisits Decision #1 (making one choice from multiple options) and Decision #2 (pursuing one course of action) and introduces new characteristics:

- Decision #1: Examine situations where the timelines are not equal in length.
- Decision #2: Determine another profit-focused method of reaching the same decision.

Making Choices for Unequal Timelines

Consider the situation in which you need to buy only one of two \$50,000 machines that solve the same problem. Machine #1 has savings of \$20,000 per year, while machine #2 has savings of \$14,000 per year. However, machine #1 has a life expectancy of four years, while machine #2 has a life expectancy of eight years. Calculating the net present value on these two machines at a 15% cost of capital reveals the NPV of machine #1 equals \$7,100 while the NPV of machine #2 equals \$12,823. Is this a fair comparison? Should machine #2 be selected based on its higher NPV? The answer is no. The NPV analysis does not factor in that if you chose machine #1 it must be replaced after four years and would once again have the opportunity to produce more savings over the subsequent four years, thereby offsetting the original difference in NPV.

To fairly choose between the alternatives, you need a calculation that can equate two (or more) timelines of different lengths. This requires you to convert the net present value for each alternative into its **equivalent annual cash flow**. This is an annual annuity payment amount that, when present valued using the cost of capital, arrives at the same NPV as all of the original cash flows.



The Formula

To arrive at the equivalent annual cash flow, you need to apply two formulas:

1. Formula 15.1 calculates the net present value for each alternative you are considering.
2. To convert each calculated NPV into an annual cash flow payment, convert the net present value into an annuity possessing an annual cost of capital and annual end-of-interval payments—or an ordinary simple annuity! Formula 11.4 is reprinted below to illustrate how you can adapt this formula to suit the purposes of the equivalent annual cash flow.

PV_{ORD} is Net Present Value: The NPV from Formula 15.1 represents the present value of the ordinary annuity.

PMT is Equivalent Annual Cash Flow: The ordinary annuity payment amount is the equivalent annual cash flow. If you perform a present value calculation on these payment amounts using the cost of capital as the interest rate, you compute the net present value. As with NPV itself, since future numbers are imprecise, you round this number to the nearest whole number.

N is Number of Cash Flow Payments: This is the total number of annual payments in the timeline for the project being considered. Calculate it from Formula 11.1.

Formula 11.4 - Ordinary Annuity Present Value: $PV_{ORD} = PMT$

i is Periodic Cost of Capital: Calculated from Formula 9.1, this is the matching periodic rate for the cost of capital; conceptually it is the same as the periodic interest rate. Since the cost of capital is typically annual, the periodic rate and nominal rate are often equal to each other.

CY and PY are Compound / Payment Frequency respectively: This is the compounding on the cost of capital, which is typically annual, so CY = 1. Since you are trying to figure out the equivalent annual cash flow, you are looking for annual payments. Therefore, you always have PY = 1. If CY = 1 as usual, then a simple annuity is formed.

$$PV_{ORD} = PMT \left[\frac{1 - \frac{1}{(1+i)^{PY}}}{i} \right]^{\frac{CY}{PY}}$$



How It Works

Follow these steps to calculate the equivalent annual cash flow:

Step 1: For each alternative project, draw a timeline and calculate the net present value using Formula 15.1 and the techniques discussed in Section 15.1.

Step 2: For each alternative project, calculate the periodic interest rate using Formula 9.1 and the number of annuity payments using Formula 11.1. Then solve Formula 11.4 for PMT.

Step 3: Compare the equivalent annual cash flow of each alternative and make the best decision.



Important Notes

Factoring in the unequal life expectancies of projects is important for situations in which you can choose only one out of many mutually exclusive projects. This is the basis for Decision #1. With respect to the other decisions:

- There is only one timeline to consider in Decision #2 (pursuing one course of action), so the issue of unequal life expectancies does not apply.
- In Decision #3 (making multiple choices under constraints), the net present value ratio provides an adequate means of equating different timelines. You do not need the equivalent annual cash flow.



Give It Some Thought

For each of the following decisions where alternative projects solve the same problem and only one can be chosen (a Decision #1 situation), indicate whether the decision should be made through the comparison of NPV or equivalent annual cash flows.

1. Project #1 with a seven-year life; Project #2 with a seven-year life
2. Project #1 with a five-year life; Project #2 with a seven-year life

Solutions:
 1. NPV; each alternative has the same time frame.
 2. Equivalent annual cash flow; each alternative has a different time frame.

Example 15.2A: Which Machine to Purchase when Timelines Are Unequal

Recall the earlier situation in which you can buy only one of two equal \$50,000 machines. Machine #1 has savings of \$20,000 per year, while machine #2 has savings of \$14,000 per year. However, machine #1 has a life expectancy of four years, while machine #2 has a life expectancy of eight years. The cost of capital is 15%. Determine which machine should be purchased and the annual benefit of your choice.

Plan	Notice that you are in a Decision #1 situation and need to choose from these two alternatives. Also notice that the cost of capital is known but the timelines are different. You need to use the equivalent annual cash flow to make the decision.	
Understand	What You Already Know Step 1: The timelines for machine #1 and machine #2 appear below, respectively. Machine #1: $PV = -\\$50,000$, $IY = 15\%$, $CY = 1$, $PMT = \\$20,000$, $PY = 1$, $FV = \\$0$, $Years = 4$ Machine #2: $PV = -\\$50,000$, $IY = 15\%$, $CY = 1$, $PMT = \\$14,000$, $PY = 1$, $FV = \\$0$, $Years = 8$	How You Will Get There Step 1 (continued): Each timeline represents an ordinary simple annuity. Calculate the net present value by applying Formulas 9.1, 11.1, 11.4, and 15.1 to each alternative. Step 2: For each alternative, calculate the equivalent annual cash flow using Formula 11.4 (rearranging for PMT). Step 3: Make the decision. Step 4: Determine on an annual basis how much better your decision is by taking the equivalent annual cash flow of your chosen alternative and subtracting the equivalent annual cash flow of the other alternative.

Perform																														
		Machine #1	Machine #2																											
	Step 1	$i = 15\%/1 = 15\%; N = 1 \times 4 = 4 \text{ payments}$ $PV_{ORD} = \$20,000 \left[\frac{1 - \left[\frac{1}{(1 + 0.15)^{\frac{1}{1}}} \right]^4}{(1 + 0.15)^{\frac{1}{1}} - 1} \right] = \$57,099.56725$ $NPV = \$57,099.56725 - \$50,000 = \$7,099.56725$ $\Rightarrow \$7,100$		$i = 15\%/1 = 15\%; N = 1 \times 8 = 8 \text{ payments}$ $PV_{ORD} = \$14,000 \left[\frac{1 - \left[\frac{1}{(1 + 0.15)^{\frac{1}{1}}} \right]^8}{(1 + 0.15)^{\frac{1}{1}} - 1} \right] = \$62,822.50111$ $NPV = \$62,822.50111 - \$50,000 = \$12,822.50111$ $\Rightarrow \$12,823$																										
	Step 2	$PMT = \frac{\$7,100}{\left[\frac{1 - \left[\frac{1}{(1 + 0.15)^{\frac{1}{1}}} \right]^4}{(1 + 0.15)^{\frac{1}{1}} - 1} \right]} = \$2,486.883996$ $\Rightarrow \$2,487$		$PMT = \frac{\$12,823}{\left[\frac{1 - \left[\frac{1}{(1 + 0.15)^{\frac{1}{1}}} \right]^8}{(1 + 0.15)^{\frac{1}{1}} - 1} \right]} = \$2,857.606699$ $\Rightarrow \$2,858$																										
	Step 3: The best choice is machine #2 because it has a higher equivalent annual cash flow of \$2,858. Step 4: Annual benefit = Machine #2 – Machine #1 = \$2,858 – \$2,487 = \$371																													
Present	Calculator Instructions																													
	<table><tr><th colspan="5">Cash Flows</th></tr><tr><th></th><th colspan="2">Machine #1</th><th colspan="2">Machine #2</th></tr><tr><th>Cash Flow</th><th>Amount (CXX)</th><th>Frequency (FXX)</th><th>Amount (CXX)</th><th>Frequency (FXX)</th></tr><tr><td>CF0</td><td>-50000</td><td>N/A</td><td>-50000</td><td>N/A</td></tr><tr><td>C01 & F01</td><td>20000</td><td>4</td><td>14000</td><td>8</td></tr></table>				Cash Flows						Machine #1		Machine #2		Cash Flow	Amount (CXX)	Frequency (FXX)	Amount (CXX)	Frequency (FXX)	CF0	-50000	N/A	-50000	N/A	C01 & F01	20000	4	14000	8	
	Cash Flows																													
		Machine #1		Machine #2																										
	Cash Flow	Amount (CXX)	Frequency (FXX)	Amount (CXX)	Frequency (FXX)																									
CF0	-50000	N/A	-50000	N/A																										
C01 & F01	20000	4	14000	8																										
<table><tr><th colspan="3">NPV</th></tr><tr><th></th><th>Machine #1</th><th>Machine #2</th></tr><tr><td>I</td><td>15</td><td>15</td></tr><tr><td>NPV</td><td>Answer: 7,099.567254</td><td>Answer: 12,822.50111</td></tr></table>				NPV				Machine #1	Machine #2	I	15	15	NPV	Answer: 7,099.567254	Answer: 12,822.50111															
NPV																														
	Machine #1	Machine #2																												
I	15	15																												
NPV	Answer: 7,099.567254	Answer: 12,822.50111																												
<table><tr><th>Machine</th><th>Mode</th><th>N</th><th>I/Y</th><th>PV</th><th>PMT</th><th>FV</th><th>P/Y</th><th>C/Y</th></tr><tr><td>1</td><td>END</td><td>4</td><td>15</td><td>-7100</td><td>Answer: 2,486.883996</td><td>0</td><td>1</td><td>1</td></tr><tr><td>2</td><td>✓</td><td>8</td><td>✓</td><td>-12823</td><td>Answer: 2,857.606699</td><td>0</td><td>✓</td><td>✓</td></tr></table>				Machine	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y	1	END	4	15	-7100	Answer: 2,486.883996	0	1	1	2	✓	8	✓	-12823	Answer: 2,857.606699	0	✓	✓
Machine	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y																						
1	END	4	15	-7100	Answer: 2,486.883996	0	1	1																						
2	✓	8	✓	-12823	Answer: 2,857.606699	0	✓	✓																						
The smart decision is to purchase machine #2 because it produces the highest equivalent annual cash flow of \$2,858, which represents savings of \$371 more per year than machine #1.																														

Internal Rate of Return

Another method of reaching a decision when choosing whether to pursue a single course of action (Decision #2) involves percentages. While the NPV calculations in Section 15.1 provide an exact monetary magnitude of the project, the common mindset in business focuses on profitability as a percentage and not a dollar amount. Thus, decisions are based on the **internal rate of return** for a project, or IRR for short. The IRR is the annual percentage rate of return on the investment being made such that the net present value of all cash flows in a particular project equals zero.

To interpret the IRR, examine the NPV decision criteria and the relationship to the IRR:

1. If the net present value is greater than or equal to \$0, pursue the project.

1. If the NPV is more than zero, the definition of IRR requires you to find a rate of return such that your present value becomes zero. Mathematically this means that a higher discount rate must be used to calculate your present value. In other words, the IRR is greater than the cost of capital.
2. If the NPV equals zero, by definition the cost of capital and the IRR are the same value.
2. If the net present value is less than \$0, do not pursue the project. The IRR requires you to find a rate of return where the present value becomes zero. Mathematically this means that a lower discount rate must be used to calculate your present value. In other words, the IRR is less than the cost of capital.

This table summarizes how to decide whether to pursue a single course of action using the IRR method instead of the NPV method.

If...	Then...	So...	Decision
NPV > 0	IRR > Cost of Capital	It is profitable since it makes enough money to cover the costs	Pursue the project
NPV = 0	IRR = Cost of Capital	It breaks even and just pays the bills	This is the minimum financial condition to pursue the project
NPV < 0	IRR < Cost of Capital	It is unprofitable and does not make enough money to cover the costs	Do not pursue the project



The Formula

Solving for the internal rate of return requires you to calculate the annually compounded interest rate for the project. For annuities, substituting and rearranging Formula 15.1 produces:

$$\text{NPV} = (\text{Present Value of All Future Cash Flows}) - (\text{Initial Investment})$$

$$\$0 = \text{PMT} \left[\frac{1 - \left[\frac{1}{(1+i)^{\frac{\text{CY}}{\text{PY}}}} \right]^N}{\frac{\text{CY}}{\text{PY}} - 1} \right] - (\text{Initial Investment})$$

$$\text{Initial Investment} = \text{PMT} \left[\frac{1 - \left[\frac{1}{(1+i)^{\frac{\text{CY}}{\text{PY}}}} \right]^N}{\frac{\text{CY}}{\text{PY}} - 1} \right]$$

The only algebraic method to solve this general formula for the periodic interest rate is through trial and error, which is time consuming and inefficient. The same algebraic problem exists if your cash flows consist of multiple lump-sum amounts at different points in time. Assume you have inflows of \$15,000 and \$10,000 at the end of years one and two, respectively. Taking Formula 15.1 you have:

$$\text{NPV} = (\text{Present Value of All Future Cash Flows}) - (\text{Initial Investment})$$

$$\$0 = (\$15,000/(1+i)^1 + \$10,000/(1+i)^2) - (\text{Initial Investment})$$

$$\text{Initial Investment} = \$15,000/(1+i)^1 + \$10,000/(1+i)^2$$

It is algebraically difficult to solve this formula for the periodic interest rate.

Therefore, using the same process as in Section 11.6, you should let the BAII Plus calculator perform the trial and error and arrive at the solution.



How It Works

Follow these steps to solve for the internal rate of return:

Step 1: Draw a timeline to illustrate the cash flows involved in the project.

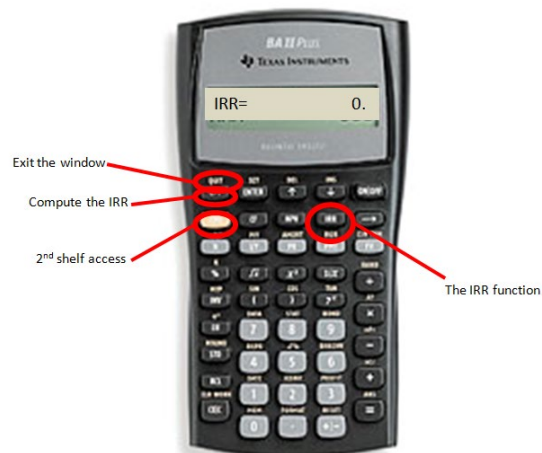
Step 2: If using manual trial and error, set up the appropriate algebraic formula to arrive at an NPV of \$0 and start the sequence of iterations to generate the solution. Alternatively, use technology such as the BAII Plus calculator by entering the cash flows and solving for the IRR.

Step 3: Compare the IRR to the cost of capital and make a decision.



Important Notes

Using the IRR Function on the BAII Plus Calculator. Use the IRR function in conjunction with the CF (cash flow) function. Once you have entered all cash flows, activate the IRR function by pressing the IRR key followed by the CPT button to perform the calculation. The output is a percentage in percent format. To exit the window, press 2nd Quit. Recall that because of the trial-and-error method required, the calculator may briefly hesitate before displaying the solution.



Things To Watch Out For

When making decisions, you use the internal rate of return only to figure out if one particular path should be followed or not (which is Decision #2). The internal rate of return should not be used when making one choice from multiple options (Decision #1) or when making multiple choices under constraints (Decision #3). This rule is in place for two reasons:

1. **The Cost of Capital Is Ignored.** The IRR does not factor in the cost of capital in its computations. Recall that the fundamental concept of time value of money requires all money to be on the same date using an appropriate rate of interest—the cost of capital—before any decision can be made. Therefore, if you have not factored in the cost of capital, then your analysis is incomplete and choosing between different alternatives based solely on the IRR is flawed.
2. **The Magnitude of the Decision Is Ignored.** It can easily happen that an alternative has a high IRR but a low NPV. For example, using a cost of capital of 10% consider two alternatives. Alternative A invests \$1 and one year later returns \$1.50. The IRR is 50%, while the NPV is \$0.36. Alternative B invests \$1,000 and one year later returns \$1,250. The IRR is 25%, while the NPV is \$136.36. If choosing between these two options based on the IRR, you select Alternative A, resulting in a net present value that is \$136 lower than for Alternative B.



Give It Some Thought

In each of the following situations, determine whether the project should be pursued or not.

1. Cost of capital = 15%; IRR = 17%
2. Cost of capital = 12%; IRR = 9%
3. Cost of capital = 14%; IRR = 14%

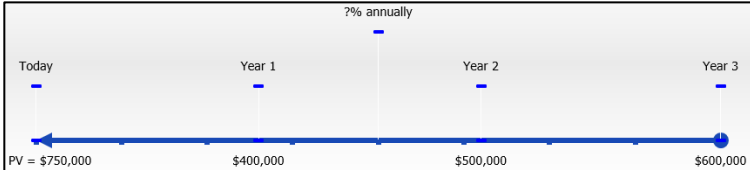
Solutions:

1. Pursue it; IRR > cost of capital
 2. Do not pursue it; IRR < cost of capital
 3. Minimum condition to pursue it; IRR = cost of capital (breaks even)

Example 15.2B: Pursuing a Project Using the IRR Criterion

Tim Hortons has purchased the lease on a three-year onsite concession space in the cafeteria at a local college for \$750,000. The franchise is expected to earn \$400,000, \$500,000, and \$600,000 in profits per year for the first three years, respectively.

- a. What is the investment's internal rate of return?
- b. If the cost of capital is 20%, did Tim Hortons make a smart financial decision?

Plan	You need to calculate the internal rate of return (IRR) for this project. Once calculated, you can compare it to the provided cost of capital to evaluate the decision.																
Understand	What You Already Know Step 1: The timeline for this project appears below. PV = \$750,000, CY = 1 C01 = \$400,000, Years = 1 C02 = \$500,000, Years = 2 C03 = \$600,000, Years = 3	How You Will Get There Step 2: This project involves multiple lump-sum cash flows, so apply Formulas 9.2 and 9.3 (rearranged for PV). Substitute into the rearranged Formula 15.1. Algebraically, this must be solved through trial and error for i (note that $i = \text{IY}$ since the compounding frequency is 1). Alternatively, use the cash flow and internal rate of return function on your calculator. Step 3: Compare the IRR to the cost of capital to make the decision. 															
	Step 2: Cash Flow 1: $N = 1 \times 1 = 1$ compound; $PV = \\$400,000 \div (1 + i)^1$ Cash Flow 2: $N = 1 \times 2 = 2$ compounds; $PV = \\$500,000 \div (1 + i)^2$ Cash Flow 3: $N = 1 \times 3 = 3$ compounds; $PV = \\$600,000 \div (1 + i)^3$ $\\$750,000 = \\$400,000 \div (1 + i)^1 + \\$500,000 \div (1 + i)^2 + \\$600,000 \div (1 + i)^3$ Through trial and error or by using the calculator (see instructions below), the calculated solution is: $\text{IY} = 40.9235\%$ Step 3: $40.9235\% > 20\% \Rightarrow$ smart decision																
Perform	Calculator Instructions <table><tr><th>Cash Flow</th><th>Amount (CXX)</th><th>Frequency (FXX)</th></tr><tr><td>CF0</td><td>-750000</td><td>N/A</td></tr><tr><td>C01 & F01</td><td>400000</td><td>1</td></tr><tr><td>C02 & F02</td><td>500000</td><td>1</td></tr><tr><td>C03 & F03</td><td>600000</td><td>1</td></tr></table> IRR Answer: 40.923472		Cash Flow	Amount (CXX)	Frequency (FXX)	CF0	-750000	N/A	C01 & F01	400000	1	C02 & F02	500000	1	C03 & F03	600000	1
Cash Flow	Amount (CXX)	Frequency (FXX)															
CF0	-750000	N/A															
C01 & F01	400000	1															
C02 & F02	500000	1															
C03 & F03	600000	1															
Present	Since the internal rate of return on this project is 40.9235%, which far exceeds the cost of capital of 20%, Tim Hortons made a very smart financial decision in pursuing this project.																



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Chapter 15 Summary

Key Concepts Summary

Section 15.1: Net Present Value (What Is the Value of My Decision?)

- The characteristics of making decisions including decision types, monetary sources, and interest rates
- Making decisions through net present value
- Handling calculations involving regular cash flows
- Making multiple choices when resources are limited

Section 15.2: Other Measures for Making Decisions (How Else Can I Do It?)

- Choosing between projects of different lengths
- Using internal rate of return to choose whether to pursue one course of action

The Language of Business Mathematics

cash flow A movement of money into or out of a particular project

cost of capital A weighted average of all of the debt and equity financing rates used to provide needed funds for a project.

equivalent annual cash flow An annual annuity payment amount that, when present valued using the cost of capital, arrives at the same NPV as all of the original cash flows.

hard capital rationing A process that requires the calculation of the net present value for each possible project and then the allocation of limited capital resources to these projects efficiently to maximize the combined net present value.

internal rate of return (IRR) The annual rate of return on the investment being made such that the net present value of all cash flows in a particular project equals zero.

net present value (NPV) The difference in today's dollars between all benefits and costs for any given project.

The Formulas You Need to Know

Symbols Used

NPV = net present value

NPV_{RATIO} = net present value ratio

CF_0 = initial cash flow

Formulas Introduced

Formula 15.1 Net Present Value: $NPV = (\text{Present Value of All Future Cash Flows}) - (\text{Initial Investment})$ (Section 15.1)

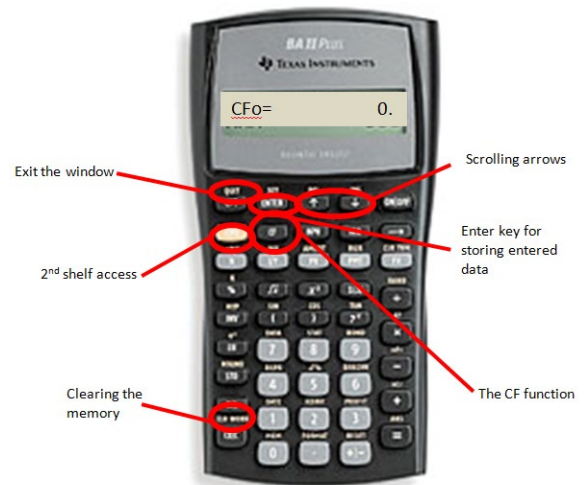
Formula 15.2 Net Present Value Ratio: $NPV_{RATIO} = \frac{NPV}{CF_0}$
(Section 15.1)

Technology

Calculator

Cash Flow Worksheet (CF)

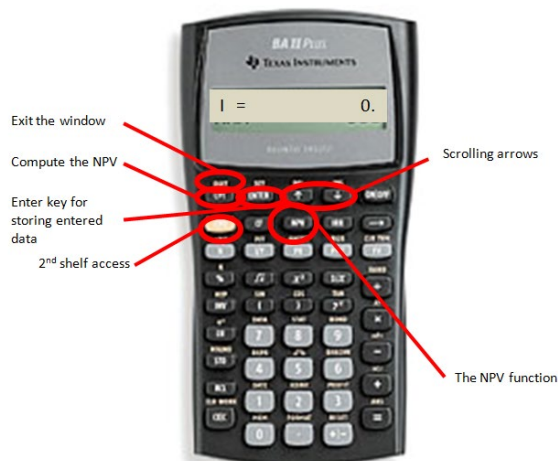
- Access the cash flow function by pressing CF on the keypad.
- Always clear the memory using 2nd CLR Work so that any previously entered data is deleted.
- Use the $\uparrow$ and $\downarrow$ arrows to scroll through the various lines.
- Strictly adhere to the cash flow sign convention when using this function and press Enter after keying in the data.



- To exit the window, press 2nd Quit.

The various lines are summarized below:

- CFo = any cash flow today.
- CXX = a particular cash flow, where XX is one of a series of cash flow numbers starting with 01. You must key in cash flows in order from the first time segment to the last. You cannot skip a time segment, even if it has a value of zero, because each time segment is a placeholder on the timeline.
- FXX = the frequency of a particular cash flow, where XX is the cash flow number. It is how many times *in a row* the corresponding cash flow amount occurs. This allows you to enter recurring amounts together instead of keying them in separately. By default, the calculator sets this variable to 1.



Net Present Value (NPV)

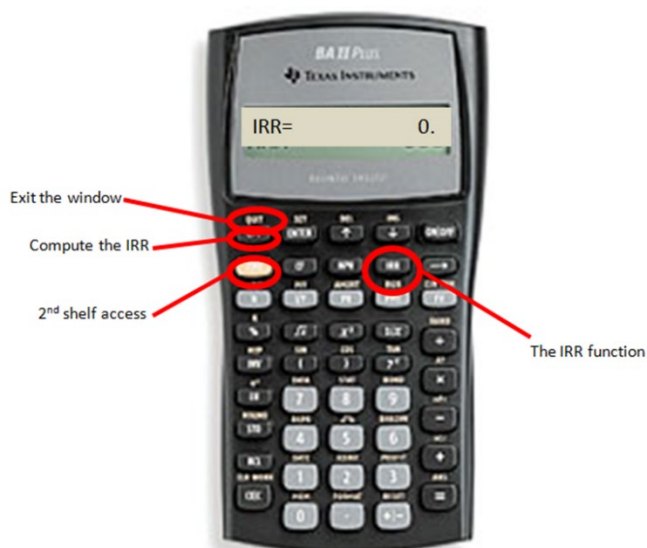
- Use this function after you have entered all cash flows.
- Press NPV on the keypad to access the function.
- Use the ↑ and ↓ arrows to scroll through the window.
- To exit the window, press 2nd Quit.

This window has two lines:

- I = the matching periodic interest rate for the interval of each time segment.
- NPV = the net present value. Press CPT to calculate this amount.

Internal Rate of Return (IRR)

- Use this function after you have entered all cash flows.
- Press IRR on the keypad to access the function.
- Press CPT button to perform the calculation. The output is in percent format.
- To exit the window, press 2nd Quit.





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Appendix A: Exercise Solutions

Note: Some solutions are too big or space consuming to be placed into this appendix such as schedules and tables. Please see the instructor manual for these complete solutions.



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Appendix C: Rounding Rules

Global Textbook Rounding Rules

1. All interim solutions never get rounded unless there is a logical reason or business process that forces the number to be rounded.
2. When writing non-terminating decimals in this textbook, only the first six (or up to six) decimals are written. The horizontal line format is used for repeating decimals. If the number is not a final solution, then it is assumed that all decimals or as many as possible are being carried forward.
3. All final numbers are rounded to six decimals in decimal format, and four decimals in percentage format unless instructions indicate otherwise.
4. Final solutions are rounded according to common business practices, practical limitations, or specific instructions.
5. Zeroes not required at the end of decimals are generally not written unless required to meet a rounding standard or to visually line-up a sequence of numbers.

Topic-Specific Rounding Rules

Section 4.1, Gross Earnings, Salaries: When calculating overtime pay, maintain all decimals in the hourly rate and only round the final pay amount.

Section 4.3, Indexes, Index Numbers: Index numbers with a 100 base are rounded to one decimal. If the index number represents dollars, then a two decimal standard is used. All other indexes require a specific rounding instruction.

Section 5.2, Break-Even Analysis, Break-even Units: All break-even units are always rounded upwards to the next integer regardless of the actual value of the decimal.

Section 7.1, Sales Taxes: The calculation of any sales tax amount is rounded to two decimals.

Section 7.2: Property Taxes: Mill rates are commonly expressed in four decimals and tax rates are commonly expressed in six decimals.

Section 7.3: Exchange Rates: In true markets, exchange rates are commonly expressed with ten decimals or more. A four decimal standard is used for mathematical purposes in this book.

Section 8.5: Demand Loan Repayment Schedules: For simplicity when writing the numbers into repayment schedules and performing all calculations, round all interest calculations to two decimals throughout the table.

Chapter 9 onwards: The calculated value of the periodic interest rate is never rounded in any interim calculation.

Section 9.5, Determining The Interest Rate, Calculating IY: Due to an imprecise single payment used in the calculation, IY is rounded to an integer or early decimal position only if it has a marginal effect. If unable to produce a marginal effect, round as per global rounding rules.

Section 9.7, Determining The Number Of Compounds, Calculating N: Due to an imprecise single payment used in the calculation, N is rounded to the nearest integer if it would be the same result as rounding to three decimals. If rounding to three decimals is not equal to rounding to an integer, then all decimals must be carried forward in further calculations.

Section 9.7, Determining The Number Of Compounds, Calculating Days: Any calculation of days is always rounded to the nearest integer since interest can only be compounded daily at minimum.

Section 11.5, Number Of Annuity Payments, Calculating N: N is rounded to the nearest integer if it would be the same result as rounding to three decimals. If rounding to three decimals is not equal to rounding to an integer, then N is always rounded up to the next nearest integer regardless of the actual value of the decimal.

Section 11.6, Annuities Interest Rates, Calculating IY: Similar to Section 9.5, due to imprecise variable(s) used in the calculation IY is rounded to an integer or early decimal position only if it has a marginal effect. If unable to produce a marginal effect, round as per global rounding rules.

Section 12.1, Deferred Annuities, Moving Money Between Accumulation and Payment Stages: Money is commonly transferred between bank accounts between the two stages requiring the interim amount to be rounded to two decimals.

Section 13.3, Amortization Schedules: When writing amounts in the table, round the answers to two decimals. However, behind the scenes all calculations must be unrounded.

Section 14.1, Bond Coupon Payment Amount: The variable PMT_{BOND} must be unrounded throughout all calculations.

Section 14.1, Bond Cash Price On A Non-Interest Date: Formula 14.5 must use the unrounded value of the Date Price in its calculation.

- Section 14.3, Sinking Fund Schedules:* When writing amounts in the table, round the answers to two decimals. However, behind the scenes all calculations must be unrounded.
- Section 15.1, Net Present Value:* Due to the imprecision of future cost and profit estimates, the net present value is rounded off to the nearest dollar unless otherwise specified.
- Section 15.1, Net Present Value Ratio:* When performing this calculation in hard capital rationing, it is round this variable to two decimals.

Part 1: Mathematics Fundamentals

- (2.1) Percentage Conversion

$$\% = \text{dec} \times 100$$

- (2.2) Rate, Portion, Base

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

- (3.1) Percent Change:

$$\Delta\% = \frac{\text{New} - \text{Old}}{\text{Old}} \times 100$$

- (3.2) Rate Of Change Over Time

$$\text{RoC} = \left(\left(\frac{\text{New}}{\text{Old}} \right)^{\frac{1}{n}} - 1 \right) \times 100$$

- (3.3) Simple Average

$$\text{SAvg} = \frac{\sum x}{n}$$

- (3.4) Weighted Average

$$\text{WAvg} = \frac{\sum wx}{\sum w}$$

- (3.5) Geometric Average

$$\text{GAvg} = \left([(1 + \Delta\%_1) \times (1 + \Delta\%_2) \times \dots \times (1 + \Delta\%_n)]^{\frac{1}{n}} - 1 \right) \times 100$$

Part 2: Business Applications

- (4.1) Salary and Hourly Gross Earnings

$$\text{GE} = \text{Regular Earnings} + \text{Overtime Earnings} + \text{Holiday Earnings} + \text{Statutory Holiday Worked Earnings}$$

- (4.2) Annual Income Tax

$$\text{Income Tax} = \sum (\text{Eligible Income In Tax Bracket} \times \text{Tax Bracket Rate})$$

- (4.3) Index Numbers

$$\text{Index Number} = \frac{\text{Chosen quantity}}{\text{Base quantity}} \times \text{Base value}$$

- (4.4) Purchasing Power Of A Dollar

$$\text{PPD} = \frac{\$1}{\text{CPI}/100}$$

- (4.5) Real Income

$$\text{RI} = \frac{\text{Nominal Income}}{\text{CPI}/100}$$

- (5.1) Unit Variable Cost

$$\text{VC} = \frac{\text{TVC}}{n}$$

- (5.2) Net Income Using A Total Revenue And Cost Approach

$$\text{NI} = n(S) - (\text{TFC} + n(\text{VC}))$$

- (5.3) Unit Contribution Margin

$$\text{CM} = S - \text{VC}$$

- (5.4) Net Income Using Total Contribution Margin Approach

$$\text{NI} = n(\text{CM}) - \text{TFC}$$

- (5.5) Contribution Rate If Unit Information Known

$$\text{CR} = \frac{\text{CM}}{S} \times 100$$

- (5.6) Contribution Rate If Aggregate Information Known

$$\text{CR} = \frac{\text{TR} - \text{TVC}}{\text{TR}} \times 100$$

- (5.7) Unit Break-even

$$n = \frac{\text{TFC}}{\text{CM}}$$

- (5.8) Dollar Break-even

$$\text{TR} = \frac{\text{TFC}}{\text{CR}}$$

- (6.1) Single Discount

$$N = L \times (1 - d)$$

- (6.2a & 6.2b) Discount Amount

$$D\$ = L \times d \quad \text{or} \quad D\$ = L - N$$

- (6.3) Multiple Discounts

$$N = L \times (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$$

- (6.4) Single Equivalent Discount

$$d = 1 - (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$$

- (6.5) The Selling Price Of A Product

$$S = C + E + P$$

- (6.6) Markup Amount

$$M\$ = E + P$$

- (6.7) The Selling Price Of A Product Using Markup Amount

$$S = C + M\$$$

- (6.8) Markup On Cost Percentage

$$\text{MoC}\% = \frac{M\$}{C} \times 100$$

- (6.9) Markup On Selling Price Percentage

$$\text{MoS}\% = \frac{M\$}{S} \times 100$$

- (6.10) The Sale Price Of A Product

$$S_{\text{onsale}} = S \times (1 - d)$$

- (6.11a & 6.11b) Markdown Amount

$$D\$ = S \times d \quad \text{or} \quad D\$ = S - S_{\text{onsale}}$$

- (6.12) Markdown Percentage

$$d = \frac{D\$}{S} \times 100$$

- (6.13) Maintained Markup

$$MM = \frac{M\$(n_1) + (M\$ - D\$(n_2))}{n_1 + n_2}$$

- (7.1) Selling Price Including Tax

$$S_{tax} = S + (S \times \text{Rate})$$

- (7.2) GST/HST Remittance

$$\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$$

- (7.3) Property Taxes

$$\text{Property Tax} = \sum(AV \times PTR)$$

- (7.4) Currency Exchange

$$\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$$

Part 3: Single Payment Financial Applications

- (8.1) Simple Interest

$$I = Prt$$

- (8.2) Simple Interest For Single Payments

$$S = P(1 + rt)$$

- (8.3) Interest Amount For Single Payments

$$I = S - P$$

- (9.1) Periodic Interest Rate

$$i = \frac{IY}{CY}$$

- (9.2) Number of Compound Periods For Single Payments

$$N = CY \times \text{Years}$$

- (9.3) Compound Interest For Single Payments

$$FV = PV(1 + i)^N$$

- (9.4) Interest Rate Conversion

$$i_{New} = (1 + i_{Old})^{\frac{CY_{Old}}{CY_{New}}} - 1$$

- (10.1) Periodic Interest Amount

$$I = PV \times i$$

- (10.2) Purchasing Power Of A Dollar (Compound Interest Method)

$$PPD = \frac{\$1}{(1 + i)^N} \times 100$$

Part 4: Annuity Payments Financial Applications

- (11.1) Number Of Annuity Payments

$$N = PY \times \text{Years}$$

- (11.2) Ordinary Annuity Future Value

$$FV_{ORD} = PMT \left[\frac{\left[(1 + i)^{\frac{CY}{PY} N} - 1 \right]}{(1 + i)^{\frac{CY}{PY}} - 1} \right]$$

- (11.3) Annuity Due Future Value

$$FV_{DUE} = PMT \left[\frac{\left[(1 + i)^{\frac{CY}{PY} N} - 1 \right]}{(1 + i)^{\frac{CY}{PY}} - 1} \right] \times (1 + i)^{\frac{CY}{PY}}$$

- (11.4) Ordinary Annuity Present Value

$$PV_{ORD} = PMT \left[\frac{1 - \left[\frac{1}{(1 + i)^{\frac{CY}{PY} N}} \right]}{(1 + i)^{\frac{CY}{PY}} - 1} \right]$$

- (11.5) Annuity Due Present Value

$$PV_{DUE} = PMT \left[\frac{1 - \left[\frac{1}{(1 + i)^{\frac{CY}{PY} N}} \right]}{(1 + i)^{\frac{CY}{PY}} - 1} \right] \times (1 + i)^{\frac{CY}{PY}}$$

- (12.1) Future Value Of A Constant Growth Ordinary Annuity

$$FV_{ORD} = PMT(1 + \Delta\%)^{N-1} \left[\frac{\left[\frac{(1 + i)^{\frac{CY}{PY} N}}{1 + \Delta\%} - 1 \right]}{\frac{(1 + i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right]$$

- (12.2) Future Value Of A Constant Growth Annuity Due

$$FV_{DUE} = PMT(1 + \Delta\%)^{N-1} \left[\frac{\left[\frac{(1 + i)^{\frac{CY}{PY} N}}{1 + \Delta\%} - 1 \right]}{\frac{(1 + i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right] \times (1 + i)^{\frac{CY}{PY}}$$

- (12.3) Present Value Of A Constant Growth Ordinary Annuity

$$PV_{ORD} = \frac{PMT}{1 + \Delta\%} \left[\frac{1 - \left[\frac{1 + \Delta\%}{(1 + i)^{\frac{CY}{PY} N}} \right]}{\frac{(1 + i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right]$$

- (12.4) Present Value Of A Constant Growth Annuity Due

$$PV_{DUE} = \frac{PMT}{1 + \Delta\%} \left[\frac{1 - \left[\frac{1 + \Delta\%}{(1 + i)^{\frac{CY}{PY}}} \right]^N}{\frac{(1 + i)^{\frac{CY}{PY}}}{1 + \Delta\%} - 1} \right] \times (1 + i)^{\frac{CY}{PY}}$$

- (12.5) Ordinary Perpetuity Present Value

$$PV_{ORD} = \frac{PMT}{(1 + i)^{\frac{CY}{PY}} - 1}$$

- (12.6) Perpetuity Due Present Value

$$PV_{DUE} = PMT \left(\frac{1}{(1 + i)^{\frac{CY}{PY}} - 1} + 1 \right)$$

Part 5: Amortization & Special Financial Concepts

- (13.1) Interest Portion Of An Ordinary Single Payment

$$INT = BAL \times ((1 + i)^{\frac{CY}{PY}} - 1)$$

- (13.2) Principal Portion Of A Single Payment

$$PRN = PMT - INT$$

- (13.3) Principal Portion For A Series Of Payments

$$PRN = BAL_{P1} - BAL_{P2}$$

- (13.4) Interest Portion For A Series Of Payments

$$INT = N \times PMT - PRN$$

- (13.5) Interest Portion Of A Due Single Payment

$$INT_{DUE} = (BAL - PMT) \times ((1 + i)^{\frac{CY}{PY}} - 1)$$

- (14.1) The Cash Price For Any Bond

$$\text{Cash Price} = PRI + AI$$

- (14.2) Bond Coupon Annuity Payment Amount

$$PMT_{BOND} = \text{Face Value} \times \frac{CPN}{CY}$$

- (14.3) Bond Price On An Interest Payment Date

$$\text{Date Price} = \frac{FV}{(1 + i)^N} + PMT_{BOND} \left[\frac{1 - \frac{1}{(1 + i)^N}}{i} \right]$$

- (14.4) Bond Premium or Discount

$$\text{Premium or Discount} = PRI - \text{Face Value}$$

- (14.5) Bond Cash Price On A Non-Interest Payment Date

$$\text{Cash Price} = (\text{Date Price})(1 + i)^t$$

- (14.6) Accrued Interest On A Non-Interest Payment Date

$$AI = PMT_{BOND} \times t$$

- (14.7) Interest Portion Of A Sinking Fund Single Payment Due

$$INT = (BAL + PMT) \times ((1 + i)^{\frac{CY}{PY}} - 1)$$

- (14.8) The Annual Cost Of The Bond Debt

$$ACD = (\text{Face Value} \times CPN) + (PMT \times PY)$$

- (14.9) The Book Value Of The Bond Debt

$$BVD = \text{Bonds Outstanding} - BAL$$

- (15.1) Net Present Value

$$NPV = (\text{Present Value Of All Future Cash Flows}) - (\text{Initial Investment})$$

- (15.2) Net Present Value Ratio

$$NPV_{RATIO} = \frac{NPV}{CFO}$$

Notation Summary J. Olivier • Business Math: A Step-by-Step Handbook, 2021 Revision C

% = percentage

$\Delta\%$ = constant growth rate per payment interval *or* the percent change from one period to another

Σ = summation symbol meaning to add everything up

ACD = annual cost of the bond debt

AI = accrued interest

AV = assessed value of a property

BAL = principal balance immediately after an annuity payment

BAL_{P1} = principal balance immediately prior to the first payment in a series of annuity payments

BAL_{P2} = principal balance immediately after the last payment in a series of annuity payments

Base = the whole quantity

BVD = book value of the bond debt

C = cost

Cash Price = bond price summing market price & accrued interest

CFo = initial cash flow

CM = unit contribution margin

CPI = Consumer Price Index

CPN = nominal bond coupon rate of interest

CR = contribution rate

Current Currency = currency being converted from

CY or C/Y = compounds per year, or compounding frequency

CY_{New} = the new compounding frequency

CY_{Old} = the old compounding frequency

D\$ = discount amount or markdown amount

d = discount rate or markdown rate

Date Price = bond price on the interest payment date preceding the sale date

dec = any number in decimal format

Desired Currency = currency you are converting to

Discount = bond discount amount

E = expenses

Exchange Rate = per unit conversion rate for current currency

Face Value = face value

FV = maturity value or future value in dollars

FV_{DUE} = future value of annuity due

FV_{ORD} = future value of ordinary annuity

GAvg = geometric average

GE = gross earnings

I = interest amount in dollars

i = periodic interest rate

i_{New} = the new periodic interest rate

INT = interest portion

INT_{DUE} = interest portion of a due single annuity payment

INT_{SFDUE} = the interest portion of any single annuity due sinking fund payment

i_{old} = the old periodic interest rate

IY or I/Y = nominal interest rate per year

L = list price

ln = natural logarithm

M\$ = markup amount

MM = maintained markup

MoC% = markup on cost percentage

MoS% = markup on selling price percentage

n = a frequency or count or total

N = in merchandising it is the net price; in single payment compound interest it is the number of compound periods; in annuity compound interest it is the number of annuity payments

New = the value that a quantity has become

NI = net income

NPV = net present value

NPV_{RATIO} = net present value equivalent annual ratio

Old = the value that a quantity used to be

P = profit; or principal

PMT = annuity payment amount

PMT_{BOND} = bond annuity payment

Portion = the part of the whole

PPD = purchasing power of a dollar

Premium = bond premium amount

PRI = bond market price

PRN = principal portion

Property Tax = property tax amount

PTR = property tax rate

PV = principal or present value

PV_{DUE} = present value of annuity due

PV_{ORD} = present value of ordinary annuity

PY or P/Y = payments per year or payment frequency

r = interest rate (in decimal format)

Rate = portion and base relationship; sales tax rate

Remit = tax remittance

RI = real income

RoC = rate of change per time period

S = regular or unit selling price; or sum of principal/interest

SAvg = simple average

SBE = the selling price at the break-even point

Sonsale = sale price

Stax = selling price including taxes

t = time or term; time ratio

Tax Collected = total tax collected through sales

Tax Paid = total tax paid through purchases

TFC = total fixed costs

TR = total revenue

TVC = total variable cost

VC = unit variable cost

w = weighting factor

WAvg = weighted average

x = any individual piece of data

Years = the term of an annuity



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